PHY 564
Advanced Accelerator Physics
Lecture 17

Effects of synchrotron radiation

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Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics

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I. Effects of synchrotron radiation

Synchrotron radiation is a side product of charged particles acceleration when their trajectory is curved. Storage rings are doing exactly this – they bend the particles trajectories in circle-like trajectories and force them to circulate for very many turns. Generating synchrotron (spontaneous bending-magnet or undulator/wiggler) radiation tuned to be on of very important application of accelerators – this radiation is very bright and has no competitive sources. Hence dozens of dedicated "so-called" light sources are built and used by broad community of scientists and engineers for fundamental and applied researches. They are typically are electron (with only few exemptions of positron rings) storage rings with low transverse emittance and small energy spread, nowadays equipped with multiple undulators, which further enhance brightness of the radiated photon beams. Typical energy is from ~ 1 GeV to 8 GeV ($\gamma \sim$ 2,000 – 16,000), e.g. electrons are ultra-relativistic. There are books and courses on generating and using synchrotron radiation – it is not part of this accelerator course. Neither is the detailed derivation of the synchrotron radiation process. Detailed description of synchrotron radiation can be found in your favorite E&M book.

Here we will need only few specific features of synchrotron radiation, which we will use without derivations. We will also assume that a) our particles are ultra-relativistic, $\gamma >> 1$; b) losses for synchrotron radiation per turn are a small portion of the particle' energy, i.e. we can treat it as a perturbation which introduces some damping, while not affecting tunes of the particle. For ultra-relativistic particles **E** and *pc* become essentially indistinguishable:

$$\mathbf{E} = pc / \sqrt{1-\gamma^{-2}} \cong pc(1+1/2\gamma^2) = pc\left(1+O(\gamma^{-2})\right) \tag{1}$$

hence we will use it where does not causes any confusion.

One of the most critical feature for the damping of the transverse of the synchrotron radiation that it is local and is confined within a solid angle $\sim 1/\gamma^2$ around the direction of the particle MECHANICAL momentum. This feature comes from the fact that in instant co-moving frame, charged particle radiates dipole radiation with energy proportional to square of acceleration $\vec{a}$, but the radiation has zero total momentum (see Fig. 1 A):

$$d\mathbf{E}_r = \frac{2e^2}{3c^3}\vec{a}^2 dt_o; \; d\vec{\mathrm{P}}_r = 0. \tag{2}$$

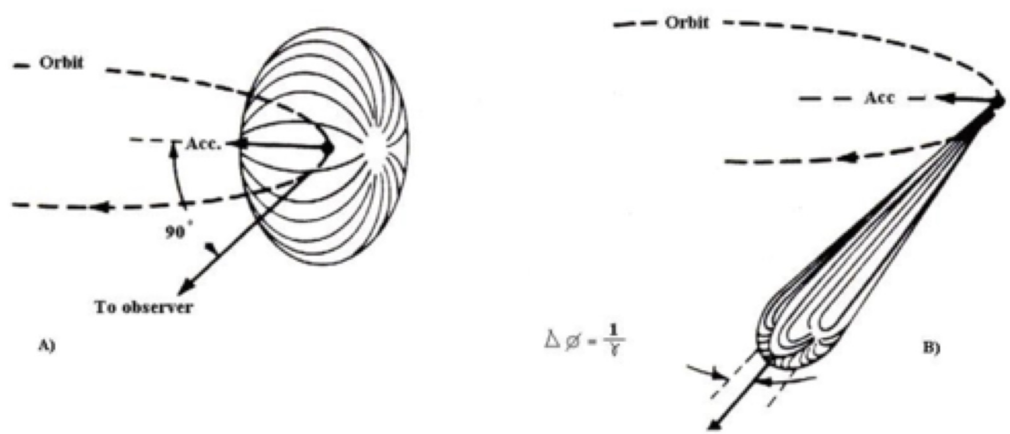


Fig. 1. Radiation of relativistic particle in a co-moving frame (A) and in the lab-frame (B)

Let's consider a photon radiated in the commoving frame with frequency ω propagating with an angle θ:

$$E_{ph} = \hbar\omega; \; \vec{p}_{ph}c = E_{ph}\left(\vec{z}\cos\theta + \vec{e}_\perp \sin\theta\right) \tag{3}$$

where $\vec{e}_\perp$ is an unit vector transverse to the particle direction of motion. In co-moving frame, a photon with opposite direction is generated with the same probability:

$$E_{ph} = \hbar\omega; \; \vec{p}_{ph}c = -E_{ph}\left(\vec{z}\cos\theta + \vec{e}_\perp \sin\theta\right) \tag{4}$$

Transformation to the lab-frame give boost to the photons as

$$E_l = \gamma\left(E_l + \beta p_z c\right) = \gamma\hbar\omega\left(1+\beta\cos\theta\right);\ p_{\perp l}c = p_{\perp}c = \hbar\omega\sin\theta;$$
$$p_{zl}c = \gamma\left(p_z c + \beta E_l\right) = \gamma\hbar\omega\left(\cos\theta + \beta\right);\ \tan\theta_l = \frac{p_{\perp l}}{p_{zl}} = \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta+\beta}; \tag{5}$$

It means that photon radiated in forward direction, $\cos\theta = 1$, have

$$\gamma\left(1+\beta\right) \cong 2\gamma$$

boost in their energy (momentum), while photon radiated back-wards $\cos\theta = -1$, have their energy chewed-up by Lorentz transformation to minuscular level:

$$\gamma\left(1-\beta\right) \cong \frac{1}{2\gamma}$$

Notice that photons radiated at 90 degrees, $\cos\theta = 0$, will have about ½ of the energy boost

$$E_l = \gamma\hbar\omega;\ p_{\perp l}c = \hbar\omega;\ p_{zl}c = \gamma\beta\cdot\hbar\omega;\ \tan\theta_l = \frac{1}{\gamma\beta};\theta_l \cong \frac{1}{\gamma}$$

and propagate in forward direction with angle ~ $1/\gamma$. In short, as shown in Fig. 22-1 (B), the most energetic photons are concentrated in $1/\gamma^2$ solid angle (cone).

It is possible to show that dipole radiation (2) in the co-moving frame, when boosted to the lab frame becomes

$$\frac{d\mathbf{E}_{rad}}{dt} = \frac{2e^2}{3c^3}\gamma^3\left(\vec{a}^2 - \left(\vec{a}\vec{\beta}\right)^2\right) = \frac{2e^2}{3c^3}\gamma^3\vec{a}^2\left(1-\beta^2\cos^2\vartheta\right) \tag{6}$$

While we are considering here storage ring with bending magnets, where $\vec{a} \perp \vec{\beta}$, it worth noting that intensity of radiation (6) strongly depends on angle ϑ between the acceleration $\vec{a}$ and beam velocity $\vec{\beta}$. Specifically, when $\vec{a} \perp \vec{\beta}$,

$$\frac{d\mathbf{E}_{rad}}{dt} = \frac{2e^2}{3c^3}\gamma^3 \vec{a}^2 \tag{7}$$

and when $\vec{a} // \vec{\beta}$:

$$\frac{d\mathbf{E}_{rad}}{dt} == \frac{2e^2}{3c^3}\gamma^3 \vec{a}^2 \left(1-\beta^2\right) = \frac{2e^2}{3c^2}\gamma \vec{a}^2 \tag{8}$$

In external EM field (6) becomes:

$$\frac{d\mathbf{E}_{rad}}{dt} = \frac{2e^4}{3m^2c^3}\gamma^2 \left\{ \left(\vec{E} + \left[\vec{\beta} \times \vec{B}\right]\right)^2 - \left(\vec{\beta} \cdot \vec{E}\right)^2 \right\} \tag{9}$$

Note that power of radiation is proportional to the charge of particles in power four (for example for ions in RHIC, ***e->Ze***!) and inverse proportional to the particle mass squared. In general, synchrotron radiation losses are growing as very high power of γ.

Still, in linear accelerators $\vec{E} // \vec{\beta}$ and radiation losses

$$\text{Linear accelerator: } \frac{d\mathbf{E}_{rad}}{dt} = \frac{2e^4 \vec{E}^2}{3m^2c^3} \tag{10}$$

are energy (γ) independent.

This is why linear accelerators are considered for a possible high-energy electron-position collider. But, let's go back to circular machines with $\vec{E}_{//}=0$ and

$$\frac{d\mathbf{E}_{rad}}{dt}=\frac{2e^4}{3m^2c^3}\gamma^2\left(\vec{E}+\left[\vec{\beta}\times\vec{B}\right]\right)^2 \tag{11}$$

For the reference particle

$$K(s)=\frac{1}{\rho(s)}=-\frac{e}{\beta_o p_o c}\left(\beta_o B_y+E_x\right);E_y=\beta_o B_x;$$

$$\vec{E}_\perp+\left[\vec{\beta}\times\vec{B}\right]=-\vec{n}\cdot K(s)\frac{\beta_o p_o c}{e}=-\vec{n}\cdot K(s)\cdot\gamma_o\beta_o\frac{mc^2}{e}$$

we can express the radiated power by reference particle through the (radius of) curvature using $cdt_o=\beta_o ds$:

$$\frac{d\mathbf{E}_{SR}}{dt}=\frac{2}{3}\gamma_o^{\,4}\beta_o^{\,3}e^2K^2(s)\equiv\frac{2}{3}\frac{\gamma_o^{\,4}\beta_o^{\,3}e^2}{\rho^2(s)}\cong\frac{2}{3}\gamma_o^{\,4}\frac{e^2}{\rho^2(s)} \tag{12}$$

Assuming that the energy change per turn is small we can calculate the energy loss per turn for reference particle to be:

$$\Delta\mathbf{E}_{SR}\cong\frac{2}{3}e^2\gamma_o^{\,4}\int_0^C\frac{ds}{\rho^2(s)}\equiv\frac{2}{3}e^2\gamma_o^{\,4}\int_0^C K(s)^2\,ds=\frac{2}{3}e^2\left(\frac{E_o}{mc^2}\right)^4\int_0^C K(s)^2\,ds \tag{13}$$

e.g. for a given geometry of a storage ring energy losses are proportional to the fourth power of relativistic factor γ or the particle energy.

It means that when electrons ($m_e c^2$ =0.511 MeV) of the same energy with protons ($m_p c^2$ =938.272 MeV) will radiate

$$\left(\frac{m_p c^2}{m_e c^2}\right)^4 \approx 1.14\times 10^{13}$$

more than the proton energy in the same geometry of the storage ring. We should note that magnetic field of proton ring has to be 1,836-fold higher than that of the electron ring, which is not always possible. Hence, this astronomical number is a bit deceiving.

If the magnetic field (here we should remember that we can generate much stronger magnetic field and electric fields) is given, than we need to use (11) to see that the local losses are proportional to $\left(E_o / mc^2\right)^2$. Since the circumference of the storage ring with grow proportionally with the energy $\sim E_o / B_o$ – thus the total losses for one turn in a ring with given magnetic field are $\sim E_o^{\,3} B_o / \left(mc^2\right)^2$ (only 3.37 x 10^6 more radiation for an electron than for a proton). Thus, with exception of LHC, where protons do radiate their full energy of 7 TeV in 10+ hours, and "very future" FCC where even proton will see significant radiation damping, the synchrotron radiation is always important feature in electron/position storage rings. It is interesting that we can add energy losses by the synchronous particle into the Hamiltonian. Since we are considering the relative energy losses to be very small per turn, $\left|\Delta E_{SR}\right| << E_o$, we can add it to an averaged longitudinal Hamiltonian from previous lectures:

$$\left\langle \mathcal{H}_s \right\rangle = \eta_\tau \frac{\pi_\tau^{\,2}}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} + \frac{1}{C}\frac{\Delta \mathbf{E}_{SR}}{p_o c}\tau;$$

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C\cdot p_o c}\left(eV_{RF}\sin\left(k_o h_{rf}\tau\right) - \Delta \mathbf{E}_{SR}\right); \tag{14}$$

Which is equivalent to a forced pendulum:

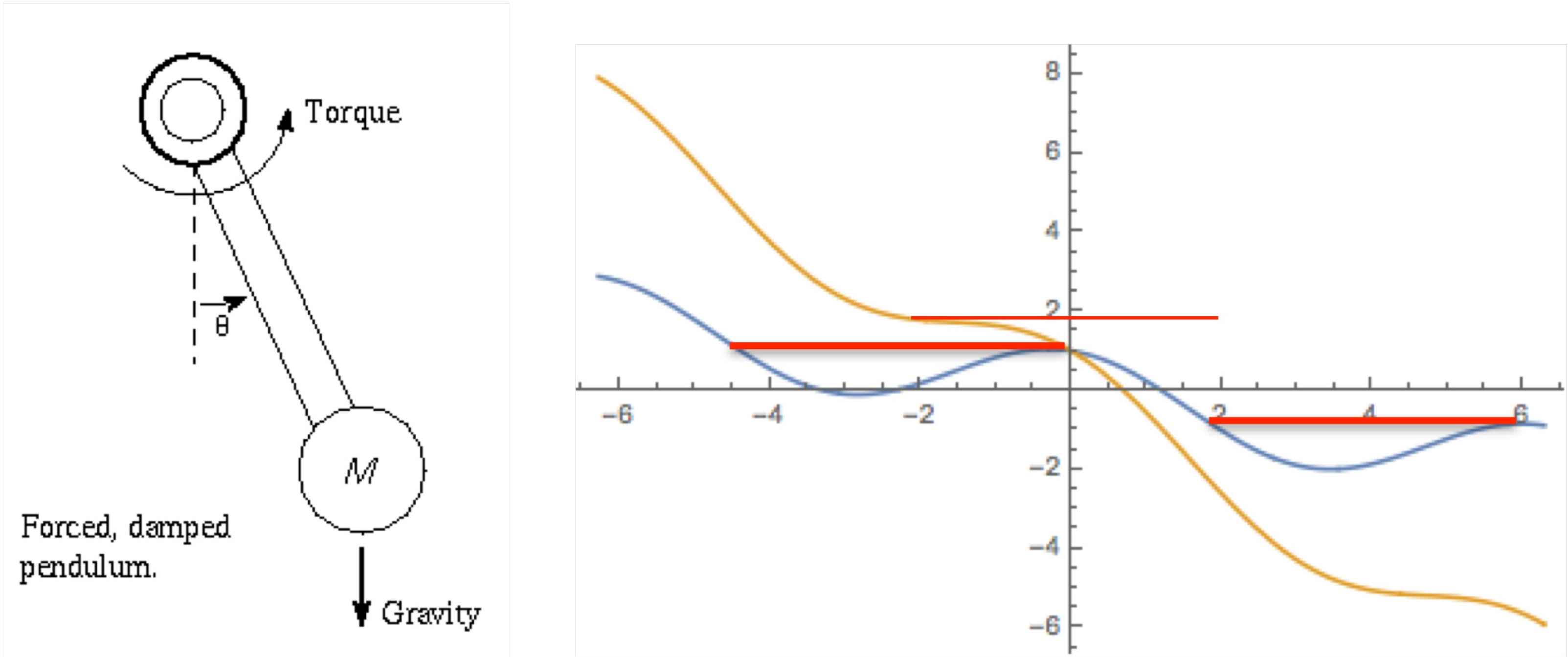


Fig. 2. A forced pendulum (left) and effective potential in Hamiltonian for $|eV_{RF}| > \Delta E_{SR}$ (blue) and for $|eV_{RF}| < \Delta \mathbf{E}_{SR}$ (yellow/orange).

If torque is too large (i.e. RF voltage is insufficient to compensate the energy losses $|eV_{RF}| < \Delta \mathbf{E}_{SR}$), the pendulum will rotate faster and faster. Otherwise, there are two stationary points in the phase space:

$$\varphi_o = \sin^{-1}\left(\frac{\Delta \mathbf{E}_{SR}}{eV_{RF}}\right), \pi_\tau = 0 \tag{15}$$

one of which (depending on the sign of η_τ) is stable and the other is the separatrix crossing (unstable) point.

Fig. 3 demonstrates the phase space trajectories

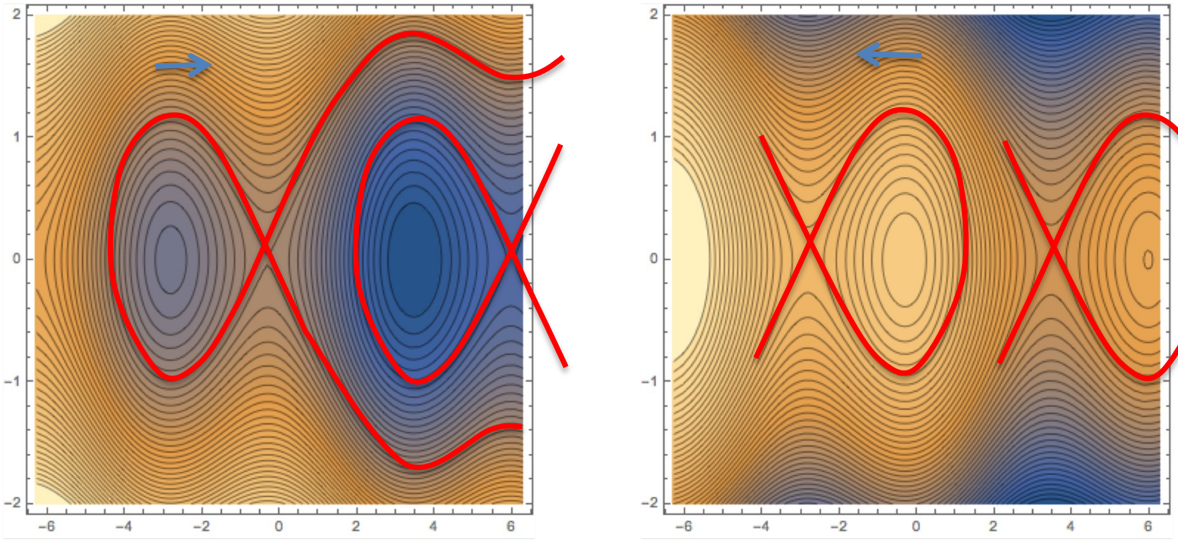


Fig. 3. Phase-space trajectories for Hamiltonian (14) with positive η_τ (a) and negative η_τ and $\Delta \mathbf{E}_{SR} = 0.3|eV_{RF}|$. There are stable (confined) areas of the synchrotron oscillations.

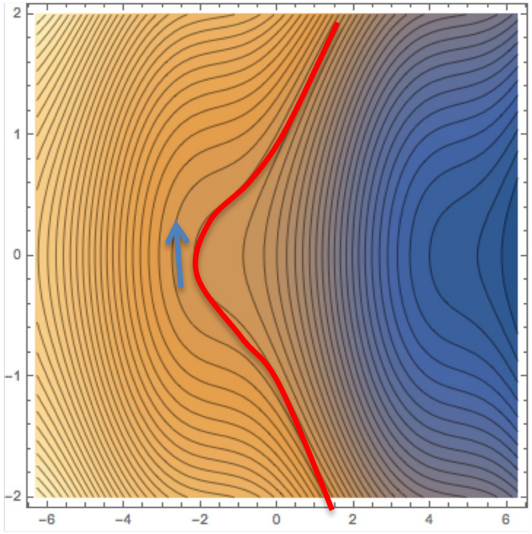


Fig. 4. Phase-space trajectories for Hamiltonian (14) with positive η_τ and $\Delta \mathbf{E}_{SR} = 1.2|eV_{RF}|$. Particles are no longer confined.

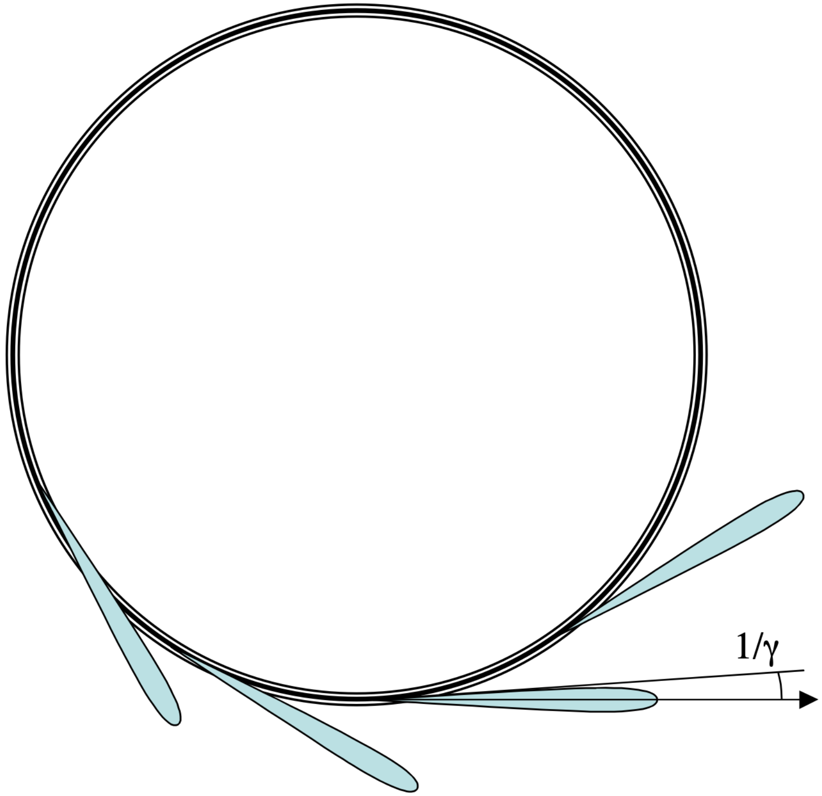
It is easy to show that expanding the Hamiltonian (15) around the stable point yields frequency of synchrotron oscillations of:

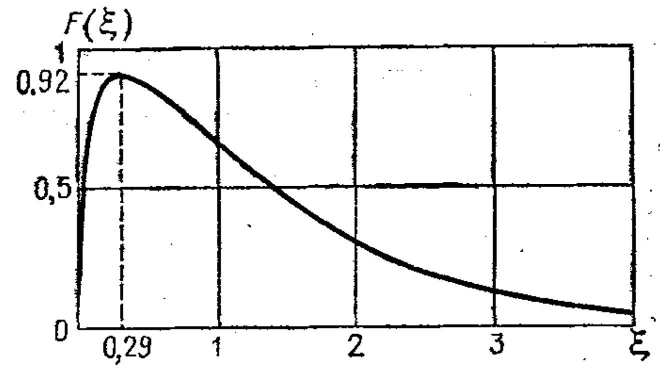
$$C\cdot\Omega_s=\sqrt{2\pi h_{rf}\frac{\left|\eta_\tau eV_{RF}\cos\varphi_o\right|}{p_oc}};Q_s=\sqrt{h_{rf}\frac{\left|\eta_\tau eV_{RF}\cos\varphi_o\right|}{2\pi p_oc}}. \tag{16}$$

To a degree, this is just a trivial shift of the accelerating phase to compensate for the radiated energy. This is the must for electron storage ring with any reasonable energy.

Now, let's switch to less trivial problem of what is happening with small betatron and longitudinal oscillations caused by synchrotron radiation? First, let's remind ourselves that the radiation is propagating in a very narrow cone into the direction of the beam motion. It means that the recoil (lost momentum) has direction opposite to that of the particle's motion. It is possible to show that this is accurate assumption with accuracy $\sim\gamma_o^{-2}$. In other words, the loss of the particle momentum is proportional to the total energy loss of the particle with recoil directed against the direction of the momentum:

$$\frac{\vec{p}'}{p}=-\frac{1}{p}\frac{dp_{rad}}{ds}\cdot\frac{\vec{p}}{p}\cong-\frac{1}{\mathbf{E}}\frac{d\mathbf{E}_{rad}}{ds}\cdot\left(\vec{\tau}+\vec{n}\cdot x'+\vec{b}\cdot y'\right)+O\left(x'^2,y'^2,\gamma^{-2}\right) \tag{17}$$





Landau, Lifshitz, Classical Theory of Fields

$$\xi = \frac{\omega}{\omega_c}; \omega_c = \frac{2}{3}\gamma^3 \frac{c}{\rho}$$

Fig. 5. Synchrotron radiation is a fan of well-directed radiation with vertical opening ~ 1/γ. Radiation at a certain point of curved trajectory directed along momentum of the particle and is confined within ~1/γ opening angle in both horizontal and vertical directions.

Synchrotron radiation has a white spectrum, which spectral power density growing as $\omega^{1/3}$ at low frequencies as $\omega < \omega_c$ and falling exponentially at high frequencies $\omega > \omega_c$ The critical frequency, $\omega_c = 2\pi c/\lambda_c$ divides radiated power by halves: the half $\omega > \omega_c$ and the other half at $\omega > \omega_c$. Critical wavelength of λ_c is ~ $R//\gamma^3$ and

$$\omega_c = \frac{2}{3}\gamma^3 \frac{c}{\rho},$$

where ρ is the radius of curvature. Hence, the most of the radiation happens in the bending magnets.

Returning to s as independent coordinate let introduce also the relative energy loss rate:

$$I_{SR} = \frac{1}{\mathbf{E}_o}\frac{d\mathbf{E}_{rad}}{ds} = \frac{2}{3\mathbf{E}_o}\cdot\frac{e^4\mathbf{E}^2}{m^4c^8}\{\cdot(\vec{E}-[\vec{\beta}\times\vec{B}])^2-(\vec{\beta}\cdot\vec{E})^2\}\cdot(1+Kx) \tag{18}$$

where we considered that $\frac{\mathrm{v}dt}{ds}=(1+Kx)$. We need to extend (18) as function of the energy and position:

$$I = I_o\cdot(1+c_x\cdot x+c_y\cdot y+c_{px}\pi_x+c_{py}\pi_y+c_\tau\cdot\tau+2\pi_\tau);\ \mathbf{E}^2=\mathbf{E}_\mathbf{o}^{\,2}\cdot\left(1+2\pi_\tau+\pi_\tau^{\,2}\right)$$
$$I_o = \frac{2}{3}\frac{e^4\mathbf{E}_\mathbf{o}}{m^4c^8}(B_y+E_x)^2_{eo}. \tag{19}$$

with

$$c_x = K-\frac{2e}{K\mathbf{E}_o}\frac{\partial}{\partial x}\left(E_x+B_y\right)+\frac{e^2B_s^{\,2}}{K\mathbf{E}_o^{\,2}};\ c_y=-\frac{2e}{K\mathbf{E}_o}\frac{\partial}{\partial y}\left(E_x+B_y\right)-\frac{e^2B_sE_s}{K\mathbf{E}_o^{\,2}};$$
$$c_{px}=\frac{2eE_s}{K\mathbf{E}_o};c_{py}=\frac{2eB_s}{K\mathbf{E}_o};c_\tau=\frac{2e}{K\mathbf{E}_o}\frac{\partial}{\partial t}\left(E_x+B_y\right). \tag{20}$$

where for consistency we use β_o=1 where is possible.

Adding linearized energy and momentum loos terms to linearized Hamiltonian equation results on

$$\frac{d}{ds}X = (\mathbf{SH} - I_o(s)\cdot\mathbf{G})X;\ \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ L & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c_x & c_{px} & c_y & c_{py} & c_\tau & 2 \end{bmatrix}. \tag{21}$$

where we need to express the in $\vec{n}\cdot x' + \vec{b}\cdot y'$ (17) using components of the canonical momenta

$$p_x = p_o x' = p_o(\pi_x - Ly);\ p_y = p_o y' = p_o(\pi_y + Lx);$$

$$\frac{d\pi_{x1}}{ds} = -\frac{\partial h}{\partial x} - I_o\cdot(\pi_x - Ly);\ \frac{d\pi_y}{ds} = -\frac{\partial h}{\partial y} - I_o\cdot(\pi_y + Lx);$$

First, the trace of the new D-matrix is no longer zero, which means that determinants of the transport matrices are no longer unit:

$$\det\mathbf{M}((s_1|s_2)) = \exp\left\{-\int_{s_1}^{s_2} Trace(I_o(s)\cdot\mathbf{G})ds\right\} = \exp\left\{-4\int_{s_1}^{s_2} I_o(s)ds\right\};$$

$$\det\mathbf{M}(s|s+C) = \det T(s) = \prod_1^6 \lambda_i = \prod_1^3(\lambda_i\lambda_i^*) = \prod_1^3|\lambda_i|^2 = \exp\left\{-4\int_0^C I_o(s)ds\right\} \tag{22}$$

The above formula is known as a theorem of sums of the decrements in storage rings: i.e. the sum of the decrements of all three eigen modes of oscillations (2 betatron and one synchrotron) is equal to four times relative loss of energy into synchrotron radiation. This is the rate with which 6D phase space volume shrinks. Surprisingly, it is not very hard to make one of the modes (usually horizontal or synchrotron, which are strongly coupled) to experience anti-damping caused by synchrotron radiation, i.e. to have exponential growth.

Before we go into full-fledged calculations, let's look at a simple picture of what's happening in the vertical plane in a ring without transverse coupling (i.e. for majority of the ring's designs). As shown in Fig. 6, the radiation reduces transverse component of the particle's momentum and

$$\frac{p'_y}{p_o} = -\frac{1}{\mathbf{E}}\frac{d\mathbf{E}_{rad}}{ds}\cdot y'; \; \alpha = \left\langle \frac{1}{2\mathbf{E}}\frac{d\mathbf{E}_{rad}}{ds} \right\rangle_C$$
$$y \cong \sqrt{a\beta_y}\cos\left(\psi_y + \varphi\right)\cdot e^{-\alpha\cdot s} \tag{23}$$

where we neglected effect of the distribution of the radiation along the circumference of the machine replacing it by an average damping. The average energy loss for synchrotron radiation is restored by an RF cavity. By design, the RF cavity boosts only longitudinal momentum of the particles (along s), while leaving transverse momenta unchanged. This feature completes the circuit, which is important for understanding of the radiation damping. Thus, we can conclude that the vertical betatron oscillations in the storage wing will damp e-fold when particle radiated twice its energy.

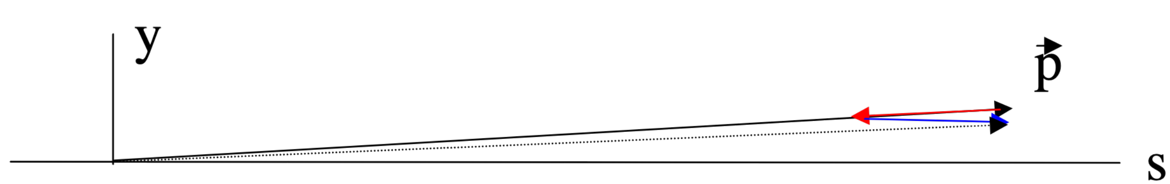


Fig. 6. Particle loses parts of its vertical momentum during radiation process. RF cavity restores (in average) only longitudinal component of the loss momentum. As the result, transverse momentum is damped.

How to handle situation in a general case? Assuming that the effect is weak, we can use perturbation theory and eigen vectors.

$$X = \mathrm{Re}\sum_{k=1}^{3} a_k Y_k e^{i(\psi+\varphi)}$$

$$\mathrm{Re}\sum_{k=1}^{3} \frac{da_k}{ds} Y_k e^{i(\psi_k+\varphi_k)} = -I_o(s)\cdot \mathbf{G}\cdot \mathrm{Re}\sum_{m=1}^{3} a_m Y_m e^{i(\psi_m+\varphi_m)} \tag{24}$$

$$\frac{da_k}{ds} = -\frac{-I_o(s)}{2i} Y_k^{*T}\mathbf{SG}\cdot e^{-i(\psi_k+\varphi_k)}\left(\sum_{m=1}^{3} a_m Y_m e^{i(\psi_m+\varphi_m)} + c.c.\right).$$

where $c.c.$ stands for complex conjugate. With exception of one term, $a_k Y_k$ in the sum, the rest are fast oscillating and average to zero. The remaining term yields us

$$\frac{da_k}{ds} = -\left\langle \frac{I_o}{2i} Y_k^{*T}\mathbf{SG}Y_k \right\rangle a_k;$$

$$a_k = a_{ko}\exp\left(-\zeta_k \frac{s}{C}\right);\zeta_k = \frac{1}{2i}\int_o^C I_o(s) Y_k^{*T}(s)\mathbf{SG}(s) Y_k(s)\, ds. \tag{25}$$

In practice we are interested in real part of the decrement (increment) of radiation damping ξ_k. Imaginary part gives us addition tune shift, which we can usually neglect. We can use either exact eigen vectors, or an approximate one we derived for slow synchrotron motion.

To make rather large expressions (25) into a manageable, let's consider a case of plane orbits with uncoupled transverse motion:

$$c_x = K - \frac{2e}{KE_o}\frac{\partial}{\partial x}\left(E_x + B_y\right);\ c_y = 0;\ c_{px} = 0; c_{py} = 0; c_\tau = 0;$$

$$\mathbf{SG} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c_x & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; Y_y \to \begin{bmatrix} 0 \\ Y_y \\ 0 \end{bmatrix}; Y_y = \begin{bmatrix} \mathrm{w}_y \\ \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \end{bmatrix}$$

$$Y_y^{*T}\mathbf{SG}Y_y = \begin{bmatrix} \mathrm{w}_y \\ \mathrm{w}'_y - \dfrac{i}{\mathrm{w}_y} \end{bmatrix}^T \begin{bmatrix} \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \\ 0 \end{bmatrix} = i; \tag{26-v}$$

as expected giving us:

$$\zeta_y = \frac{1}{2}\frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o}. \tag{27-v}$$

Getting decrements for horizontal betatron oscillations and synchrotron oscillations is the just a bit more work:

$$Y_x^T = \left[\begin{array}{cccccc} \mathrm{w}_x & \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} & 0 & 0 & \mathrm{w}_{x\tau} & 0 \end{array}\right];$$

$$Y_s^T = \frac{1}{i\mathrm{w}_s}\left[\begin{array}{cccccc} \eta_x & \eta' & 0 & 0 & \mathrm{w}^2_s & i \end{array}\right];$$

$$\mathrm{w}_{x\tau} = \eta_x\left(\mathrm{w}'_x + \frac{i}{\mathrm{w}_x}\right) - \eta'_x \mathrm{w}_x; \tag{26-h}$$

$$Y_x^{*T}\mathbf{SG}Y_x = \mathrm{w}_x\left(\mathrm{w}'_x + \frac{i}{\mathrm{w}_x}\right) + c_x \mathrm{w}^*_{x\tau} = i(1 - c_x\eta_x) + \mathrm{Re}\, Y_x^{*T}\mathbf{SG}Y_x.$$

which give damping coefficient for horizontal betatron oscillations and synchrotron oscillations:

$$\mathrm{Re}\,\zeta_x = \frac{1}{2}\frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o}(1 - \xi_{xs});\ \ \zeta_{xs} = \frac{\left\langle K^3\eta_x\left(1 - \dfrac{2}{K^2}\dfrac{e}{p_o c}\dfrac{\partial(E_x + B_y)}{\partial x}\right)\right\rangle}{\langle K^2\rangle}; \tag{27-h}$$

$$\mathrm{Re}\,\zeta_s = \frac{3}{2}\frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o} - \mathrm{Re}\,\zeta_x = \frac{1}{2}\frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o}(2 + \xi_{xs}).$$

where we simply used the theorem for the sum of the decrements.

$$\xi_k = \mathrm{Re}\,\zeta_k \Big/ \left(\frac{1}{2}\frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o}\right); \sum_{k=1}^{3} \xi_k = 4;$$

$$2\zeta_k = \xi_k \frac{\Delta\mathbf{E}_{SR}}{\mathbf{E}_o} = \xi_k \frac{2}{3}\frac{e^2\gamma^3}{mc^2}\int_0^C K(s)^2\, ds; \tag{28}$$

Finally, for the actions of the oscillation we can write:

$$\frac{dI_k}{ds} = \frac{1}{2}\frac{d|a_k|^2}{ds} = -2\,\mathrm{Re}\,\zeta_k \cdot I_k; \quad I_k = I_{ko} e^{-\xi_k \frac{\Delta E_{SR}}{E_o}\cdot\frac{s}{C}} \tag{29}$$

just to see that it damps with twice of the amplitude decrement.

Let's now discuss what stops beam oscillations to completely decay? In fact, there are quantum fluctuations of the radiation process, i.e. the fact that charged particles radiate energy by quanta in a random time – e.g. the fact that radiation process is almost instantaneous and uncorrelated with previous radiation. It is definitely and directly exciting synchrotron oscillations. Let's consider a small but a sudden jump of particle's energy of $\Delta\mathbf{E}$: we have to expand it using our eigen modes:

$$\Delta X = \mathrm{Re}\sum_{k=1}^{3}\Delta\left(a_k e^{i\varphi}\right)Y_k e^{i\psi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\Delta E}{E_o} \end{bmatrix}; \Delta\left(a_k e^{i\varphi}\right) = \frac{e^{-i\psi}}{i} Y_k^{*T}\mathbf{S}\Delta X = \frac{e^{-i\psi}}{i}\frac{\Delta E}{E_o} Y_k^*(5); \tag{30}$$

$$I_k = \frac{|a_k|^2}{2};\ \Delta I_k = \mathrm{Re}\, a^*_k \frac{e^{-i\psi}}{i}\frac{\Delta\mathbf{E}}{\mathbf{E}_o} Y_k^*(5) + \frac{1}{2}\left(\frac{\Delta\mathbf{E}}{\mathbf{E}_o}\right)^2 \left|Y_k^*(5)\right|^2$$

where $Y_k^*(5)$ is the fifth element (tau) of *k-th* eigen vector. Expressed through 4-component vectors for betatron eigen vectors it is.

$$Y_k(5) = \eta^T S Y_k$$

If energy jumps are occurring at random time (positions), than first term in ΔI_k averages out, while the second remains. Let's now consider quantum fluctuations of spontaneous radiation. To calculate the growth of the action, we will need statistically averages

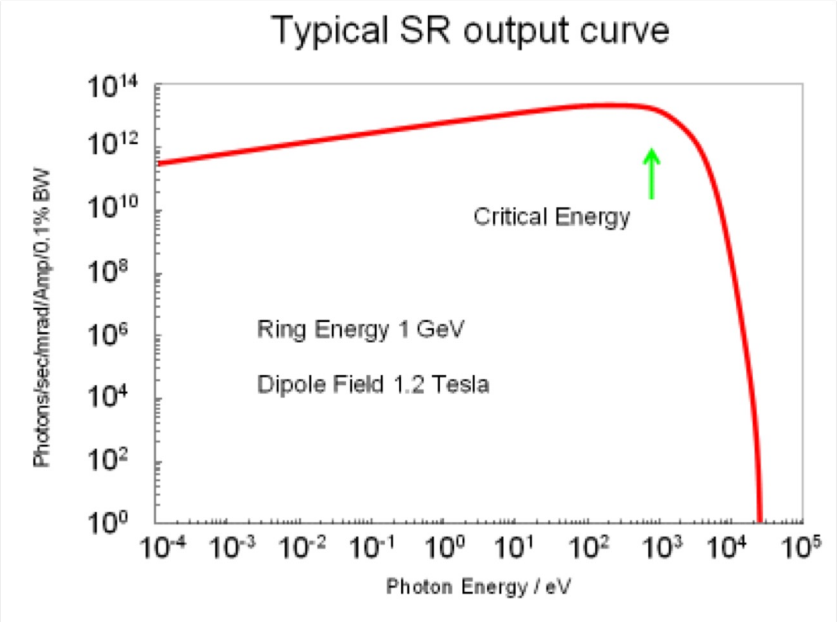
$$\left\langle \left(\frac{\Delta \mathbf{E}}{\mathbf{E}_o} \right)^2 \right\rangle = \frac{\left\langle \left(\hbar \omega_{SR} \right)^2 \right\rangle}{\mathbf{E}_o^{\ 2}} \tag{31}$$

The probability of radiating the photon with energy $\hbar\omega$ is proportional to the spectral density of synchrotron radiation:

$$N'_{ph} \hbar \omega = \frac{U(u)}{c} du; u = \frac{\omega}{\omega_c}; \omega_c = \frac{2}{3} \gamma^3 \frac{c}{\rho};$$

$$U(u) = \frac{3\sqrt{3}}{4\pi} e^2 c K^2 \gamma^4 u \int_u^\infty K_{5/3}(u') du',$$

where ω_c is critical frequency of synchrotron radiation, and $K_{5/3}$ is McDonald's function of 5/3 order.



To calculate the rate of the fluctuation term

$$\sum_{photons} N'_{ph}(\hbar\omega)^2 = \int_0^\infty \frac{u\cdot\hbar\omega_c}{c} K_{5/3}(u)du = \frac{55}{24\sqrt{3}} e^2\hbar c K^3\gamma^7;$$

$$\left(\frac{dI_k}{ds}\right)_{qf} = \frac{1}{2}\sum_{photons} N'_{ph}\left(\frac{\hbar\omega}{E_o}\right)^2 \left|Y_k^*(5)\right|^2 = \frac{55}{48\sqrt{3}}\frac{e^2}{mc^2}\frac{\hbar}{mc}|K|^3\gamma^5\left|Y_k^*(5)\right|^2.$$

Hence, the quantum fluctuations cause the growth of the actions while the radiation damping causes it to decay (29). Combining two we have

$$\frac{d\langle I_k\rangle}{ds} = -2\frac{\zeta_k}{C}\langle I_k\rangle + D_k;\; D_k = \frac{55}{48\sqrt{3}}\frac{e^2}{mc^2}\frac{\hbar}{mc}K^3\gamma^5\left|Y_k^*(5)\right|^2. \quad (32)$$

Hence, there is a stationary average value of the actions of each mode:

$$\langle a_k^2\rangle = 2\langle I_k\rangle = \frac{\langle D_k\rangle C}{\zeta_k} = \frac{55}{32\sqrt{3}}\gamma^2\frac{\hbar}{mc}\frac{\left\langle |K|^3\left|Y_k^*(5)\right|^2\right\rangle}{\xi_k\left\langle K(s)^2\right\rangle}; \quad (33)$$

Instead of calculating the longitudinal action, it is traditional to calculate the stationary energy spread:

$$\frac{d}{ds}\left\langle\left(\frac{\delta\mathbf{E}}{\mathbf{E}_o}\right)^2\right\rangle - 2\frac{\zeta_s}{C}\left\langle\left(\frac{\delta\mathbf{E}}{\mathbf{E}_o}\right)^2\right\rangle + \frac{55}{48\sqrt{3}}\frac{e^2}{mc^2}\frac{\hbar}{mc}K^3\gamma^5;$$

$$\left\langle\left(\frac{\delta\mathbf{E}}{\mathbf{E}_o}\right)^2\right\rangle = \frac{55}{32\sqrt{3}}\gamma^2\frac{\hbar}{mc}\frac{\left\langle |K|^3\right\rangle}{\xi_s\left\langle K^2\right\rangle}. \quad (34)$$

For the uncoupled vertical betatron motion and plane orbit, the expressions are much simpler:

Vertical stationary action is formally zero! This is, of cause, does not happens, because of a week error coupling. This is typically a main determining factor in such storage rings. Otherwise:

$$\left\langle a_x^2 \right\rangle = \frac{55}{32\sqrt{3}}\gamma^2 \frac{\hbar}{mc} \frac{\left\langle |K|^3 \left(\left(\mathrm{w}_x \eta_x' - \mathrm{w}_x' \eta_x\right)^2 + \left(\frac{\eta_x}{\mathrm{w}_x}\right)^2 \right) \right\rangle}{\left(1-\xi_{xs}\right)\left\langle K^2 \right\rangle};$$

$$\left\langle \left(\frac{\delta \mathbf{E}}{\mathbf{E}}\right)^2 \right\rangle = \frac{55}{32\sqrt{3}}\gamma^2 \frac{\hbar}{mc} \frac{\left\langle |K|^3 \right\rangle}{\left(2+\xi_{xs}\right)\left\langle K^2 \right\rangle}; \qquad (35)$$

$$\xi_{xs} = \left\langle \eta_x K^3 (1-2\overline{n}) \right\rangle; \overline{n} = \frac{e}{K^2 E_o} \frac{\partial\left(E_x + B_y\right)}{\partial x}.$$

Note that $\Lambda_c = \hbar / mc$ is the particle's Compton wavelength. It sets scale for $\left\langle a_k^2 \right\rangle$, which we

an emittance of k-th mode. What is important to remember that stationary emittances are proportional to γ^2, the relative energy spread and beam sized are proportional to γ.

In practice vertical betatron oscillations are excited via coupling, which always exist in real world, with horizontal oscillations. Nevertheless, theoretically it is interesting what would happen in ideal accelerator with perfect plane symmetry and without coupling. Than, as we found above, the energy jumps would not excite vertical oscillations. The only remaining excitation will come from the fact that photons are emitted not exactly long the direction of particle's motion but into a angle ~ 1/γ. It means that each emitted photon will generate transverse momentum of

$$\delta p_y \cong \frac{\delta \mathbf{E}_i}{c}\theta_{iy} \to \delta y' = \frac{\delta \mathbf{E}_i}{\mathbf{E}_o}\theta_{iy}. \tag{36}$$

In order to find the diffusion coefficient for amplitude of vertical oscillations

$$\delta\left(a_y e^{i\varphi_y}\right) = \frac{e^{-i\psi}}{i}\left(\mathrm{w}_y, \mathrm{w}'_y - i/\mathrm{w}_y\right)\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \delta p_y \end{pmatrix} = \frac{e^{-i\psi}}{i}\mathrm{w}_y \frac{\delta \mathbf{E}_i}{\mathbf{E}_o}\theta_{iy}; \tag{37}$$

we need to find probability of radiating a photon at specific energy (e.g. harmonic) at specific vertical angle. This expression for k-th harmonic of radiation ($\omega_o = \rho / c$) for γ >>1 is

$$\kappa(k,\theta_v) = \frac{k^2}{4\pi^3\gamma^4}\left(g^2 K_{2/3}\left(\frac{k}{3}g^{3/2}\right) + g\sin^2\theta_v K_{1/3}\left(\frac{k}{3}g^{3/2}\right)\right); g = 1-\beta^2\cos\theta_v; \tag{38}$$

After averaging it yields necessary diffusion coefficient of

$$D_y = \frac{13\pi}{12\sqrt{3}} r_e \Lambda \gamma^3 \left\langle K \mathrm{w}^2{}_y \right\rangle; \ \zeta_y = \frac{1}{2}\frac{\Delta E_{SR}}{E_o} = \frac{1}{3} r_e \gamma^3 \left\langle K^2 \right\rangle C;$$
$$\left\langle I_y \right\rangle = \frac{\left\langle a_y{}^2 \right\rangle}{2} = \pi \frac{13\sqrt{3}}{24} \Lambda \frac{\left\langle K \beta_y \right\rangle}{C \left\langle K^2 \right\rangle}; \ \Lambda = \frac{\hbar}{mc} \cong 3.86 \cdot 10^{-13} m. \tag{39}$$

The result is rather remarkable that the RMS action in vertical direction does not depend on energy and rather weak function if other coefficient. We can estimate it as.

$$\beta_y \sim C / 2\pi Q_y; \frac{\langle K \rangle}{\left\langle K^2 \right\rangle} \sim \rho; \left\langle I_y \right\rangle \sim \frac{13\sqrt{3}}{48} \Lambda \frac{\rho}{Q_y}. \tag{40}$$

What we learned today

- Synchrotron radiation is a natural phenomenon when charged particles are accelerated
- There is dramatic difference between acceleration in the direction if particle's motion with that transverse to it (e.g. bending of particle trajectory). The transverse acceleration generates γ^2-fold more radiation and energy loss compared with the longitudinal
- Synchrotron radiation is a strong factor of particle's relativistic factor γ – wavelength of photons ~ R/γ^3 and energy loss per turn grows as γ^4! Hence we focused on the case of ultra-relativistic particles with $\gamma >> 1$
- Since in a given field, the power of synchrotron radiation is inverse proportional to the square of particle's mass. It means that light charged particles, such as electrons and positions radiate much more energy - ~ $4 \cdot 10^6$ fold more - than protons and antiprotons, which are about 2,000 heavier. Synchrotron radiation is becoming important for protons at LHC energies and above.
- We treated synchrotron radiation momentum loss as a weak perturbation to calculate damping decrements of synchrotron radiation
- We used quantum fluctuation – e.g. randomness – of radiation to calculate diffusion coefficients for energy and betatron motion
- We finished with deriving average values for actions (square of amplitudes) for each modes of oscillation
- We are leaving question about the resulting distribution function for next class, where we will be solving Fokker-Plank equation