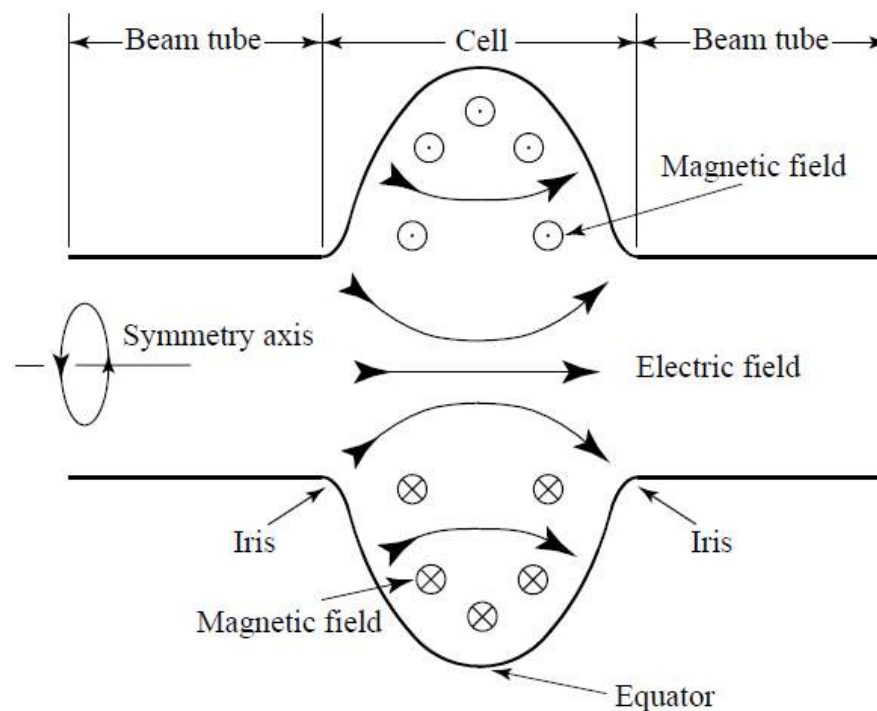


# PHY 554

## Fundamentals of Accelerator Physics

### ▫ Lecture 9: Introduction to RF accelerators

Vladimir N. Litvinenko



# Acknowledgement

Next three lectures use some materials from courses on RF and Superconducting RF (SRF) accelerators taught by Prof. S. Belomestnykh at SBU/BNL and USPAS, which can be found on the following websites:

*[http://case.physics.stonybrook.edu/index.php/Courses:\\_P554\\_Fundamentals\\_of\\_Accelerator\\_Physics,\\_Spring\\_2014](http://case.physics.stonybrook.edu/index.php/Courses:_P554_Fundamentals_of_Accelerator_Physics,_Spring_2014)*

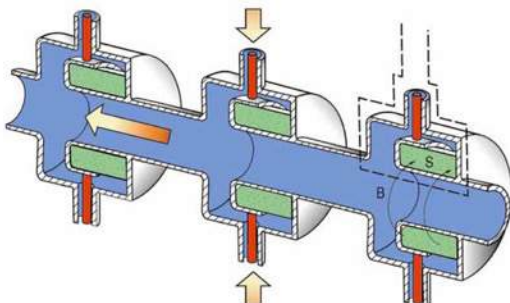
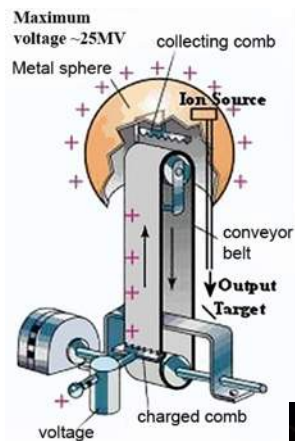
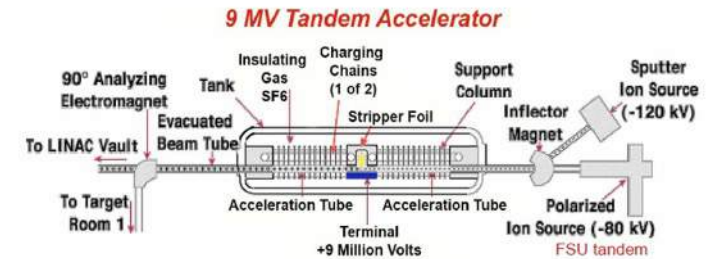
*<http://case.physics.stonybrook.edu/index.php/CASE:Courses>*

*<https://sites.google.com/site/srfsbu11/>*

*<http://uspas.fnal.gov/materials/materials-table.shtml>*

I added additional material as needed

# Linear accelerators: from electrostatic to RF

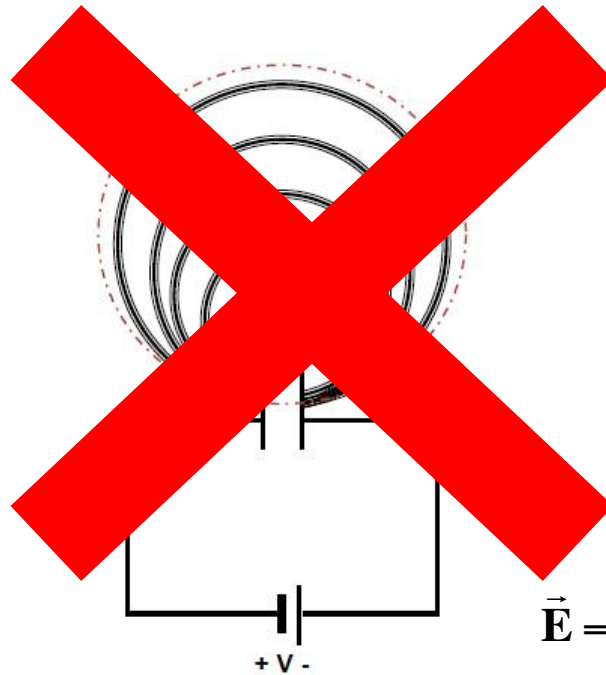


Can one gain the energy again and again by passing through a DC accelerating gap?

Electrostatic: what is the limit ?

Maxwell equations and energy conservation law!

$$\Delta E = e \oint \vec{E} \cdot d\vec{l} = -\frac{e}{c} \frac{\partial}{\partial t} \left( \int \vec{H} \cdot d\vec{s} \right)$$



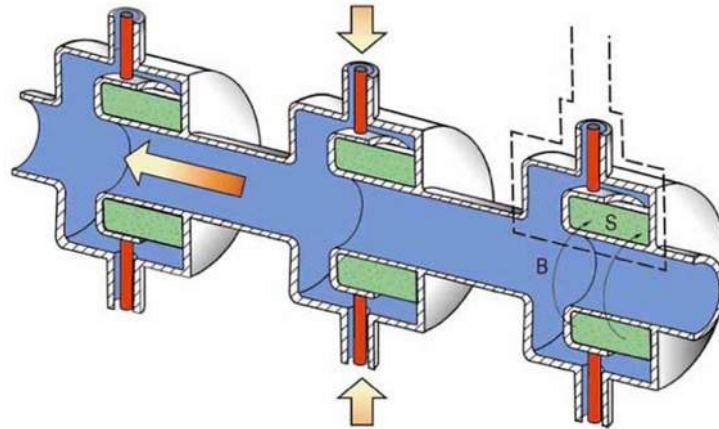
DC

$$\Delta E = e \oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{E} = -\vec{\nabla} \varphi \rightarrow E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can not cheat the Maxwell equations

# Induction linacs: linear betatrons

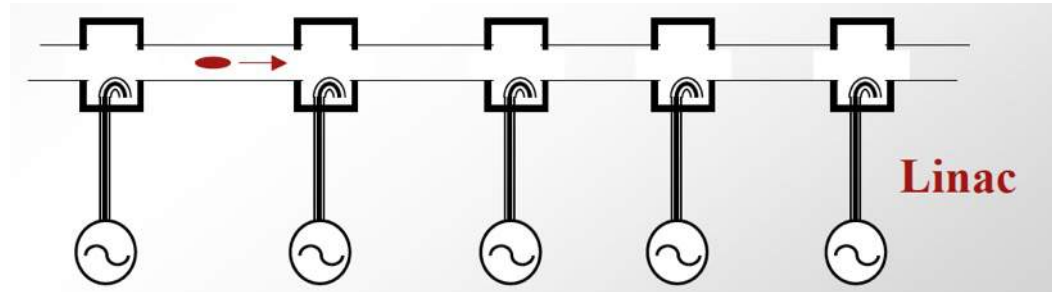


$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \left( \int \vec{H} \cdot d\vec{s} \right)$$

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatedly low rep-rate (kHz)

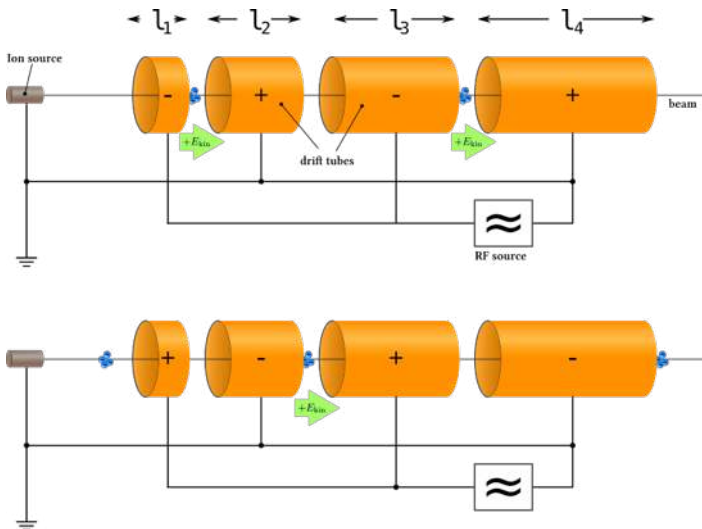
# How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle's trajectory) electric field
- It also has longitudinal structure (cells) which alternates the direction of the field
- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently

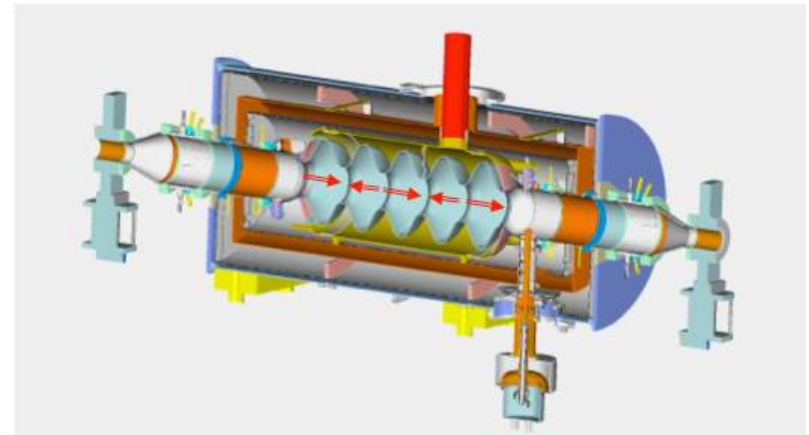


$$\frac{dE}{dt} = e\vec{E} \cdot \vec{v} \rightarrow \text{sign}(\vec{E} \cdot \vec{v}) = \text{const}$$

Wideröe's linac:  $\beta = v/c$  is changing

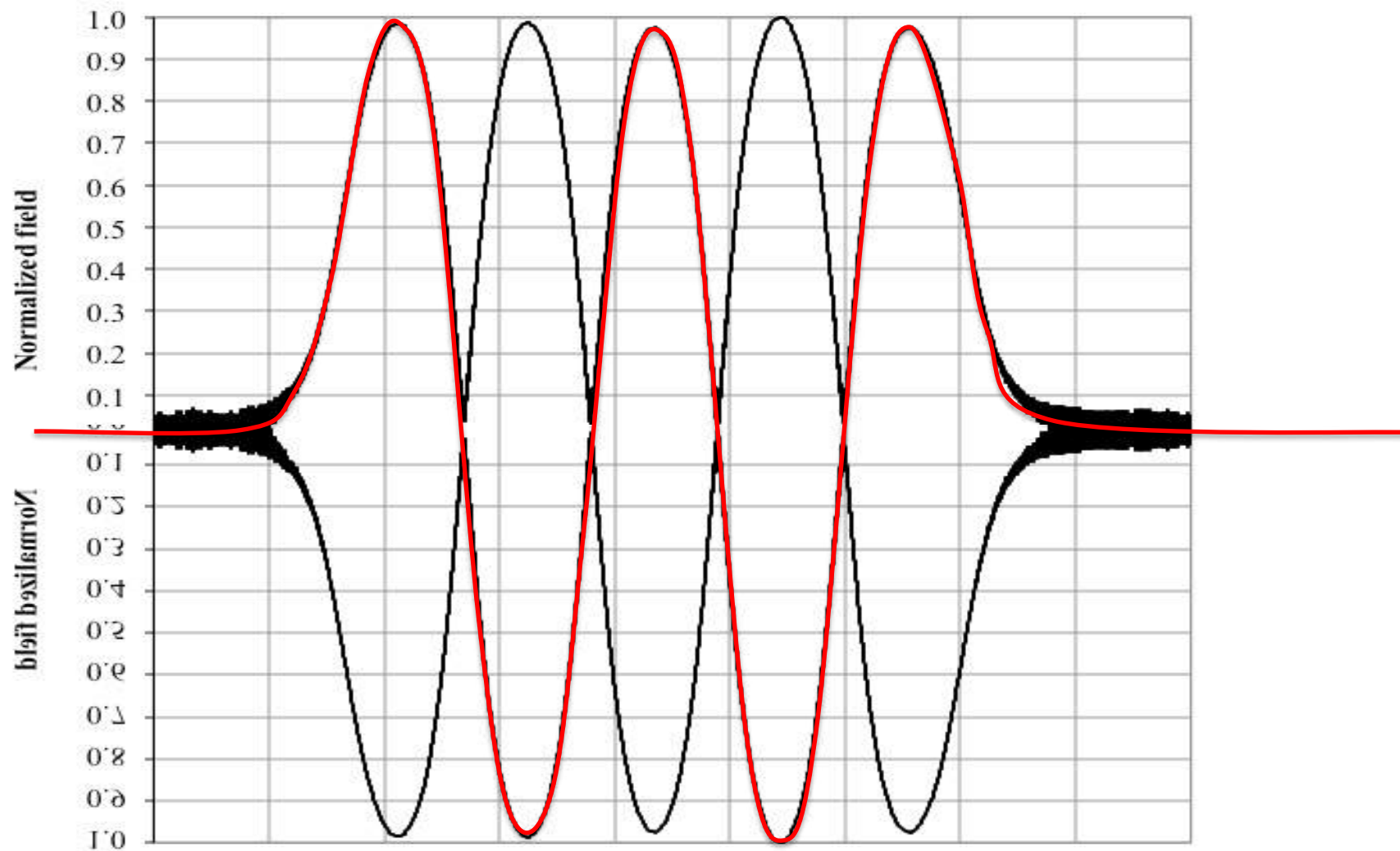


Electron linac



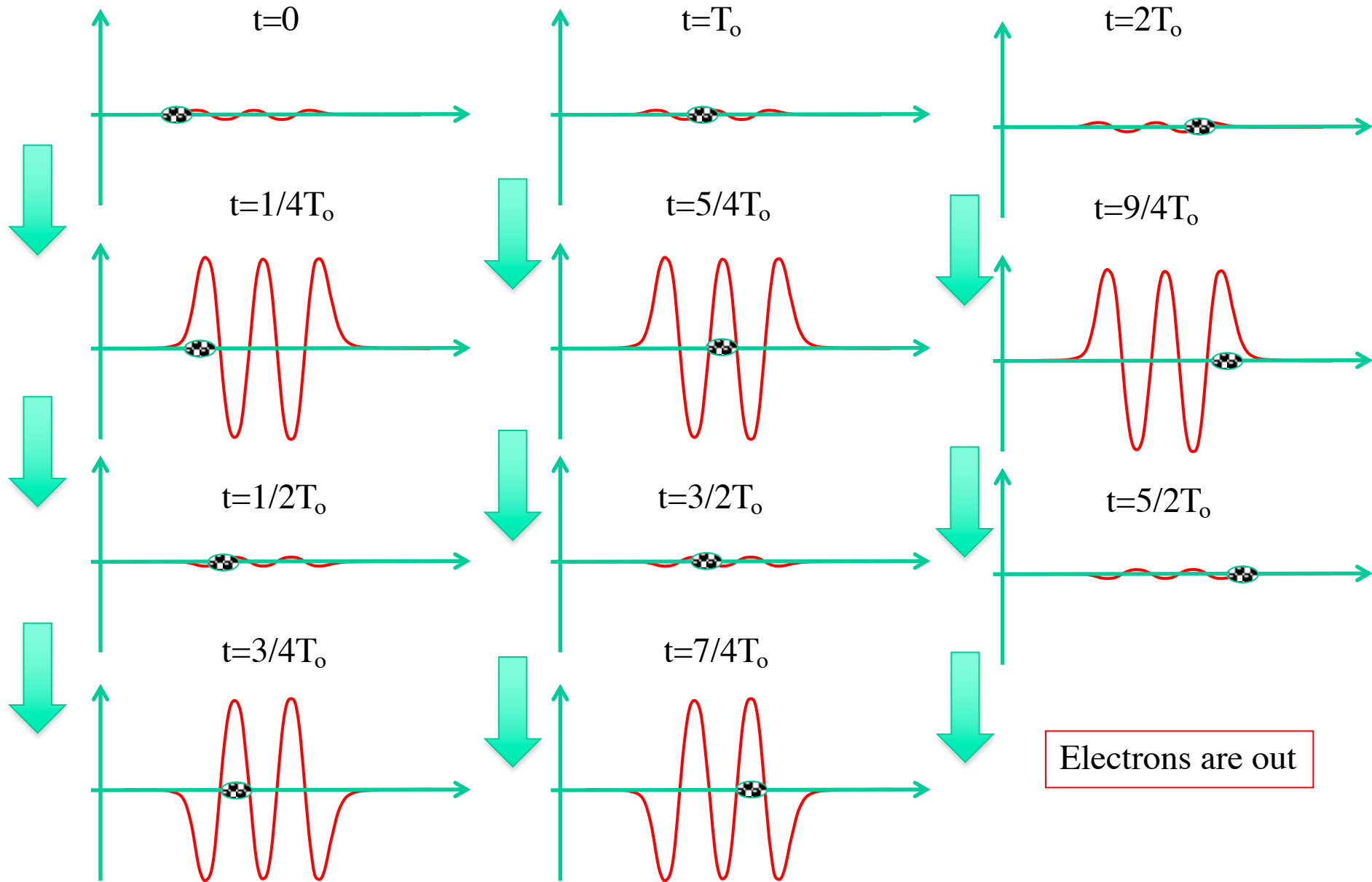
$$\beta = v/c \sim 1$$

# Wave-form in 5-cell cavity



# How $\beta=1$ RF linac works?

## Example of 5-cell cavity



# Simple things to remember



- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)
- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy
- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.
- Accurate synchronization of RF linac is important task for any linear accelerator

# A bit of EM and conducting media

$$\vec{j} = \sigma \vec{E};$$

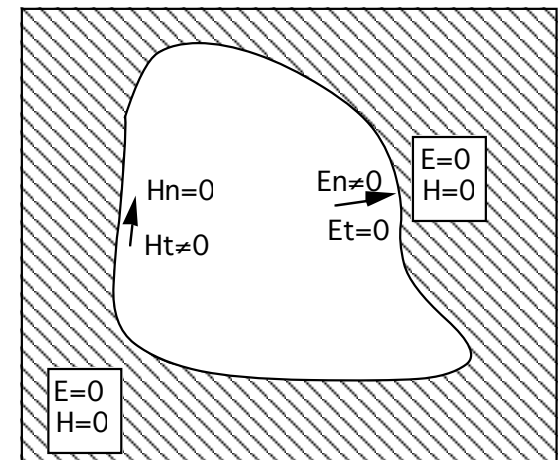
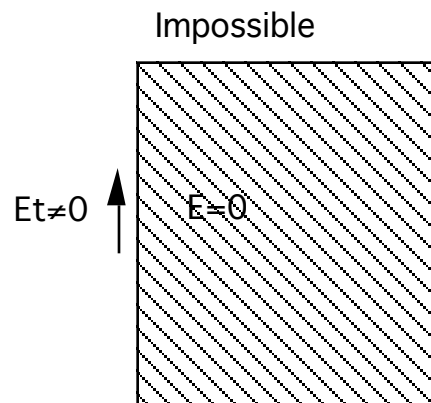
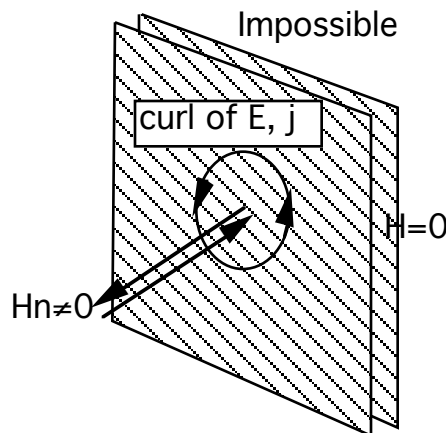
- Assuming oscillating field we can use Coulomb gauge for EM field

$$\vec{A} = \text{Re} \left\{ \vec{A}(\vec{r}) \exp(i\omega t) \right\}; \varphi = 0;$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \text{curl} \vec{A}.$$

$$|\vec{H}| \propto \left| \frac{(\alpha + i\beta)}{k_o} \right| |\vec{E}| = \left| \sqrt{1 + \frac{4\pi i\sigma}{\omega}} \right| |\vec{E}|$$

$$\sigma \rightarrow \infty$$

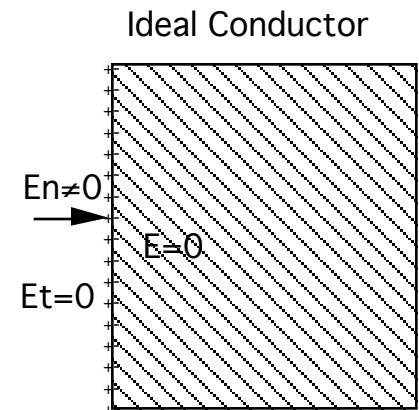
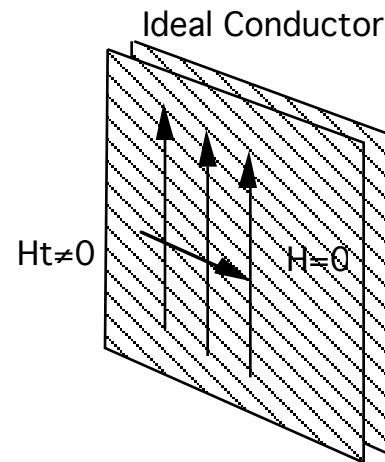


# Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM field inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivity as a perturbation later
- Q-factor:  $Q_{\text{room temp}} \sim 10^3 - 10^5$ ,  $Q_{\text{SRF}} \sim 10^9 - 10^{10}$

$$\vec{A} = \text{Re} \left\{ \vec{A}(\vec{r}) \exp(i\omega t - \alpha t) \right\};$$

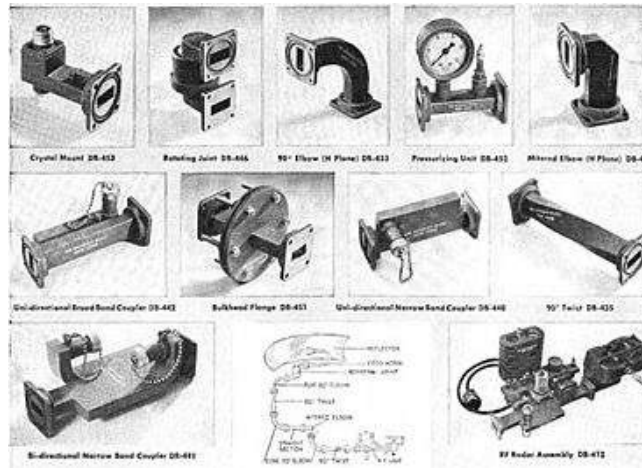
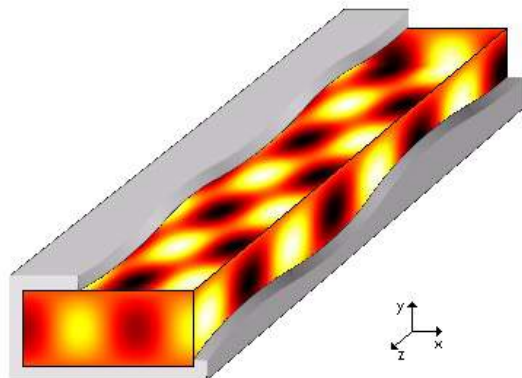
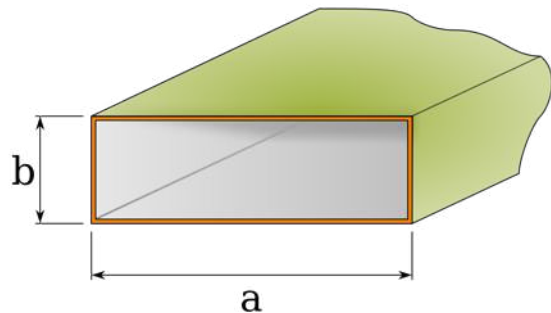
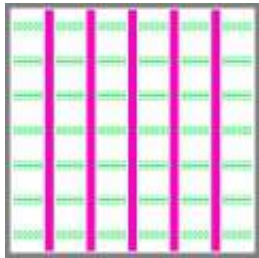
$$\alpha = \frac{2\pi\omega}{Q}$$



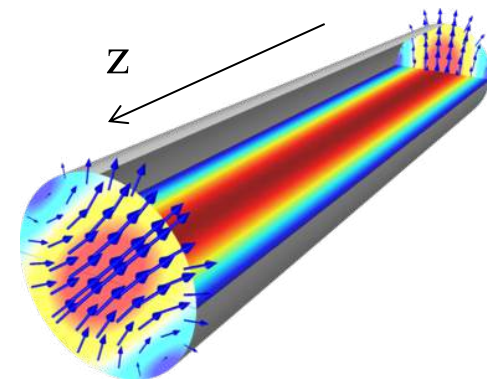
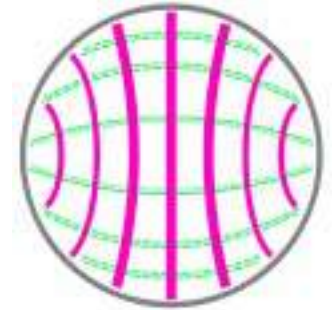
$$\vec{E} = \vec{n}(\vec{n} \cdot \vec{E}) + \vec{E}_{\parallel}; \vec{B} = \vec{n}(\vec{n} \cdot \vec{B}) + \vec{B}_{\parallel};$$

# Waveguides

## Rectangular



## Circular



$$\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A} = 0; \quad \Delta \equiv \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

$$\vec{A} = \text{Re} \left\{ \vec{A}(\vec{r}_\perp) \exp(i(\omega t - k_z z)) \right\};$$

$$\vec{\nabla}_\perp^2 \vec{A} + (k_o^2 - k_z^2) \vec{A} = 0; \quad k_o = \frac{\omega}{c}.$$

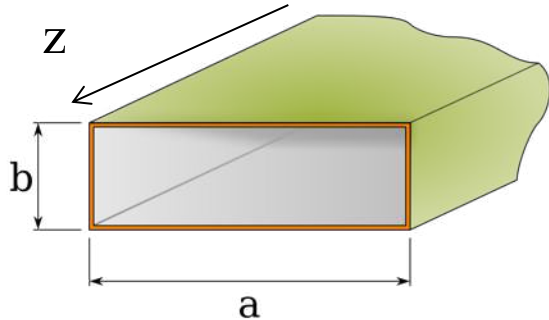
$$\vec{\nabla}_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

*At the surfaces*

$$\vec{n} \times \vec{E}|_s = 0; \quad \vec{n} \cdot \vec{B} = 0 \rightarrow E_z|_s = 0; \quad \frac{\partial B_z}{\partial n}|_s = 0$$

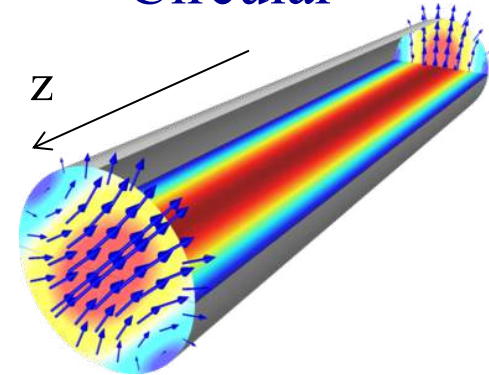
# TE and TM waves

Rectangular



- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)

Circular



$$\vec{E} = \vec{E}_z + \vec{E}_\perp; \vec{B} = \vec{B}_z + \vec{B}_\perp; \vec{A}_z \equiv \hat{z}A_z;$$

$$\vec{\nabla} \times \vec{E} = ik_o \vec{B}; \vec{\nabla} \times \vec{B} = -ik_o \vec{E}; \Rightarrow$$

$$ik_z \vec{E}_\perp + ik_o [\hat{z} \times \vec{B}_\perp] = \vec{\nabla}_\perp \vec{E}_z;$$

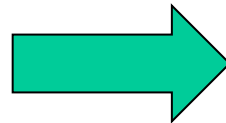
$$ik_z \vec{B}_\perp - ik_o [\hat{z} \times \vec{E}_\perp] = \vec{\nabla}_\perp \vec{B}_z;$$

At the surfaces

$$\vec{n} \times \vec{E}|_s = 0; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z|_s = 0; \left. \frac{\partial B_z}{\partial n} \right|_s = 0$$

$$TM : B_z \equiv 0; E_z|_s = 0;$$

$$TE : E_z \equiv 0; B_z|_s = 0;$$



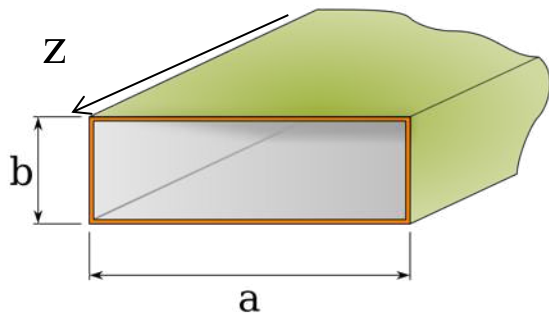
- Last two equations indicated that  $E_z$  and  $B_z$  fully determine transverse component of the EM field
- It means that we can always consider a linear combination of the fields with  $E_z = 0$  everywhere (TE) and  $B_z = 0$  everywhere (TM)
- Naturally, when we interested in accelerating particles, we will need TM mode with  $E_z \neq 0$ .

$$\vec{B}_\perp = \pm \frac{k_z}{k_o} [\hat{z} \times \vec{E}_\perp] \quad \text{for both TE and TM modes}$$

$$TM: \vec{E}_\perp = \vec{\nabla}_\perp \psi_1(\vec{r}_\perp); TE: \vec{B}_\perp = \vec{\nabla}_\perp \psi_2(\vec{r}_\perp);$$

# Cut-off frequency

## Rectangular



- EM field is a linear combination of modes with  $E_z = 0$  everywhere (TE) and  $B_z = 0$  everywhere (TM)

At the surfaces

$$\vec{n} \times \vec{E}|_s = 0; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z|_s = 0; \left. \frac{\partial B_z}{\partial n} \right|_s = 0$$

$$\vec{B}_\perp = \pm \frac{k_z}{k_o} [\hat{z} \times \vec{E}_\perp] \quad \text{for both TE and TM modes}$$

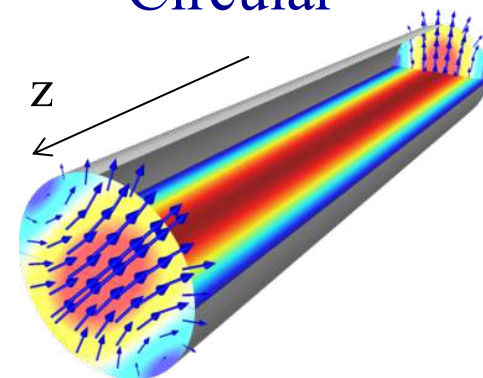
$$\text{TM: } \vec{E}_\perp = \vec{\nabla}_\perp \psi_1(\vec{r}_\perp); \quad \text{TE: } \vec{B}_\perp = \vec{\nabla}_\perp \psi_2(\vec{r}_\perp);$$

$$\vec{\nabla}_\perp^2 \psi + (k_o^2 - k_z^2) \psi = 0 + \text{boundary conditions}$$

Different boundary conditions for TE and TM modes

In general case we need to find eigen function (modes)

## Circular



$$\text{TM: } B_z \equiv 0; \quad E_z|_s = 0;$$

$$\text{TE: } E_z \equiv 0; \quad B_z|_s = 0;$$

$$\text{TM: } \psi|_s = 0; \quad \text{TE: } \left. \frac{\partial \psi}{\partial n} \right|_s = 0.$$

$$\boxed{\vec{\nabla}_\perp^2 \psi_\lambda + \gamma_\lambda^2 \psi_\lambda = 0; \quad \lambda = 1, 2, 3, \dots}$$

## Cut-off

$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

## frequency

$$k_{o\min} = \gamma_\lambda \rightarrow \omega_{\text{cut-off}} = c\gamma_\lambda$$

Below cut-off

evanescent wave:  $k_z = \pm i \sqrt{\omega_{\text{cut-off}}^2 - \omega^2} = \pm i\kappa_z$

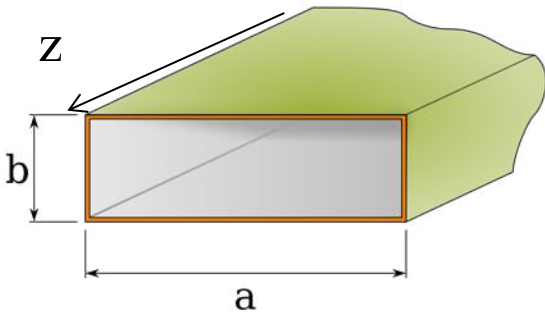
Exp decay

$$\omega < \omega_{\text{cut-off}}$$

$$\psi = \psi_0 e^{\pm \kappa_z z}$$

# Cut-off frequency

## Rectangular



$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + \gamma_{mn}^2 \psi = 0$$

$$TE : \psi_{mn}^{TE} = \psi_o \cos k_m x \cos k_n y; \quad m + n \geq 1;$$

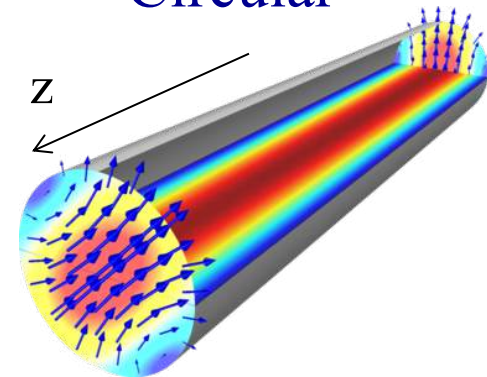
$$TM : \psi_{mn}^{TM} = \psi_o \sin k_m x \sin k_n y; \quad m \geq 1; n \geq 1;$$

$$k_m = \pi \frac{m}{a}; k_n = \pi \frac{n}{b}; \gamma_{mn}^2 = k_m^2 + k_n^2.$$

*Different boundary conditions  
for TE and TM modes*

$$TM : \psi|_s = 0; \quad TE : \left. \frac{\partial \psi}{\partial n} \right|_s = 0.$$

## Circular



$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0$$

$$\psi_{mn} = \varphi_{mn}(r) e^{in\theta} \Rightarrow r \frac{\partial}{\partial r} \left( r \frac{\partial \psi_m}{\partial r} \right) + (r^2 \gamma_{mn}^2 - n^2) = 0$$

$$\varphi_{mn} = J_n(\gamma_{mn} r)$$

## Lowest cut-off frequency

### Rectangular

$$TE : a > b; m = 1; n = 0; \omega_{cut-off} = \frac{\pi c}{a};$$

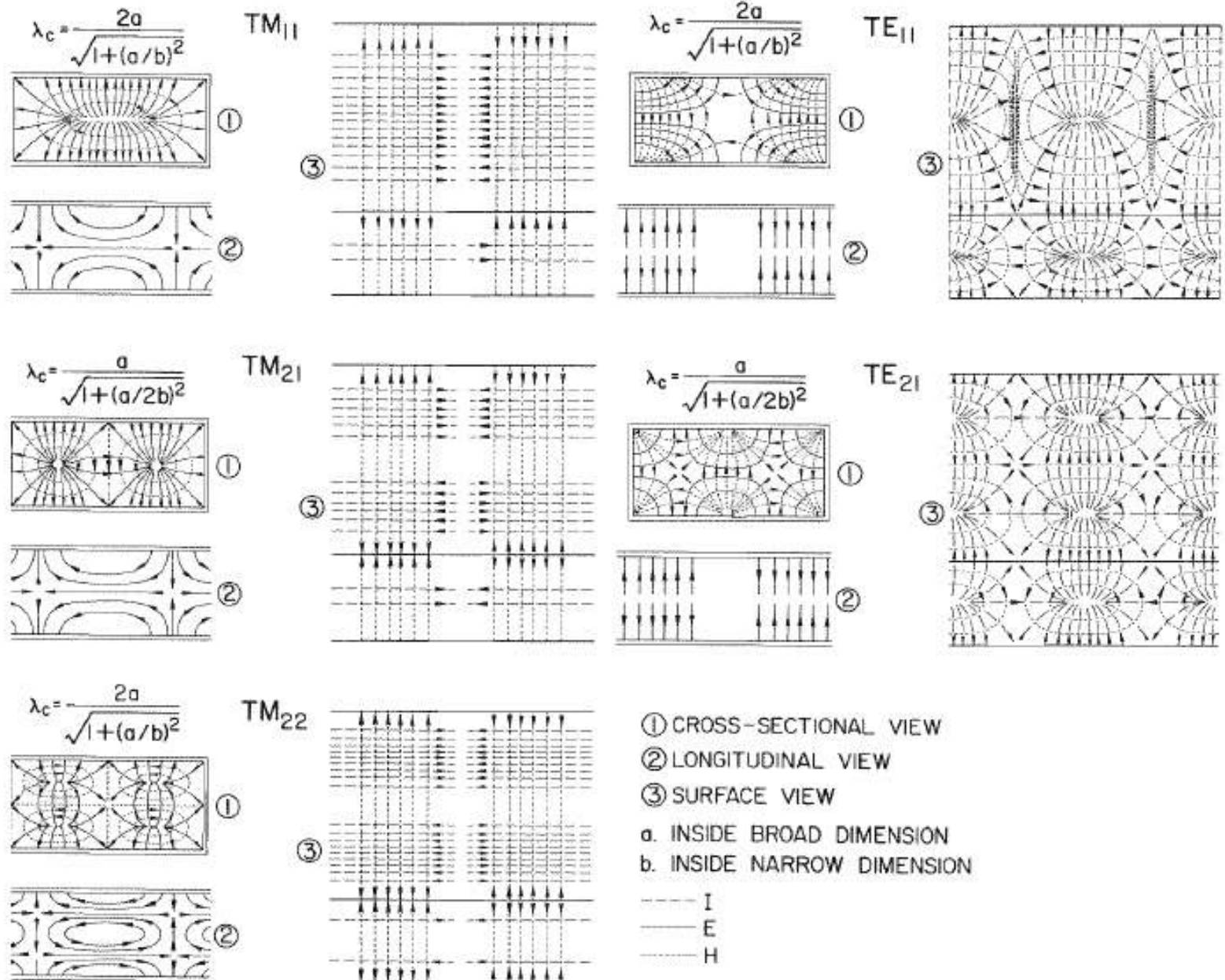
$$TM : m = 1; n = 1; \omega_{cut-off} = \frac{\pi c}{a} \sqrt{1 + \frac{a^2}{b^2}}.$$

### Circular

$$TM : J_0(\gamma_{01} R) = 0 \rightarrow \gamma_{01} \cong \frac{2.40483..}{R}; \omega_{cut-off} \cong \frac{2.40 c}{R};$$

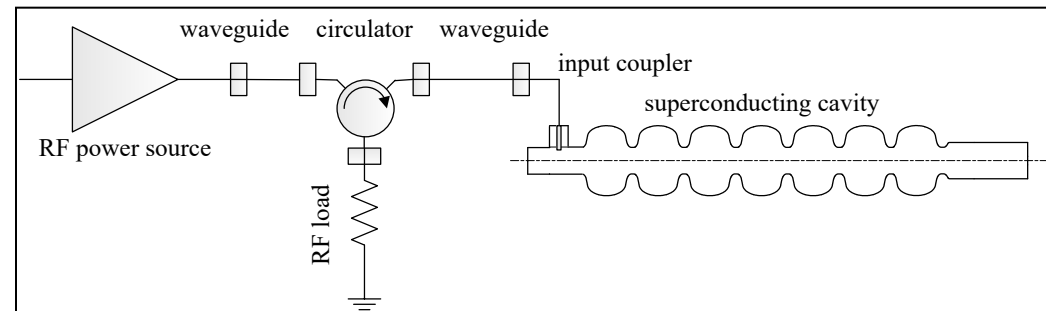
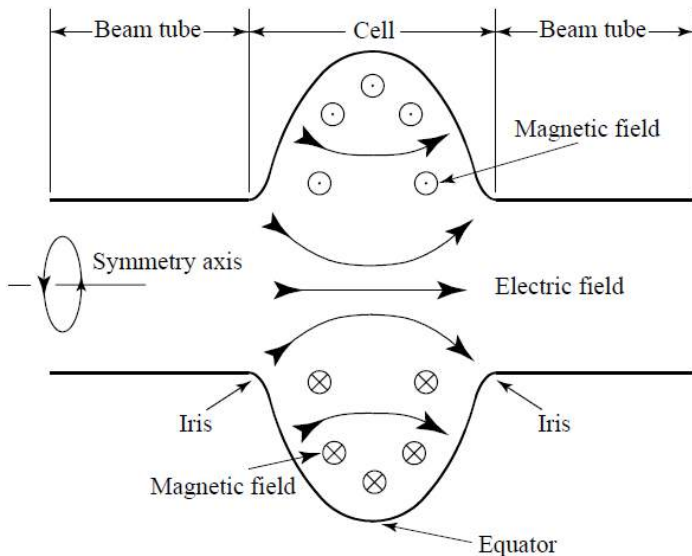
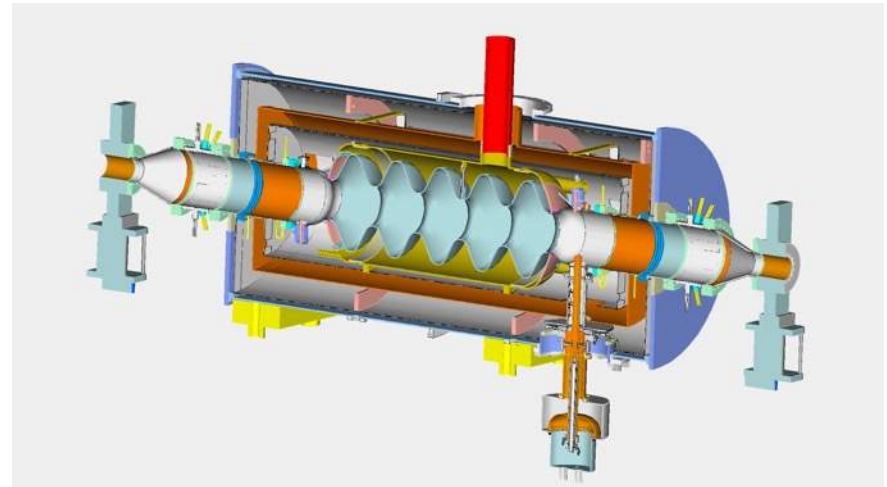
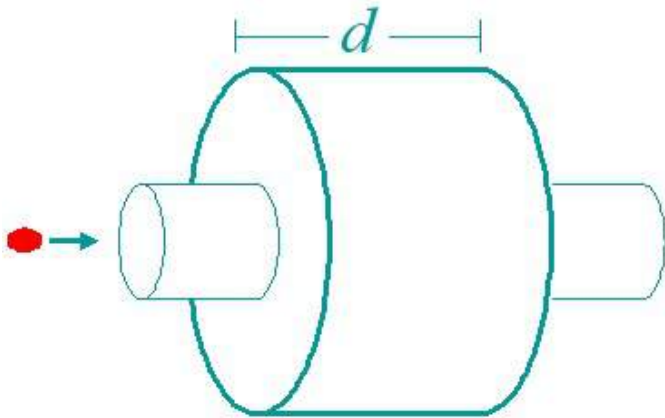
$$TE : J_1(\gamma_{11} R) = 0 \rightarrow \gamma_{11} \cong \frac{1.84118....}{R}; \omega_{cut-off} \cong \frac{1.84 c}{R}.$$

# Modes in rectangular waveguide



# RF cavities

are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them



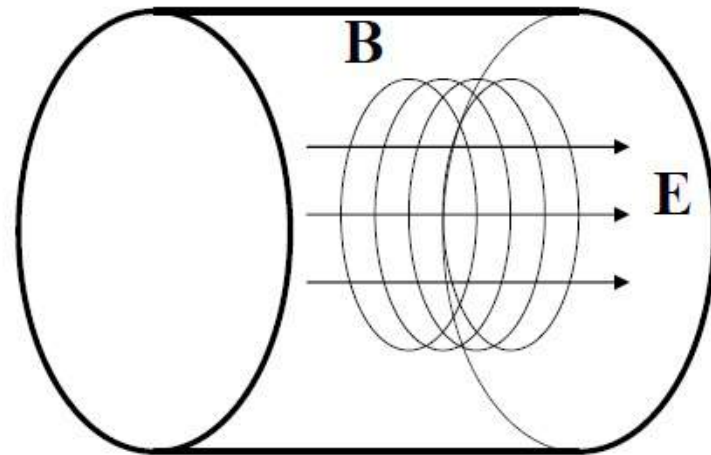
# RF Cavity Modes:

the lowest accelerating is  $\text{TM}_{010}$  mode

- Fields in the cavity are solutions of the equation
- Subject to the boundary conditions  $\hat{n} \times \mathbf{E} = 0, \hat{n} \cdot \mathbf{H} = 0$
- Two extra surfaces ( $z=0$  and  $z=d$ ): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency  $\text{TM}_{010}$  mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.

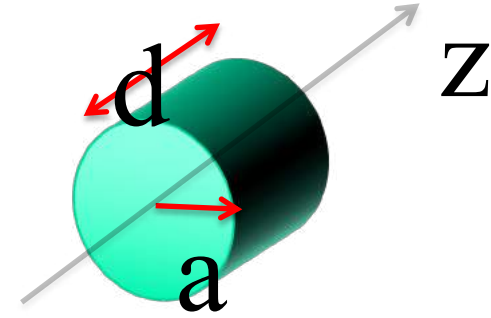
$$\left( \nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} \right) \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = 0$$

$$E_z = E_0 J_0 \left( \frac{2.405r}{R} \right) e^{i\omega t}$$
$$H_\phi = -iE_0 J_1 \left( \frac{2.405r}{R} \right) e^{i\omega t}$$
$$\omega_{010} = \frac{2.405c}{R}, \lambda_{010} = 2.61R$$



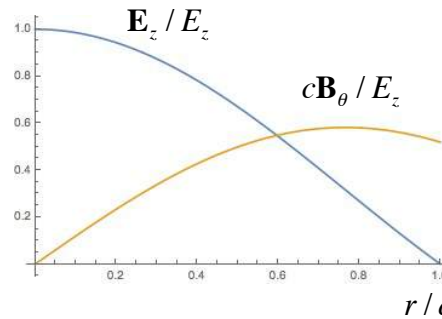
# Pillbox Cavity

- Similarly to a previous exercise, we need to pick TM mode to have non-zero  $E_z$  component
- We also select TEM01 waveguide mode and  $k_z=0$

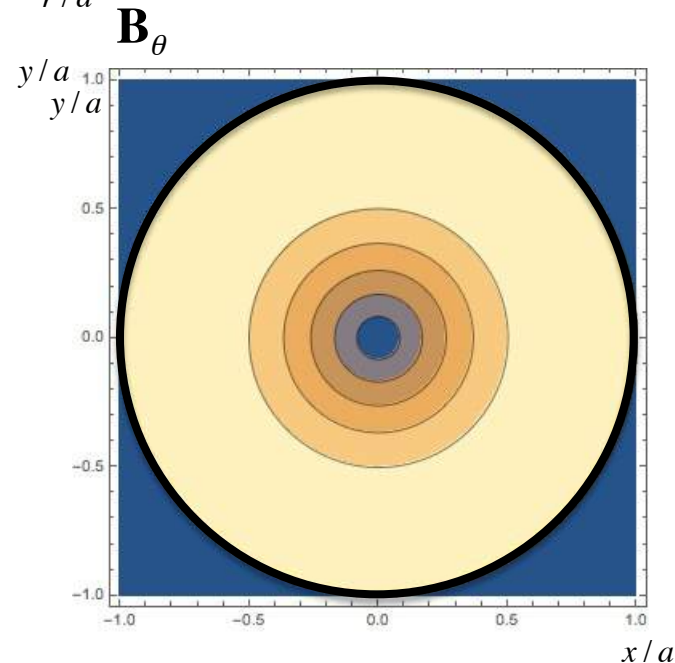
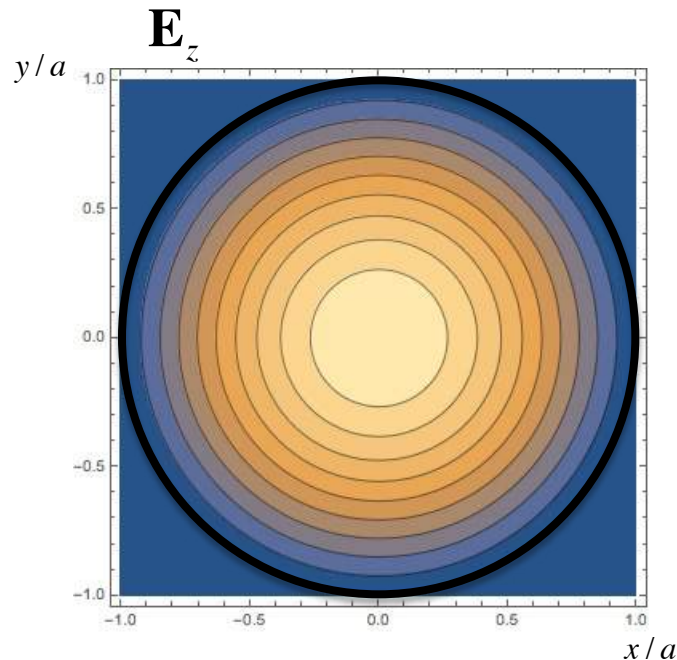


$$\mathbf{E}_z = E_z \cdot J_0\left(2.405 \frac{x}{a}\right) \sin(\omega t);$$

$$\mathbf{B}_\theta = \frac{1}{c} E_z \cdot J_1\left(2.405 \frac{x}{a}\right) \cos(\omega t)$$

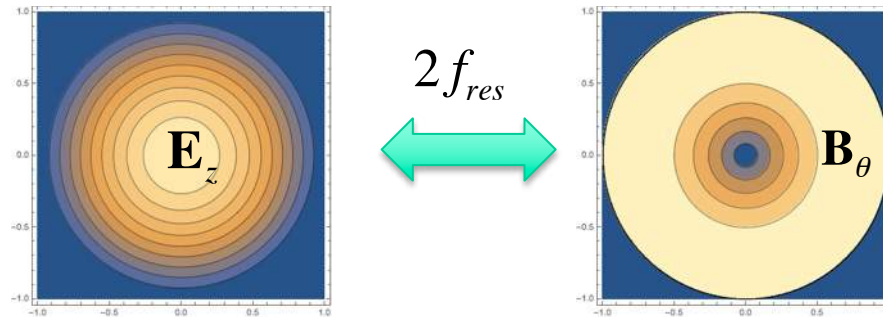


EM energy oscillates between  
electric field magnetic field:  
Peaking at the same value:  
total energy



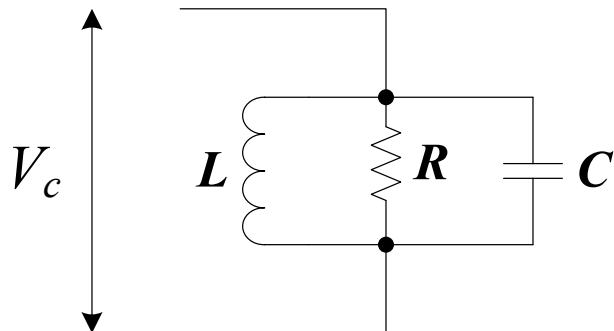
# EM Cavity

$$\begin{aligned}\vec{E} &= \vec{E}_o(\vec{r}) \cos(\omega t + \varphi(\vec{r})) \\ \vec{B} &= \vec{B}_o(\vec{r}) \sin(\omega t + \psi(\vec{r})) \\ \int \vec{E}_o^2 dV &= \int \vec{B}_o^2 dV \quad (\cdot c^2 \text{ for SI})\end{aligned}$$

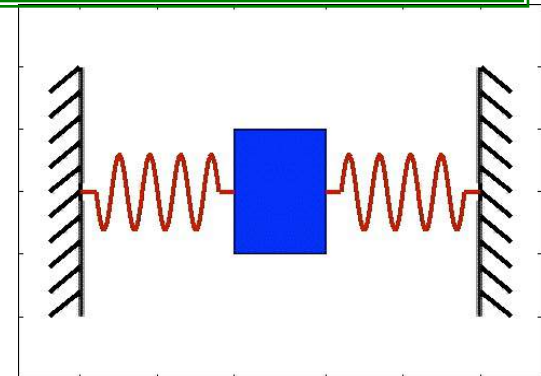


- Each mode has full analogy with a resonant LC circuit or a mechanical oscillator: energy stored in electric field can be compared to potential energy, and energy stored in magnetic field – to kinetic energy
- Typical energy stored in 5 cell, 700 MHz SRF cavity operating at 20 MV/m is  $\sim 500$  J
- What much more impressive is the intra-cavity power of about 2,000 GW!

$$E_M = \frac{LI^2}{2}; E_E = \frac{Q^2}{2C}; \omega_o = \frac{1}{\sqrt{LC}}$$



$$E_K = \frac{Mv^2}{2}; E_P = \frac{Kx^2}{2}; \omega_o = \sqrt{\frac{K}{M}}$$



# Quality factor - definition

- Let's consider a stand-alone cavity without any external couplers

- Energy stored in the cavity 
$$W = \int \left( \epsilon_o \frac{\vec{E}^2}{2} + \mu_o \frac{\vec{H}^2}{2} \right) dV = \mu_o \int \vec{H}_o^2 dV$$

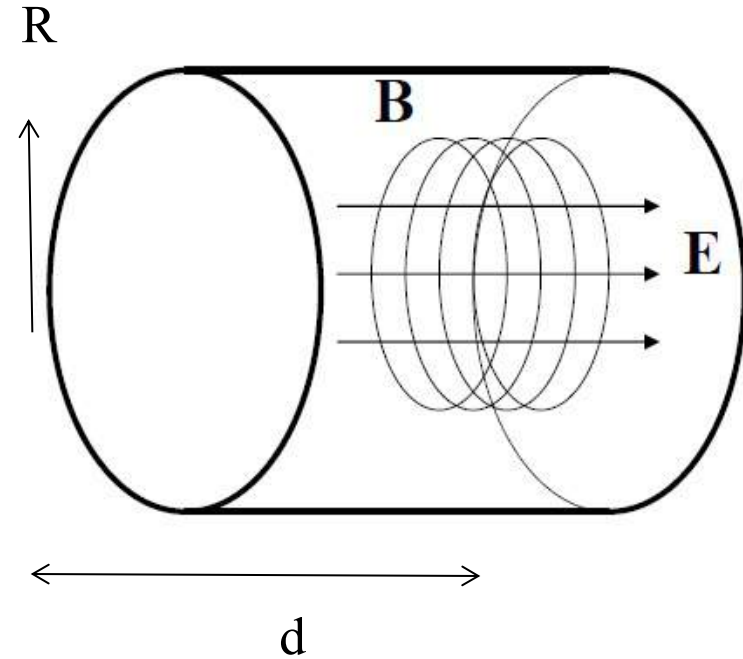
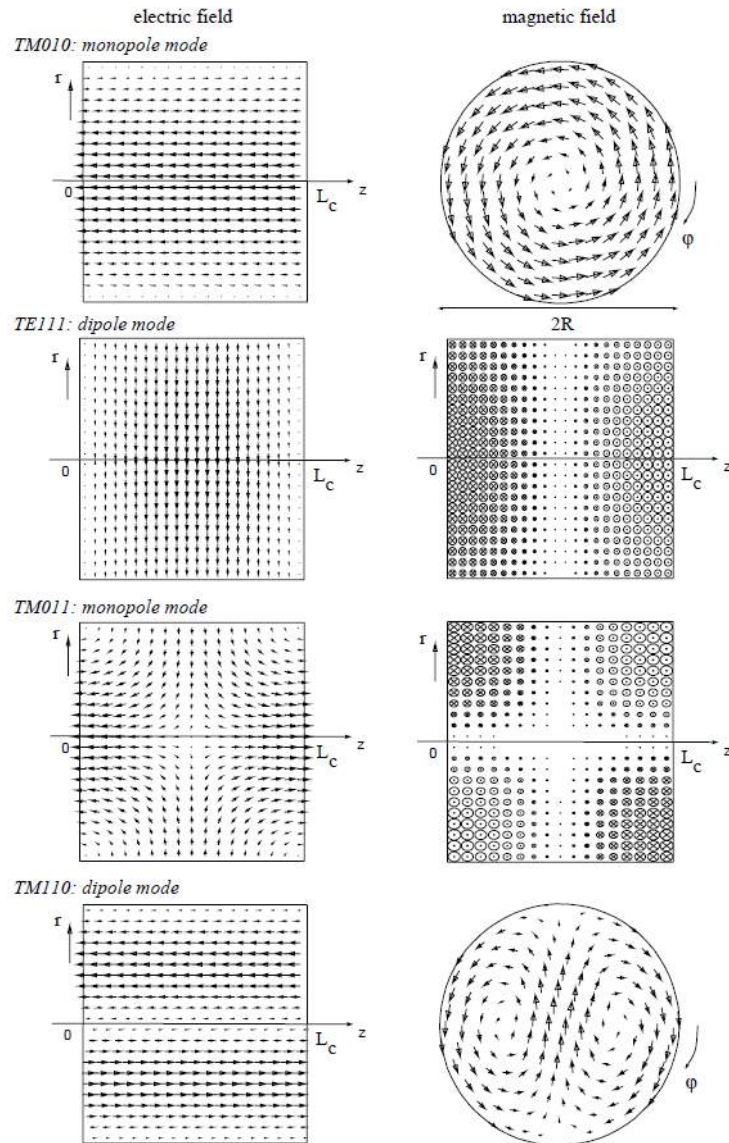
- Losses in the power

$$P_{loss} = \frac{W\omega}{Q}$$

- Quality factor - Q - it is the number of RF oscillation times  $2\pi$  required for energy inside the cavity to reduce e-fold.

# Fundamental and high order modes (HOMs)

Eigenmodes in a Pill-box cavity



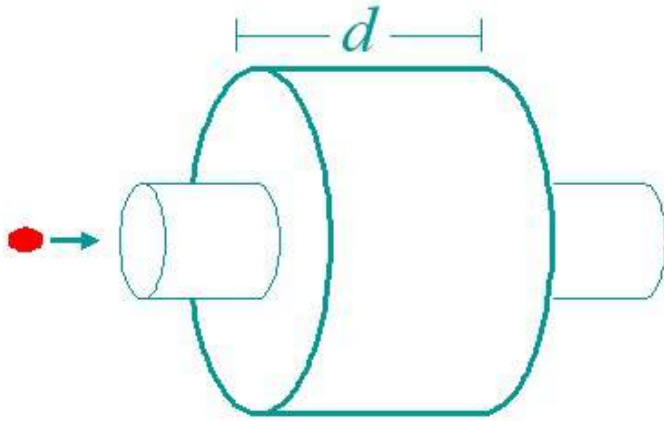
$$TM : \varphi_{mnl} = J_n(\gamma_{mn}r) \cos k_{z,l}z; \quad k_{z,l} = l \frac{\pi}{d}; \quad J_n(\gamma_{mn}R) = 0;$$

$$\omega_{res} = c \sqrt{\gamma_{mn}^2 + l^2 \frac{\pi^2}{d^2}}; \quad l = 0, 1, 2, \dots$$

$$TE : \varphi_{mnl} = J_n(\kappa_{mn}r) \sin k_{z,l}z; \quad k_{z,l} = l \frac{\pi}{d}; \quad J'_n(\kappa_{mn}R) = 0;$$

$$\omega_{res} = c \sqrt{\kappa_{mn}^2 + l^2 \frac{\pi^2}{d^2}}; \quad l = 1, 2, \dots$$

# Accelerating voltage & transit time



- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho=0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where  $T$  is the transit time factor.

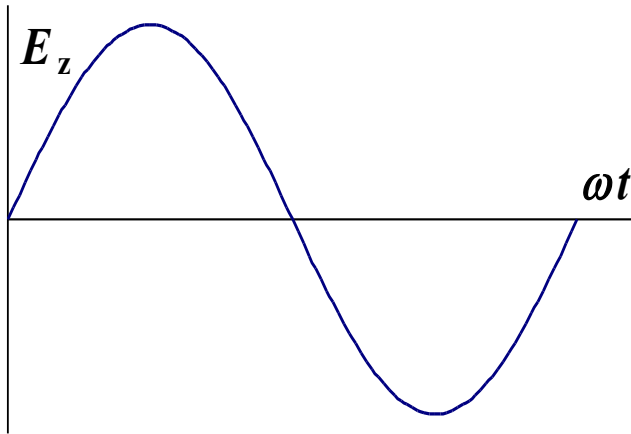
- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow V_c = \frac{2}{\pi} E_0 d$$

Thus for the pillbox cavity  $T = 2/\pi$ .

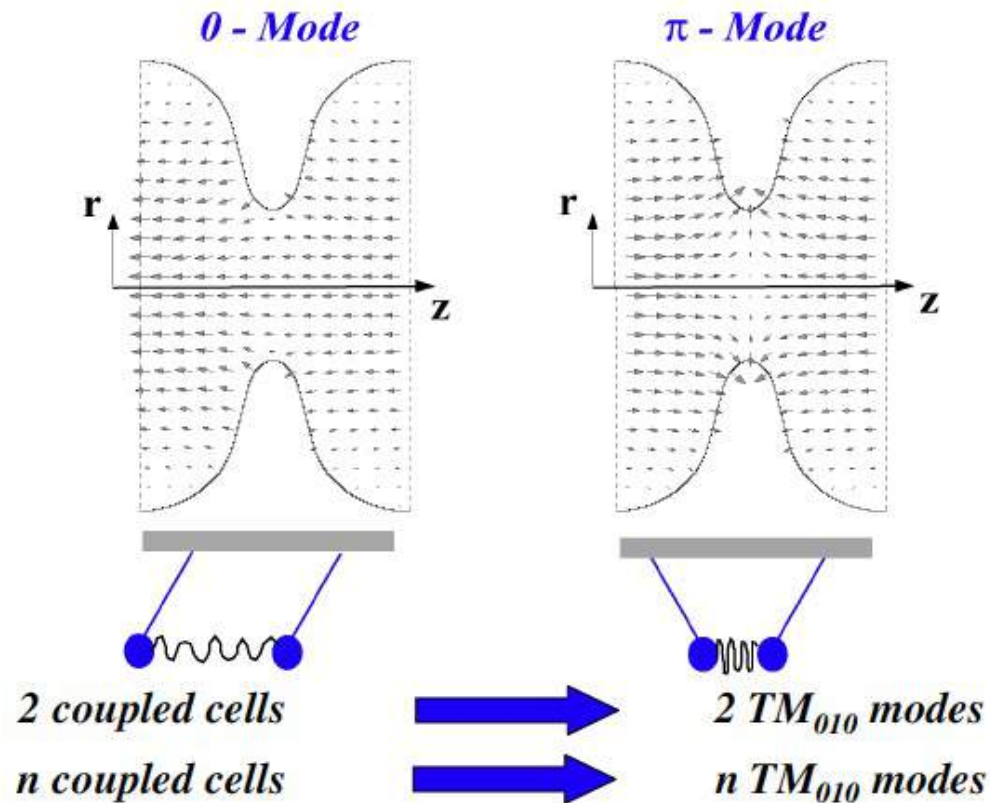
- The accelerating field  $E_{acc}$  is defined as  $E_{acc} = V_c/d$ .

Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be  $d = \beta\lambda/2$ . This works OK for multi-cell cavities, but poorly for single-cell ones.



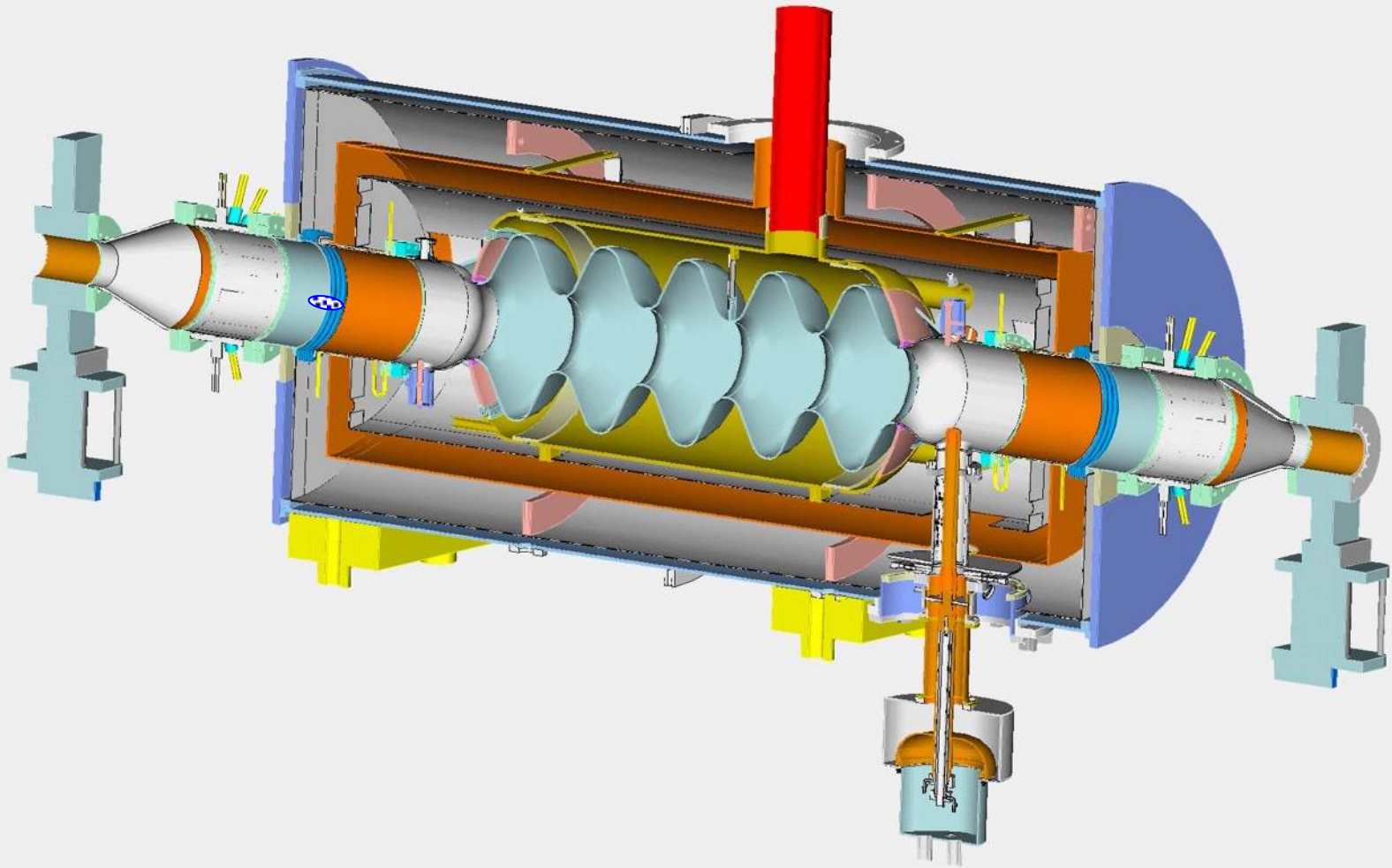
# Multicell cavities: coupled oscillators

- Several cells can be connected together to form a multicell cavity.
- Coupling of  $TM_{010}$  modes of the individual cells via the iris (primarily electric field) causes them to split:



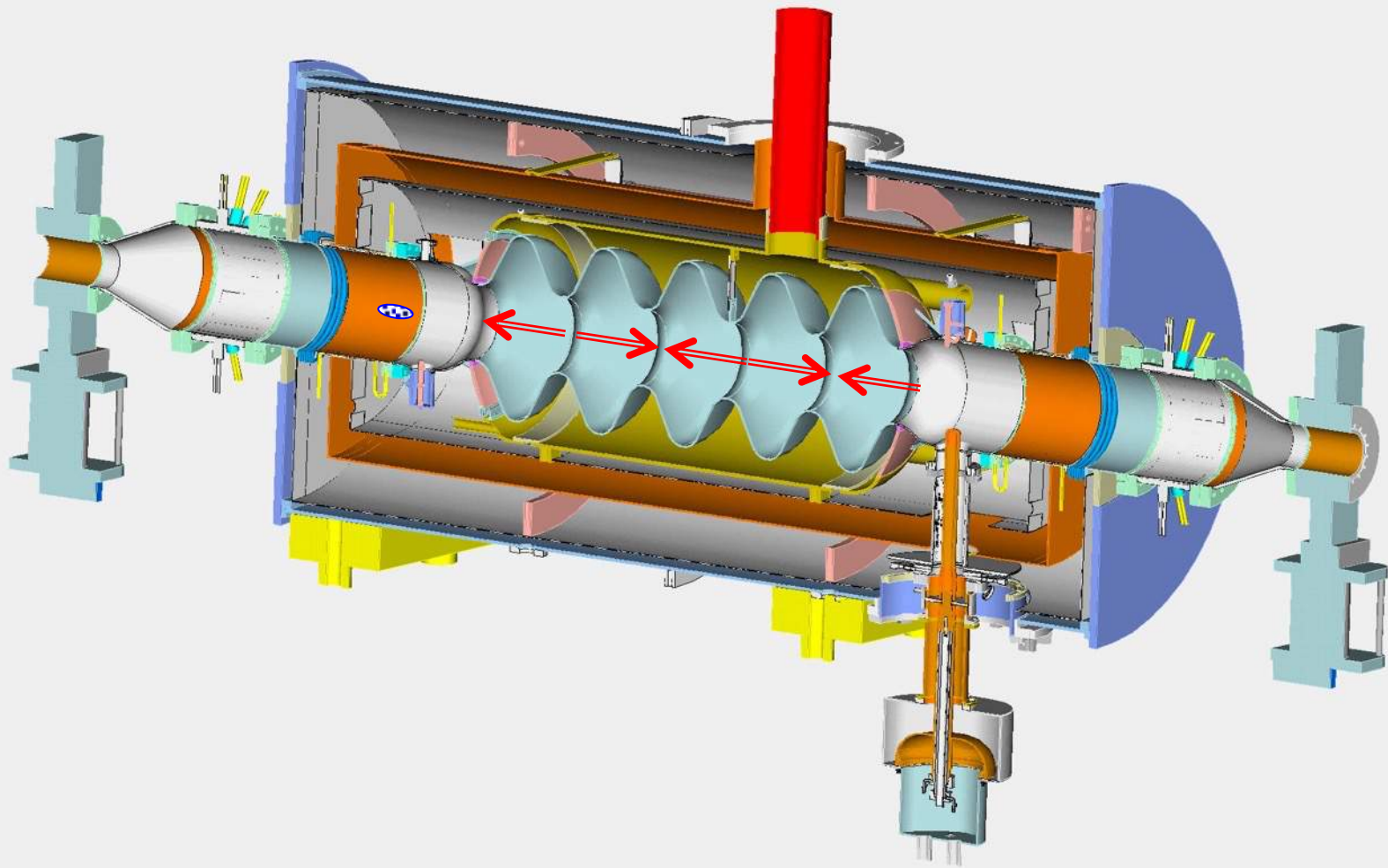
# How $\beta=1$ RF accelerator works?

## In pictures



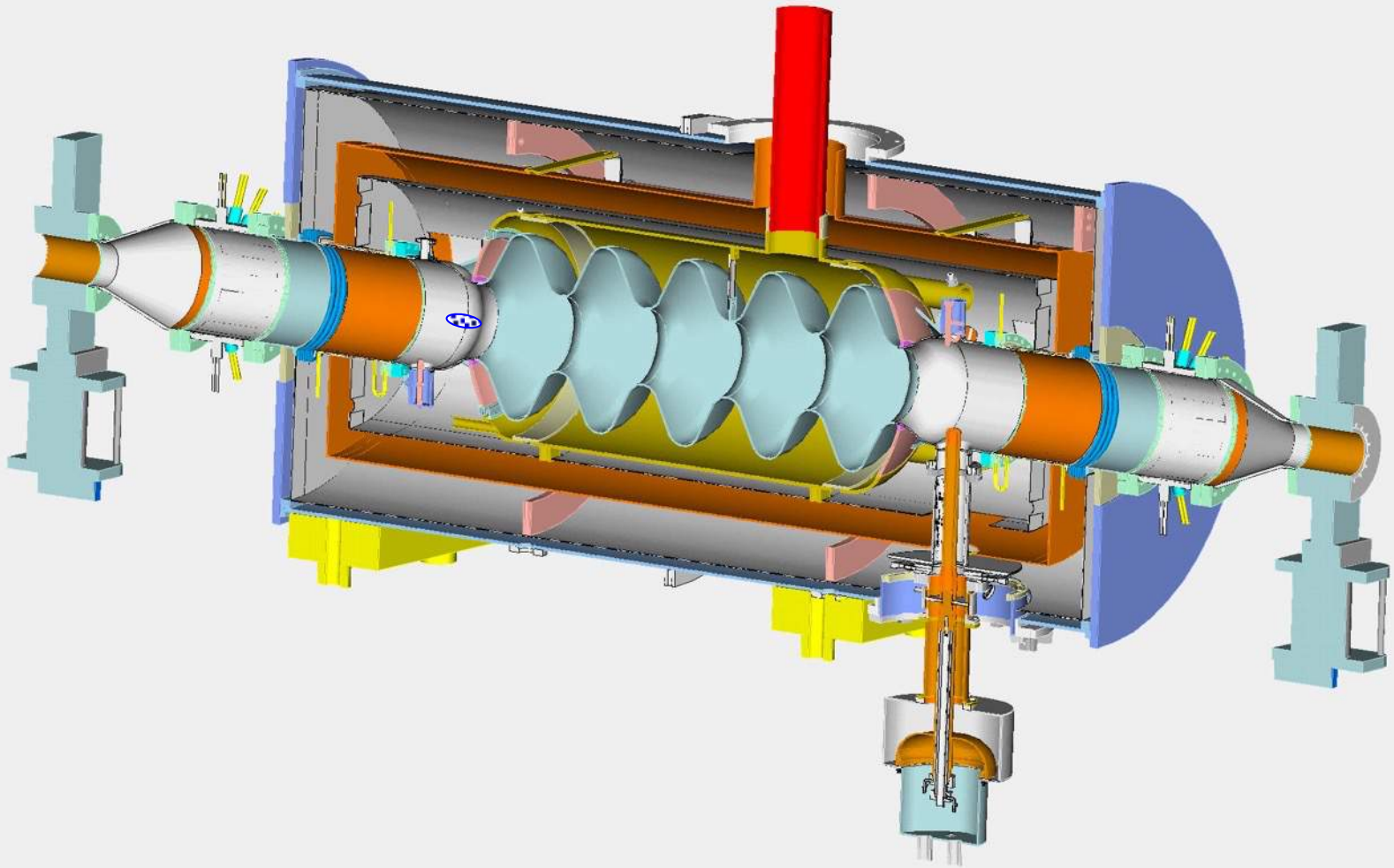
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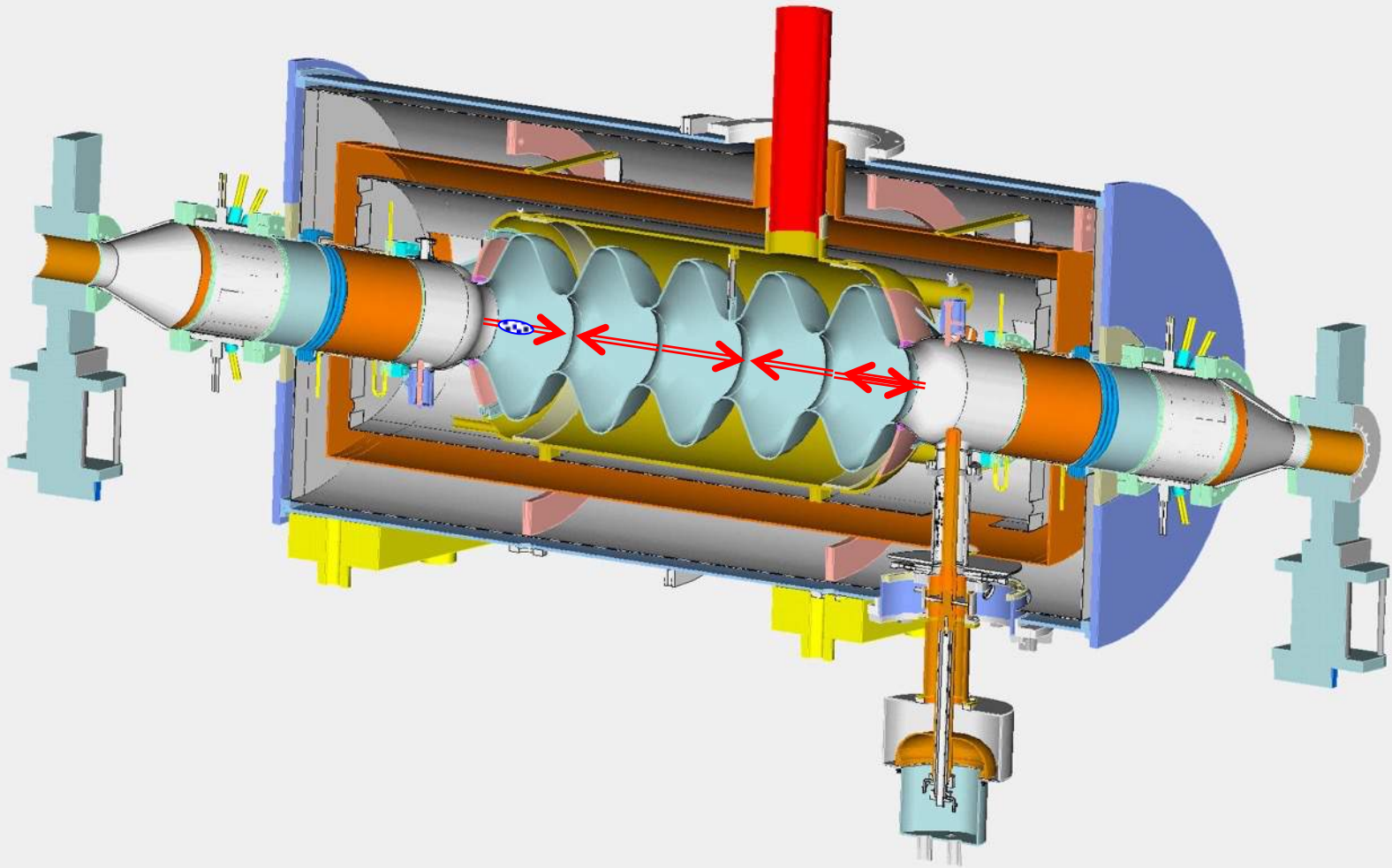
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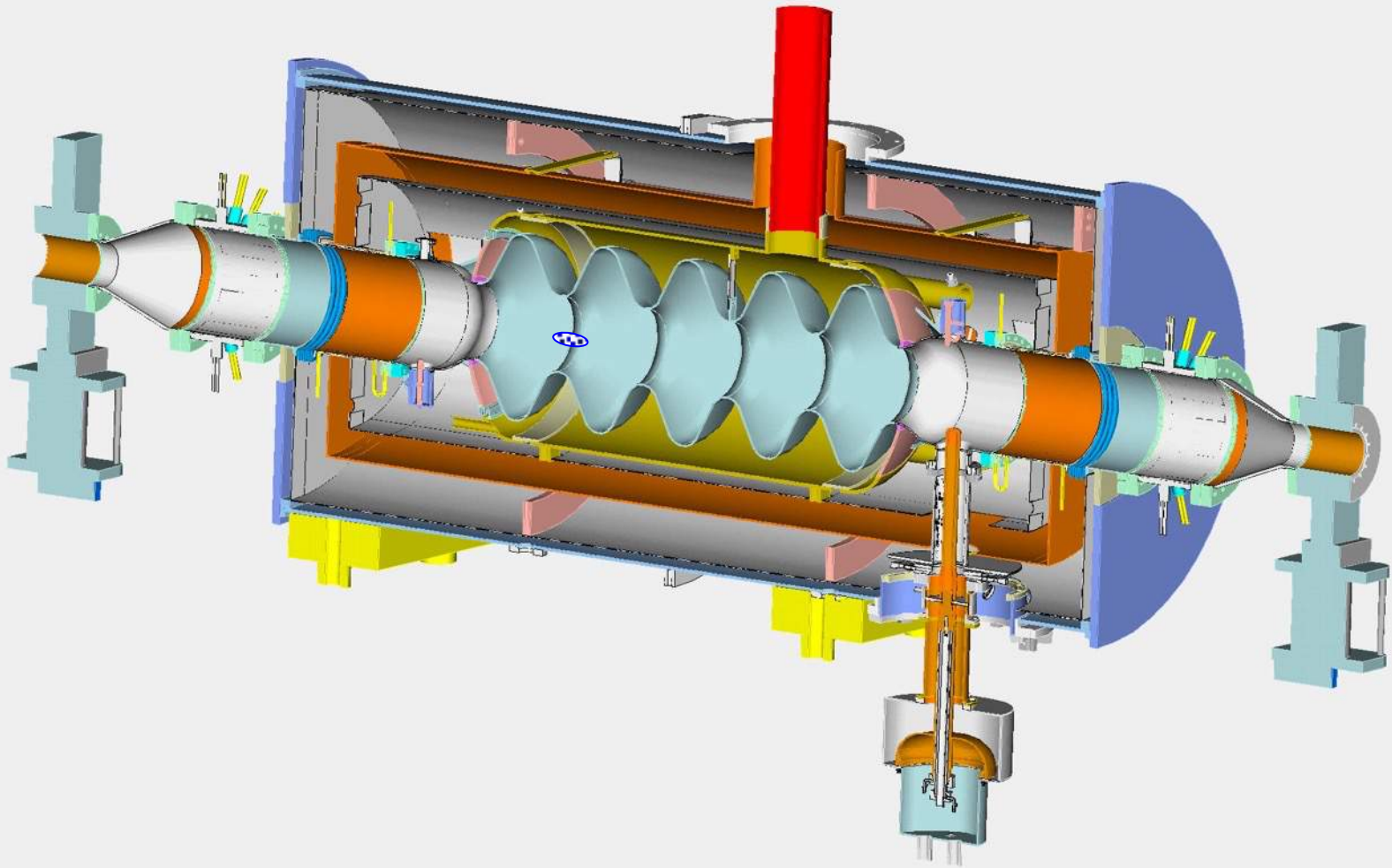
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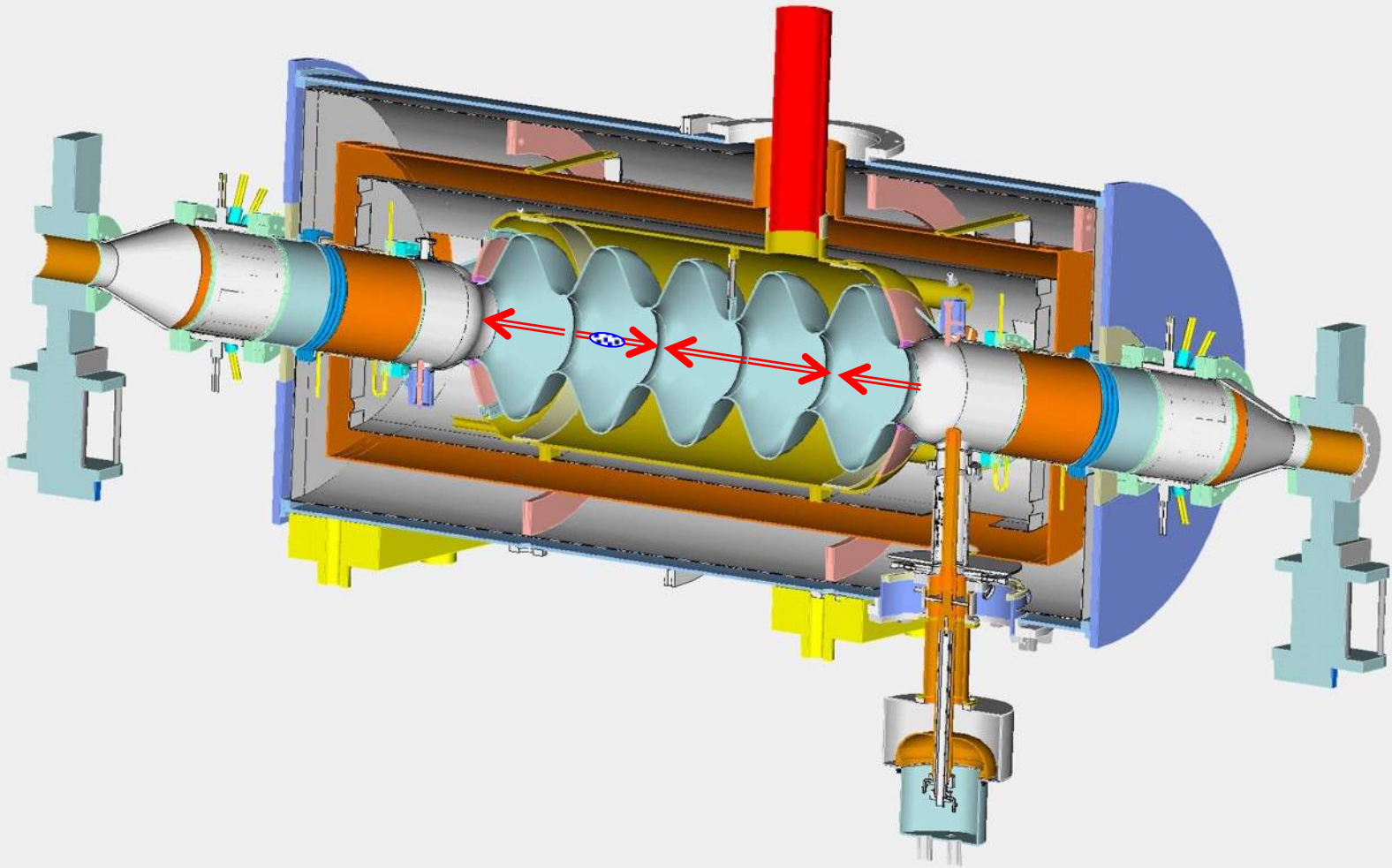
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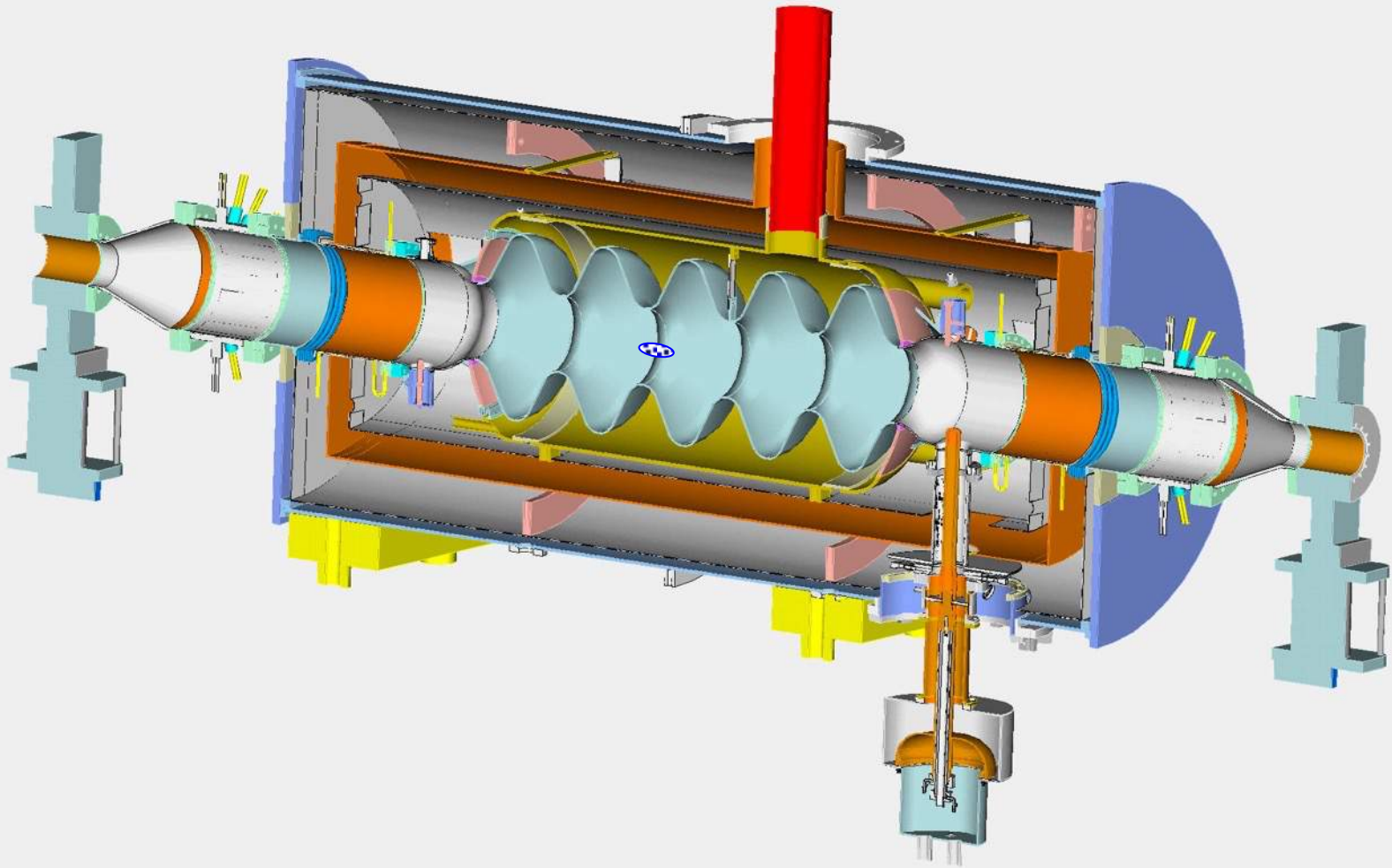
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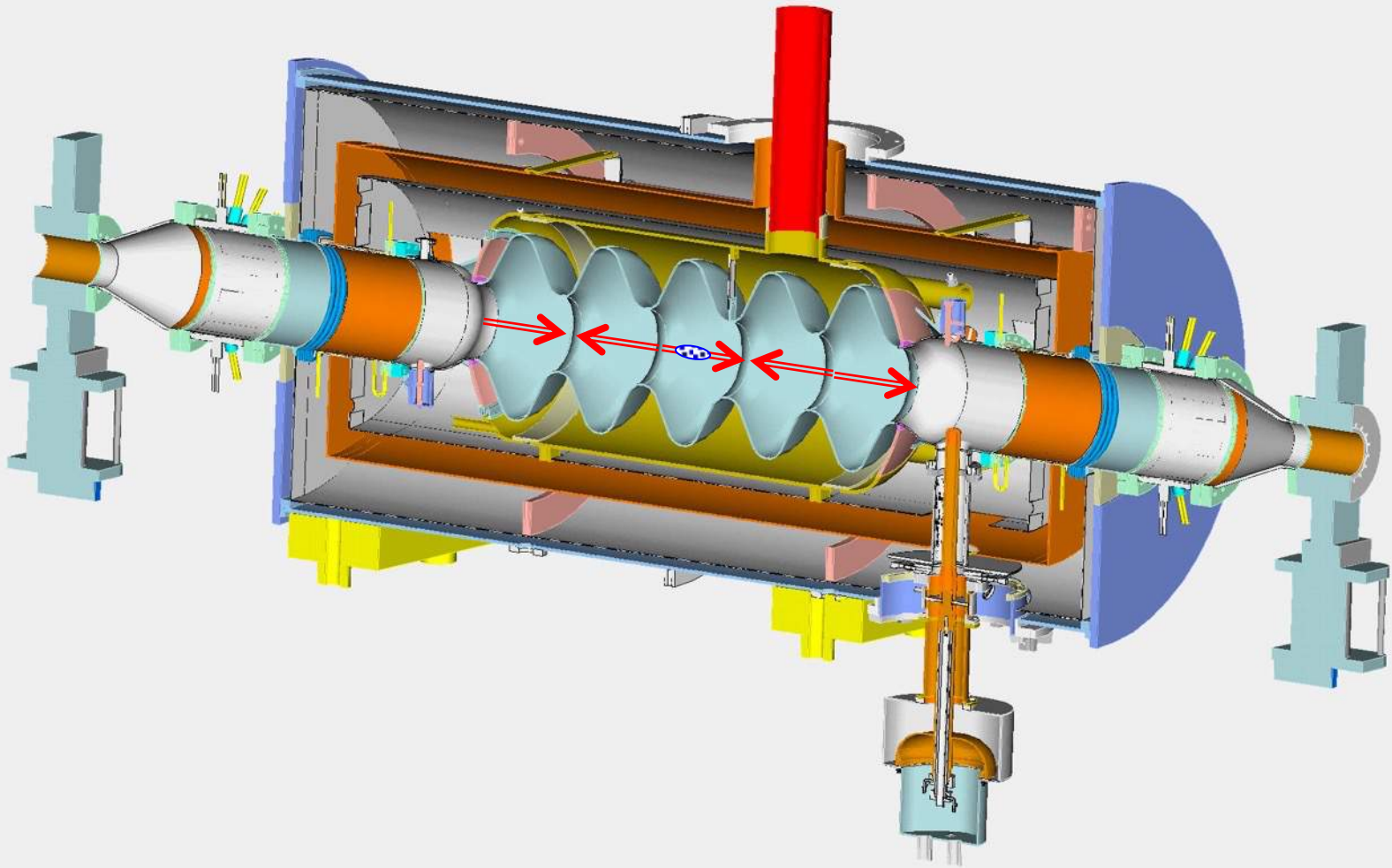
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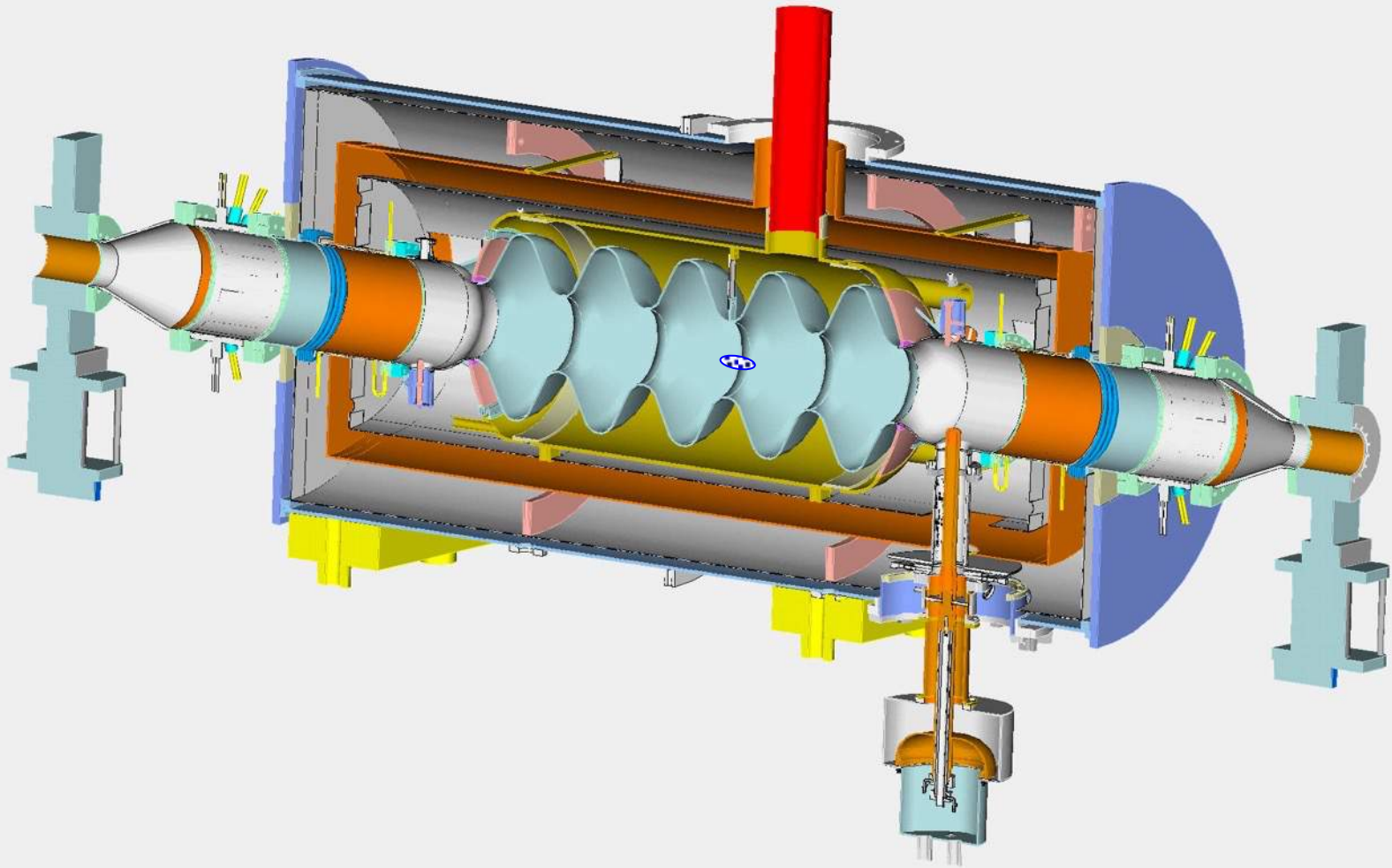
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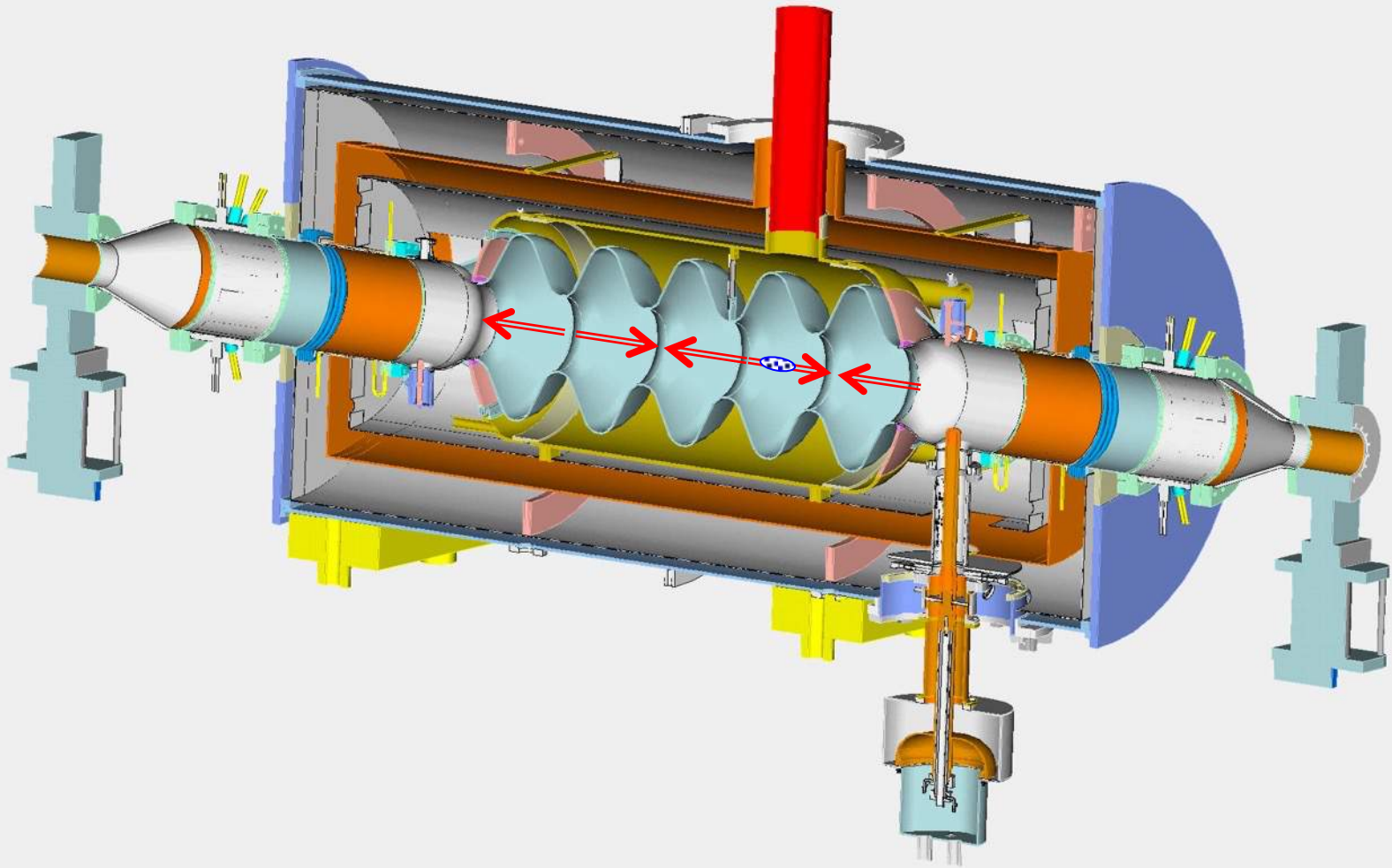
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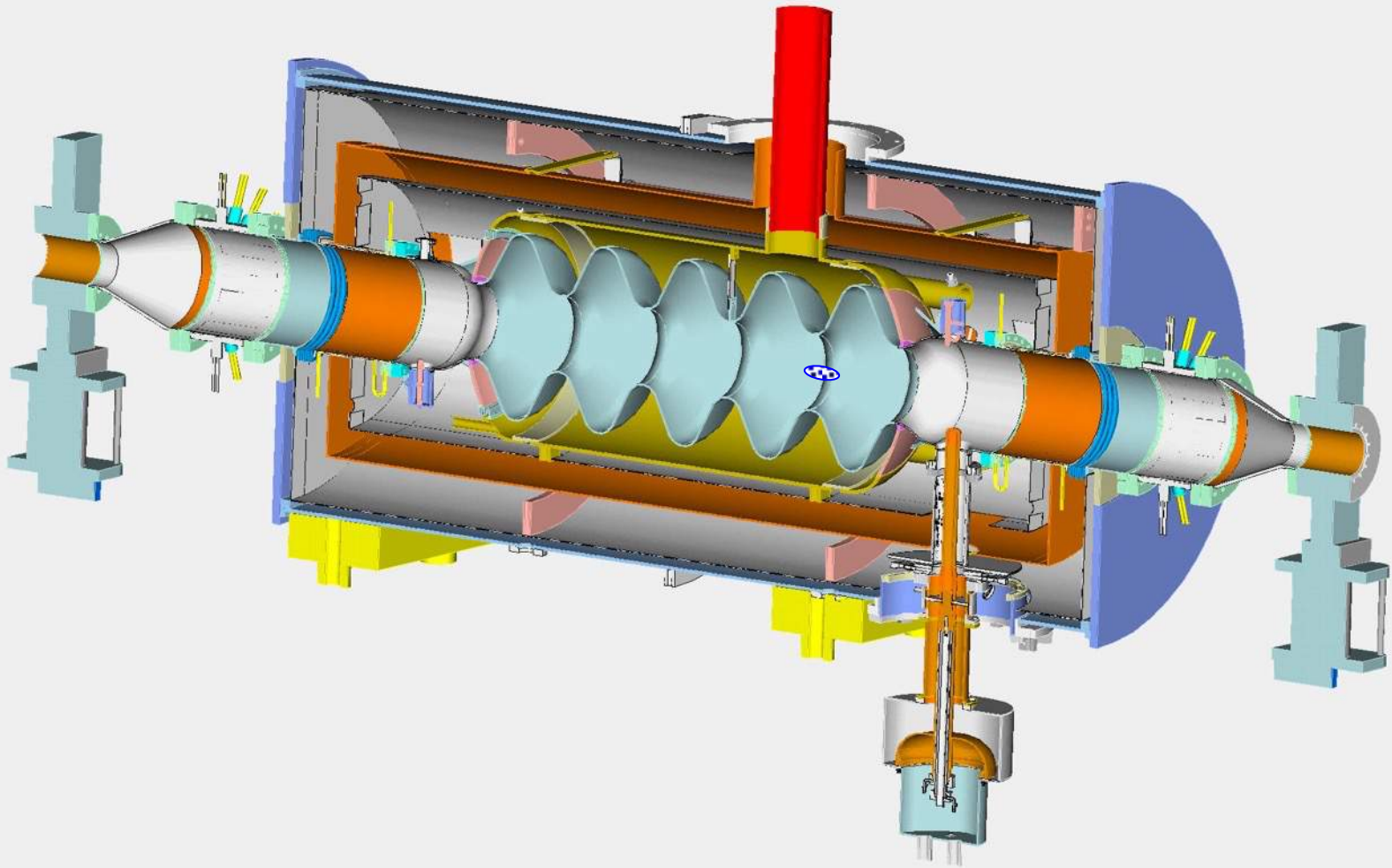
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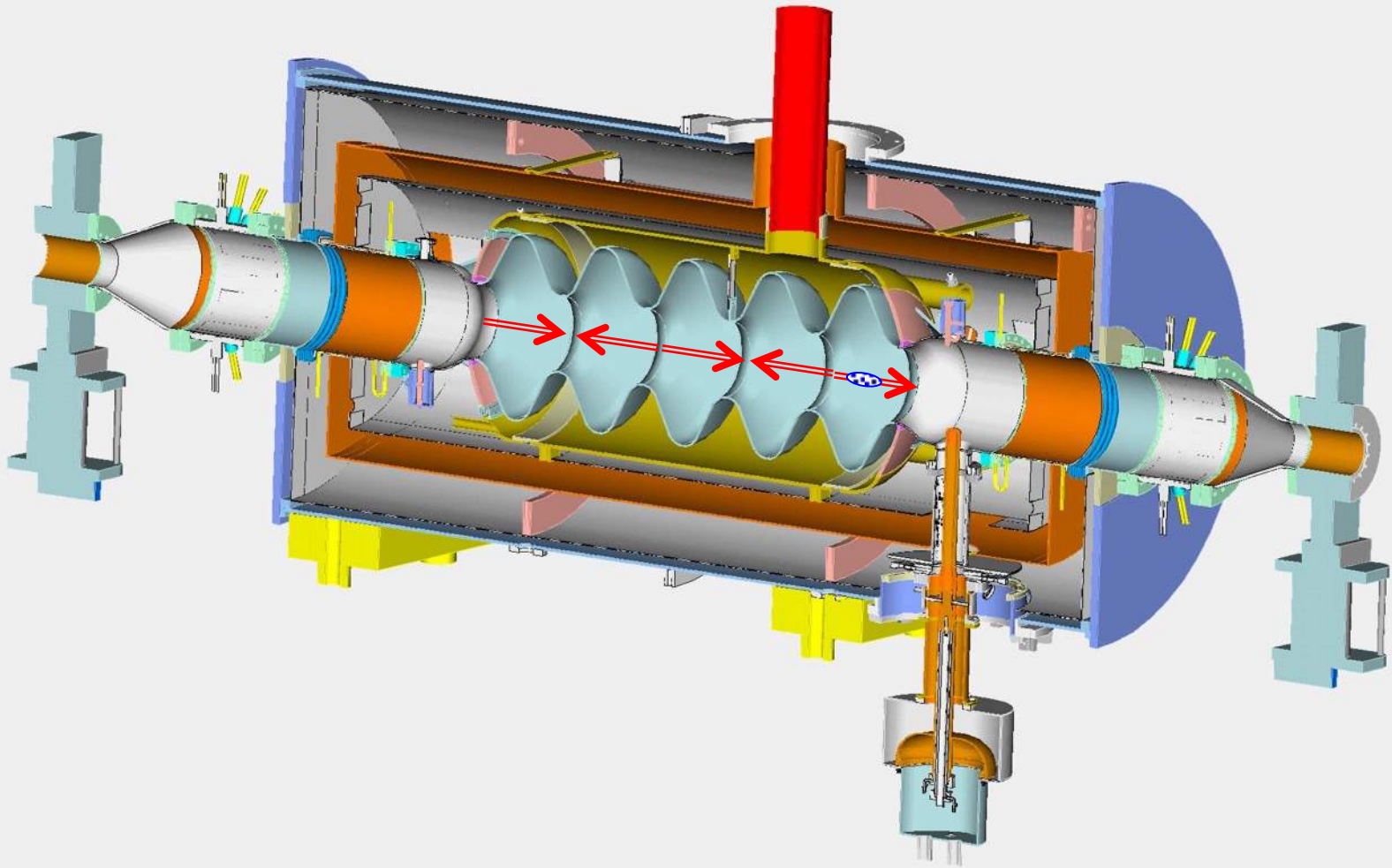
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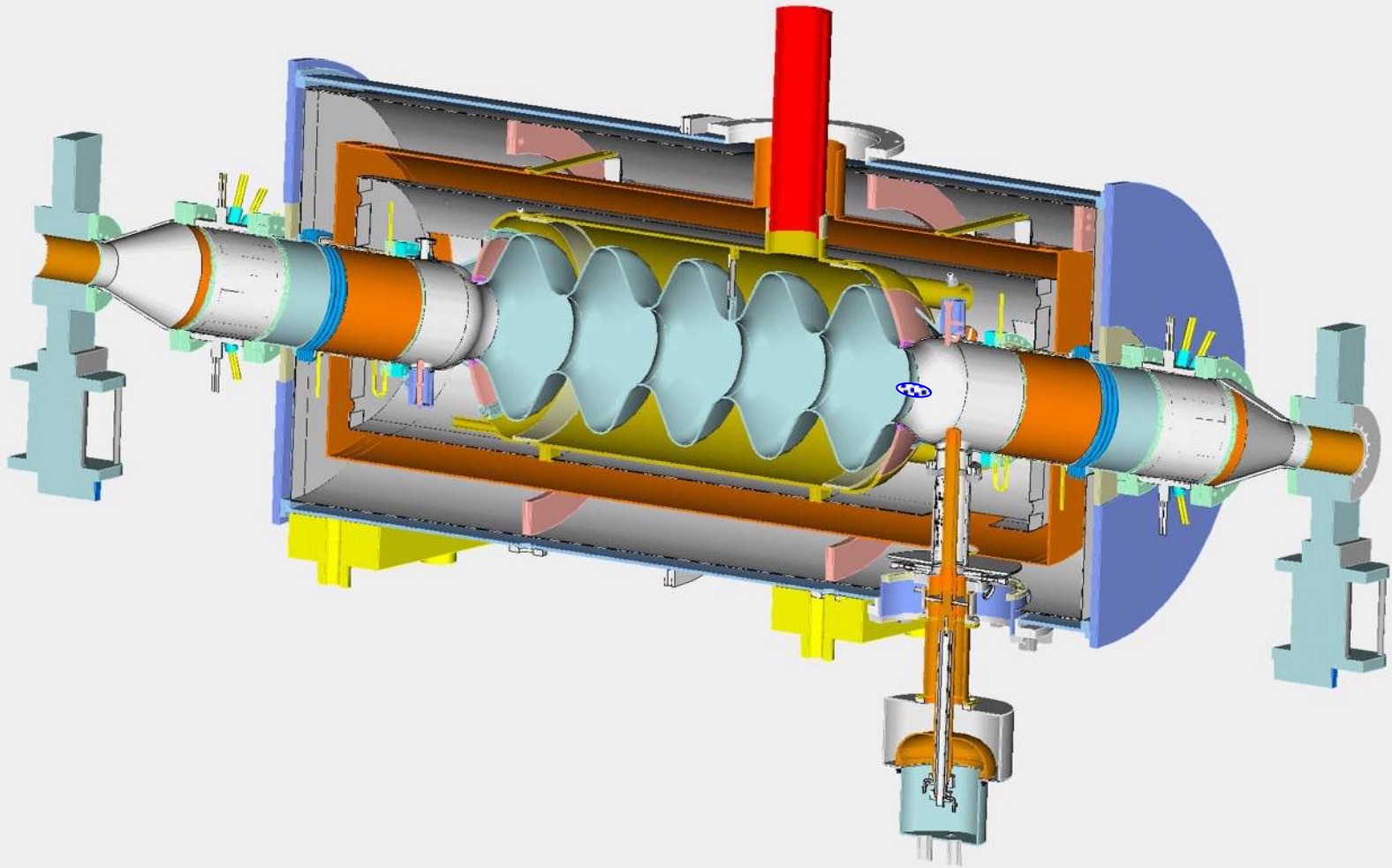
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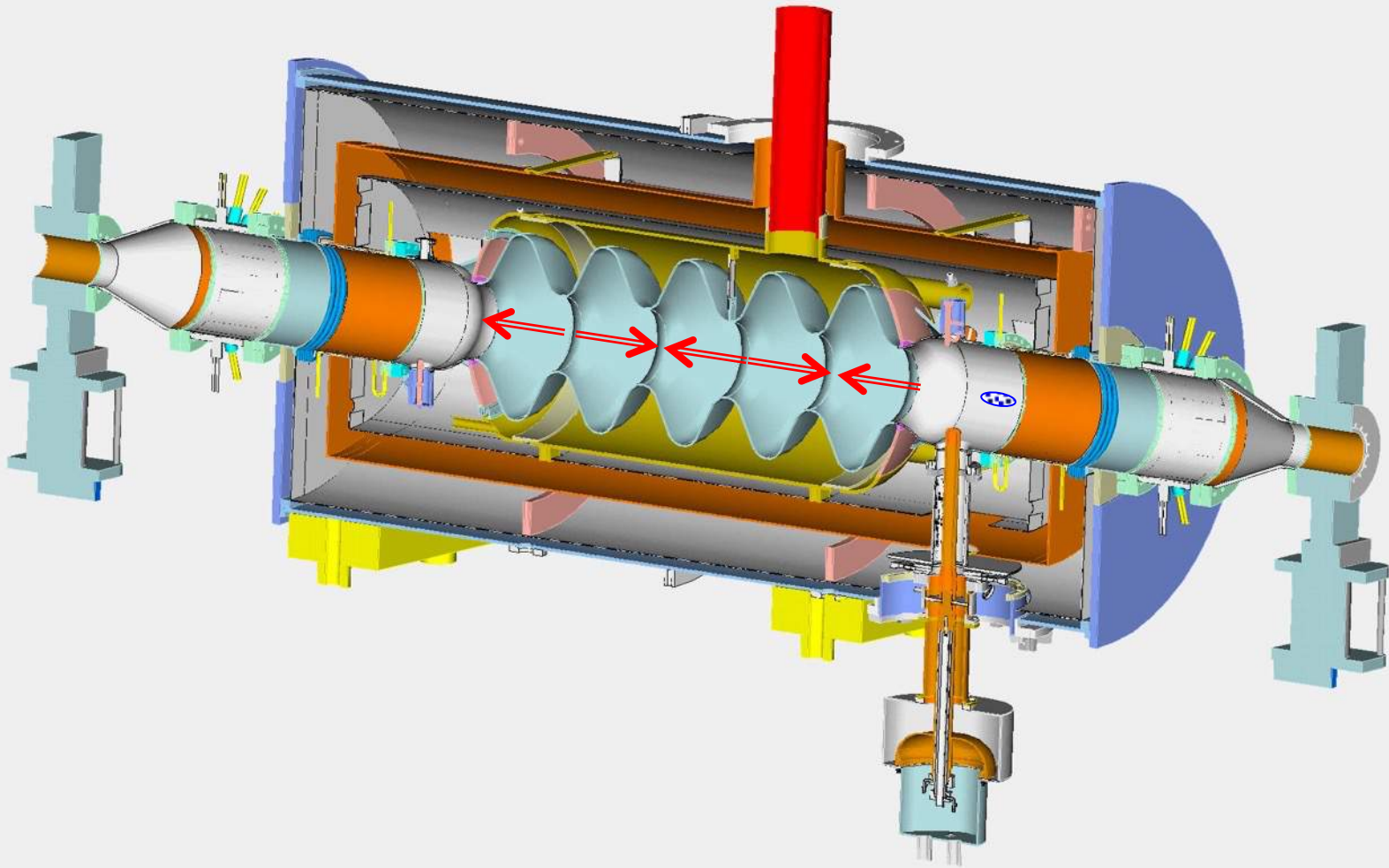
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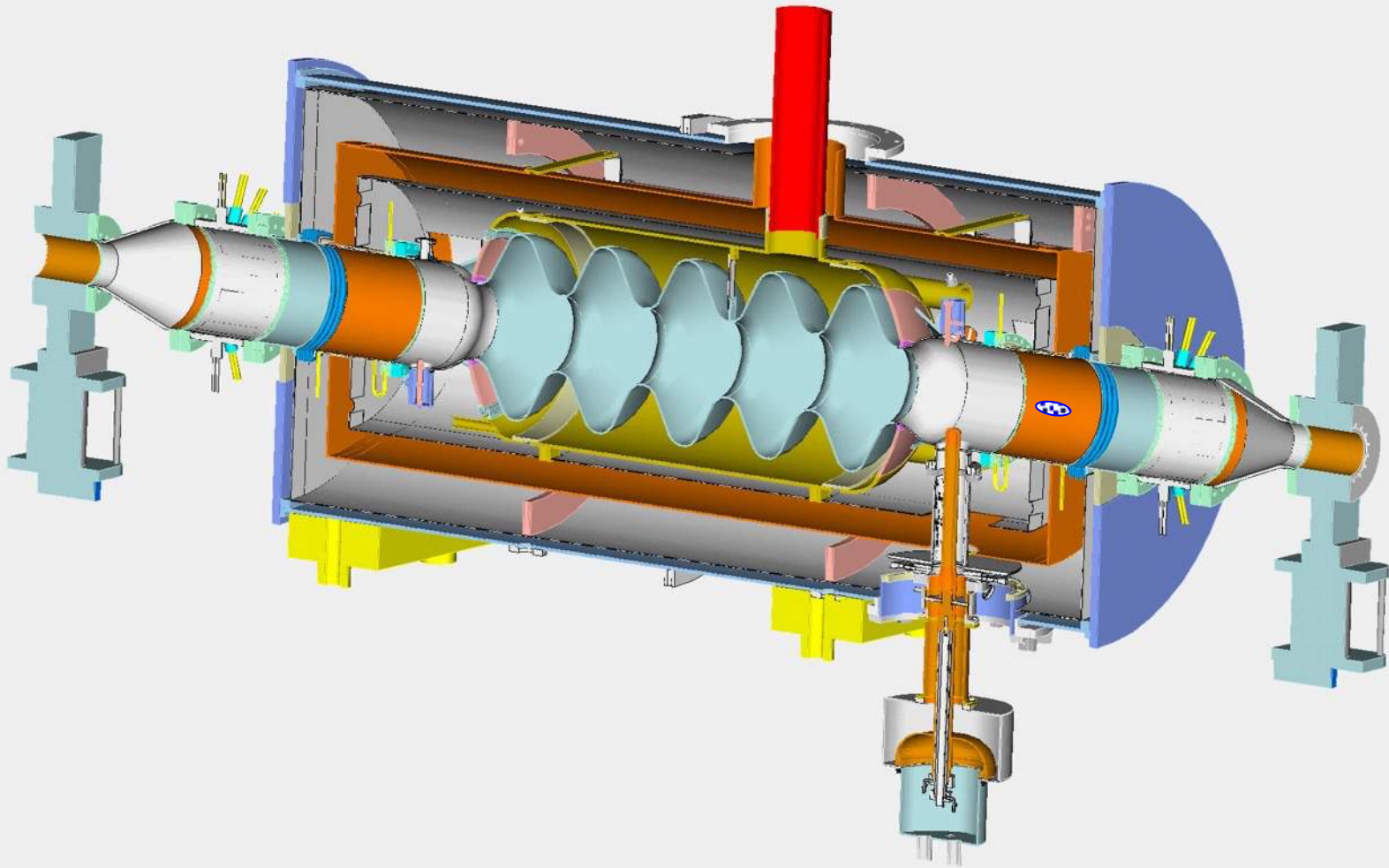
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# Multi-cell cavities

- We learned so far that single cell RF cavity has limited accelerating voltage

$$Max(V_{RF}) = \frac{E_o \lambda_{RF}}{\pi}$$

- To gain more energy we can either use more individual cells or use multi-cell cavities
- The first path, while feasible, is expensive (each cavity would need individual transmitter, waveguide, controls, etc.) and less effective – the average accelerating gradient (energy gain per meter of real estate) would be low
- Thus, where the acceleration gradient is important, the accelerator community uses multi-cell cavities



9-cell Tesla design



7-cell



5-cell



# What we learned

- Resonant modes in a cavity resonator belong to two families: TE and TM.
- There is an infinite number of resonant modes.
- The lowest frequency TM mode is usually used for acceleration.
- All other modes (HOMs) are considered parasitic as they can harm the beam.
- Several figures of merits are used to characterize accelerating cavities: main are accelerating voltage, transit time and Q-factor.
- In a multi-cell cavity every mode splits into a pass-band.
- The number of modes in each pass-band is equal to the number of cavity cells.
- The width of the pass-band is determined by the cell-to-cell coupling.
- Accelerating cavities operate at frequency below the cut-off frequency of vacuum pipes connected to them. The RF field decay exponentially along the pipes and reduces to a negligible level at length  $\sim$  few beam-pipe radii (assuming  $R \ll \lambda_{\text{RF}}$ )
- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities
- Homework is posted on the website this evening: due in one week, March 2