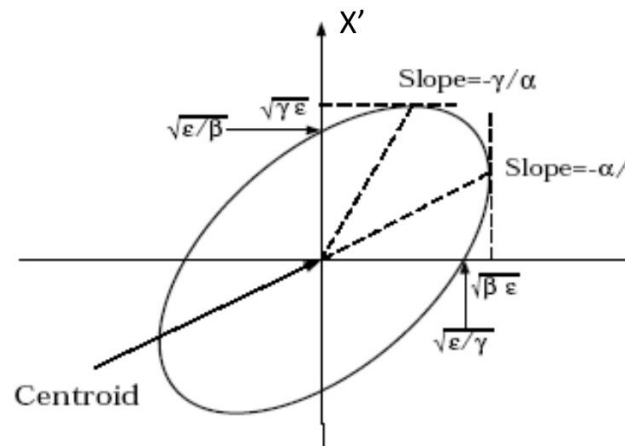


PHY 554

Fundamentals of Accelerator Physics

Lecture 5: Floquet Transformation



From Hill's equation to tune, Twiss functions, normalized phase space, and the Courant-Snyder invariant.

Today's class: turn matrices into optics

- Review Hill's equation, transfer matrices and the FODO cell.
- Clarify when the thin-lens approximation is useful and when it is not.
- Use Floquet theorem for periodic focusing systems.
- Connect Floquet functions to Courant-Snyder parameters.
- Define betatron tune and phase advance.
- Transform to normalized phase space and introduce the Courant-Snyder invariant.

Hill's equation → Floquet form → tune → invariant

Goal: assign physics meanings to beta, alpha and more.

Floquet transformation: $X'' + K(s)X = 0$

$$X(s) = aw(s)e^{i\psi(s)} \quad w'' + K(s)w - \frac{1}{w^3} = 0 \quad \psi' = \frac{1}{w^2}$$

What is the transfer matrix $M(s_2, s_1)$?
 $\psi_2 = \psi_1 + \mu$, where μ is the phase advance.

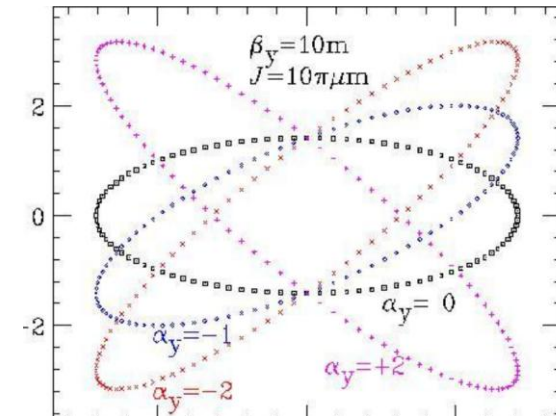
$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1} \cos \mu - w_2 w_1' \sin \mu & w_1 w_2 \sin \mu \\ -\frac{1+w_1 w_2' w_1' w_2'}{w_1 w_2} \sin \mu - (\frac{w_1'}{w_2} - \frac{w_2'}{w_1}) \cos \mu & \frac{w_1}{w_2} \cos \mu + w_1 w_2' \sin \mu \end{pmatrix}$$

$$w_1 = w_2, \quad w_1' = w_2', \quad \psi_2 - \psi_1 = \mu$$

$$M(s) = \begin{pmatrix} \cos \mu - ww' \sin \mu & w^2 \sin \mu \\ -(1+w^2) \sin \mu & \cos \mu + ww' \sin \mu \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \quad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^s \frac{1}{\beta} ds$$



What we learned:

Frenet-Serret coordinates (x,y,s)

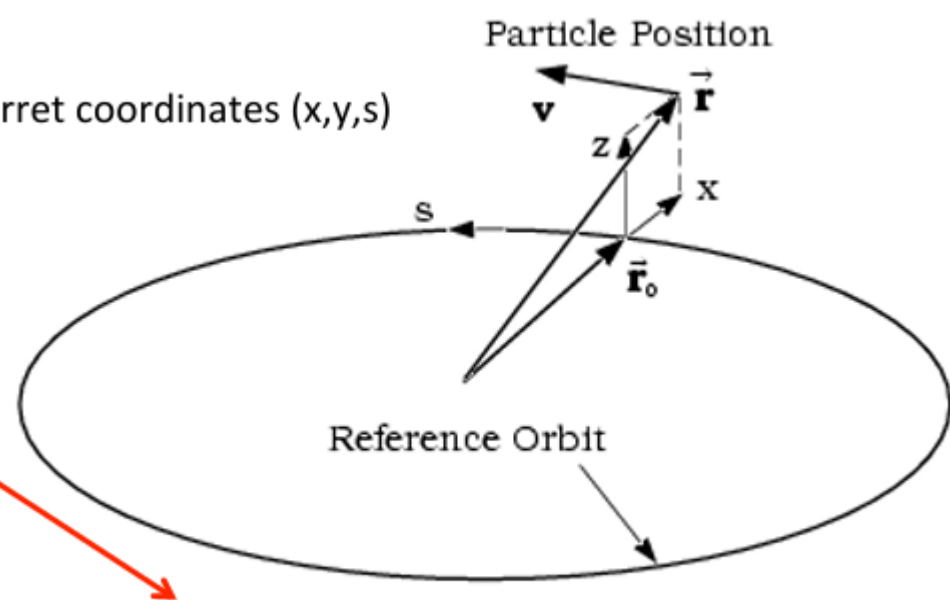
Hill's equations (derivatives w.r.t. s)

$$x'' + K_x(s)x = \pm \frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp \frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \quad K_y(s) = \pm \frac{B_1}{B\rho}$$

Natural focusing from
dipoles (curvature)

Focusing from
quadrupoles



Higher order magnet,
usually field errors

$$\theta = \frac{s}{R} = \frac{\beta ct}{R}$$

Solution of Hill's equations $X(s)$, $X'(s)$ form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s, s_0) \begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

X can be x or y

$$|M(s, s_0)| = 1$$

$$|\text{Trace}(M(s, s_0))| \leq 2$$

Stable solution conditions

Courant-Snyder parameterization

$$M(s) = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix} = I \cos \Phi + J \sin \Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

Where $\alpha, \beta, \gamma, \phi$ are functions of s and describes position dependent beam properties.

Focusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cos \sqrt{K} \ell & \frac{1}{\sqrt{K}} \sin \sqrt{K} \ell \\ -\sqrt{K} \sin \sqrt{K} \ell & \cos \sqrt{K} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Defocusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cosh \sqrt{|K|} \ell & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} \ell \\ \sqrt{|K|} \sinh \sqrt{|K|} \ell & \cosh \sqrt{|K|} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

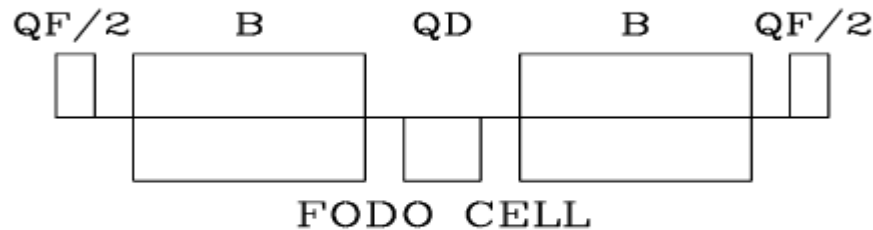
Dipole: $K=1/\rho^2$

$$M(s, s_0) = \begin{pmatrix} \cos \frac{\ell}{\rho} & \rho \sin \frac{\ell}{\rho} \\ -\frac{1}{\rho} \sin \frac{\ell}{\rho} & \cos \frac{\ell}{\rho} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Drift space: $K=0$

$$M(s, s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

$$M(s) = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

This lecture now generalizes from one ideal cell to arbitrary periodic focusing.

$$\cos \Phi = \frac{1}{2} \text{Tr}(\mathbf{M})$$

$$\cos \Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin \frac{\Phi}{2} = \frac{L_1}{2f}$$

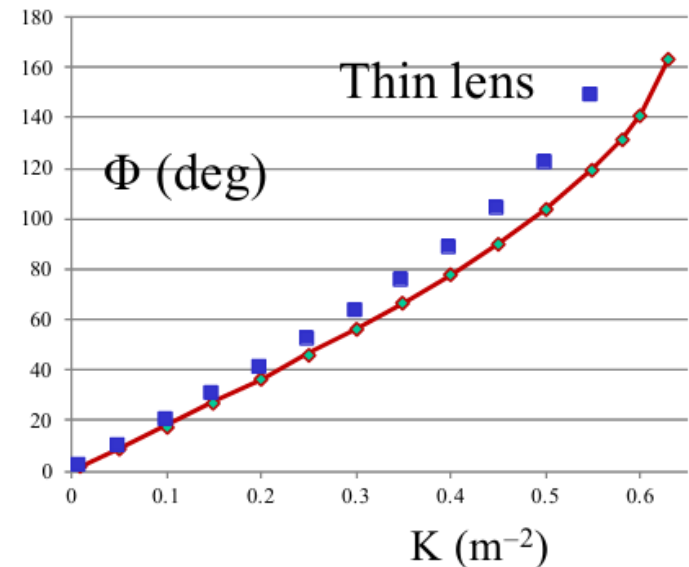
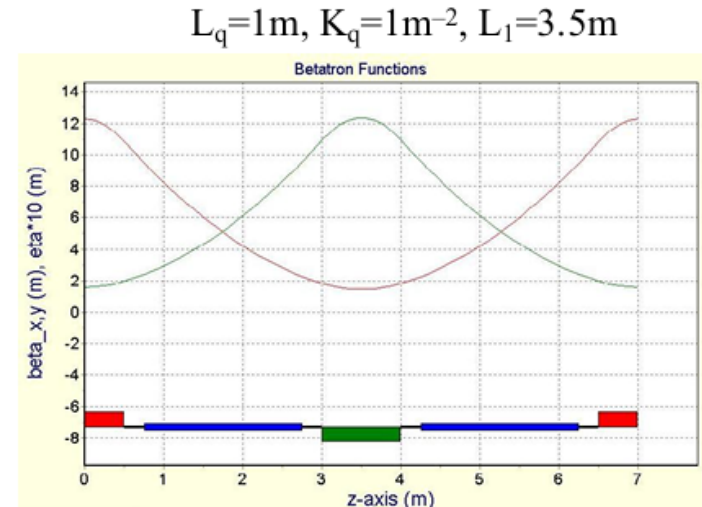
$$\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin \Phi} = \frac{2L_1(1 + \sin \frac{\Phi}{2})}{\sin \Phi}$$

$$\alpha = 0$$

How good is the thin-lens approximation?

- Thin-lens formulas are excellent for scaling and intuition.
- They assume the magnet length is negligible compared with the cell length.
- Finite-length magnets and strong focusing require thick-lens matrices or numerical optics.
- Use thin lenses to see trends; use full matrices for design and comparison with simulation.

In this FODO cell with $L_q=1.0$ m, $L_{\text{dipole}}=2.0$ m, drift length of 0.25 m, and thus $L_1=3.5$ m. Thin lens approximation is good except when the focusing strength is high. The percentage error in phase advance at high focusing gradient can be larger than 11%



Floquet Theorem

We consider the linear Hill's equation of motion $\mathbf{X}'' + \mathbf{K}(s)\mathbf{X} = \mathbf{0}$, where $X(s)$ and $X'(s)$ are conjugate coordinates, $K(s)$ is the focusing function, and the prime is the derivative with respect to the independent variable s . In many accelerator applications, $K(s)$ is a periodic function of s with period L , i.e. $\mathbf{K}(s+L)=\mathbf{K}(s)$. Floquet theorem states we can express the solution in amplitude and phase functions which satisfy a periodic boundary condition similar to that of the potential function $K(s)$, i.e.

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

where the phase advance μ in one period is independent of s . Using the Floquet transformation on Hill's equation, we get the differential equation

$$2w'\psi' + w\psi'' = 0, \quad w'' + K(s)w - w\psi'^2 = 0$$

$$\psi' = \frac{1}{w^2}, \quad \psi = \int_{s_0}^s \frac{ds}{w^2}, \quad w'' + K(s)w - \frac{1}{w^3} = 0$$

Floquet transformation: $X'' + K(s)X = 0$

$$X(s) = aw(s)e^{j\psi(s)} \quad w'' + K(s)w - \frac{1}{w^3} = 0 \quad \psi' = \frac{1}{w^2}$$

What is the transfer matrix $M(s_2, s_1)$?

$\psi_2 = \psi_1 + \mu$, where μ is the phase advance.

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1} \cos \mu - w_2 w_1' \sin \mu & w_1 w_2 \sin \mu \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2} \sin \mu - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right) \cos \mu & \frac{w_1}{w_2} \cos \mu + w_1 w_2' \sin \mu \end{pmatrix}$$

$$w_1 = w_2, \quad w_1' = w_2', \quad \psi_2 - \psi_1 = \mu$$

$$M(s) = \begin{pmatrix} \cos \mu - ww' \sin \mu & w^2 \sin \mu \\ -(1+w'^2) \sin \mu & \cos \mu + ww' \sin \mu \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \quad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^s \frac{1}{\beta} ds$$

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}[1 + (\frac{\beta'}{2})^2] = 0, \quad \text{or} \quad \alpha' = K\beta - \frac{1}{\beta}[1 + \alpha^2]$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1} \cos \psi - w_2 w_1' \sin \psi & w_1 w_2 \sin \psi \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2} \sin \psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right) \cos \psi & \frac{w_1}{w_2} \cos \psi + w_1 w_2' \sin \psi \end{pmatrix}$$

$$w_1 = w(s_1), \quad w_2 = w(s_2), \quad w_1' = w'(s_1), \quad w_2' = w'(s_2)$$

The transfer matrix from s_1 to s_2 in any beam transport line becomes

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \mu + \alpha_1 \sin \mu) & \sqrt{\beta_1 \beta_2} \sin \mu \\ -\frac{1+\alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu & \sqrt{\frac{\beta_2}{\beta_1}} (\cos \mu - \alpha_1 \sin \mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

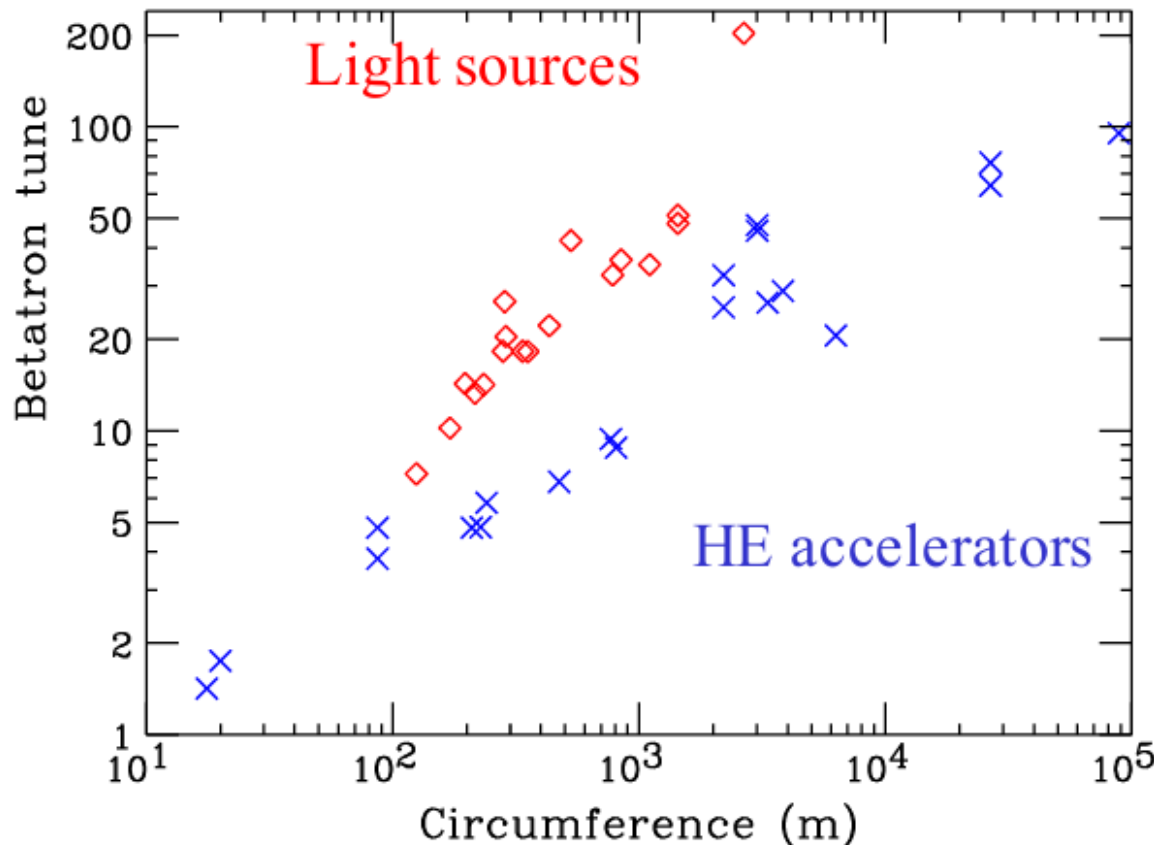
- Between any two locations, the map can be written using the Twiss parameters at both locations and the phase advance between them.
- This separates local scale factors from a simple rotation.
- This is the practical reason beta functions are so useful.

The amplitude function is $w(s) = \sqrt{\beta(s)}$

$$y(s) = a\sqrt{\beta_y(s)} \cos[\psi_y(s) + \xi_y] \quad \text{with} \quad \psi_y(s) = \int_0^s \frac{ds}{\beta_y(s)}$$

Betatron tune (ν_y , or Q_y): The number of betatron oscillations in one revolution.

$$Q_y \equiv \nu_y = \frac{1}{2\pi} \oint \frac{ds}{\beta_y}$$



While beta and alpha are local functions of s , the tune is a global quantity.

Floquet theorem: Many accelerator components obey the periodic condition: $K(s+L)=K(s)$. The solution of Hill's equation is periodic. In matrix representation, we obtain



$$M_1 M_2 \dots M_n \quad M_1 M_2 \dots M_n$$

$$M(s_1+L|s_1)=M_n M_{n-1} M_{n-2} \dots M_2 M_1 = \mathbf{M}(s_1)$$

$$M(s_2+L|s_2)=M_1 M_n M_{n-1} M_{n-2} \dots M_2 = \mathbf{M}(s_2) = M_1 \mathbf{M}(s_1) M_1^{-1}$$

Each matrix is a product of identical number of matrices. They are related by **similarity** transformation. The eigen-values of the periodic matrix $\mathbf{M}(s_i)$ are identical.

$$M(s_2) = M(s_2 | s_1) M(s_1) [M(s_2 | s_1)]^{-1}$$

Local Twiss functions change; phase advance per period does not.

With the similarity transformation of the transfer matrix,

$$M(s_2) = M(s_2 | s_1) M(s_1) [M(s_2 | s_1)]^{-1}$$

the values of the Courant–Snyder parameters $\alpha_2, \beta_2, \gamma_2$ at s_2 are related to $\alpha_1, \beta_1, \gamma_1$ at s_1 by

$$M(s_2) = I \cos \Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin \Phi = I \cos \Phi + J_2 \sin \Phi$$

$$M(s_1) = I \cos \Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin \Phi = I \cos \Phi + J_1 \sin \Phi$$

$$J_2 = M(s_2 | s_1) J_1 [M(s_2 | s_1)]^{-1}$$

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

M_{ij} is the ij -th component of the matrix $M(s_2, s_1)$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

1. The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 \left(s - \frac{\alpha_1}{\gamma_1} \right)^2 = \beta^* + \frac{(s - s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s - s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Note that γ is constant in a drift space, and $s^* = \alpha_1 / \gamma_1$ is the location for an extremum of the betatron amplitude function with $\alpha(s^*) = 0$.

2. Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

where f is the focal length of the quadrupole. Thus a thin-lens quadrupole gives rise to an angular kick to the betatron amplitude function without changing its magnitude.

Floquet transformation:

$$y'' + Ky = \frac{\Delta B}{B\rho}$$

$$x'' + K_x(s)x = \pm \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = \mp \frac{\Delta B_x}{B\rho}$$

Define the normalized coordinate $\eta = \frac{y}{\sqrt{\beta}}$, $\varphi = \frac{1}{\nu} \int^s \frac{1}{\beta} ds$

$$\begin{aligned} \frac{d^2}{d\varphi^2} \eta &= \nu\beta \frac{d}{ds} \nu\beta \frac{d}{ds} \frac{y}{\sqrt{\beta}} = \nu^2 \beta^{3/2} y'' + \nu^2 \left\{ -\frac{1}{2} \beta\beta'' + \frac{\beta'^2}{4} \right\} \frac{y}{\sqrt{\beta}} \\ &= \nu^2 \beta^{3/2} \frac{\Delta B}{B\rho} - \nu^2 \frac{y}{\sqrt{\beta}} \end{aligned}$$

$$\frac{d^2}{d\varphi^2} \eta + \nu^2 \eta = \nu^2 \beta^{3/2} \frac{\Delta B}{B\rho}$$

If $\Delta B=0$, we find

$$\frac{d^2}{d\varphi^2} \eta + \nu^2 \eta = 0$$

A simple harmonic oscillator

$$X'' + K(s)X = 0$$

Since $X(s) = a\sqrt{\beta(s)} \cos(\psi(s) + \psi_0)$ with $\psi(s) = \int_0^s \frac{ds}{\beta(s)}$

Thus $X' = -\frac{X}{\beta} \left(\tan \psi - \frac{\beta'}{2} \right)$

$$\frac{1}{2\beta} [X^2 + (\beta X' + \alpha X)^2] = \frac{X^2}{2\beta} \sec^2 \psi = \frac{a^2}{2} \equiv J$$

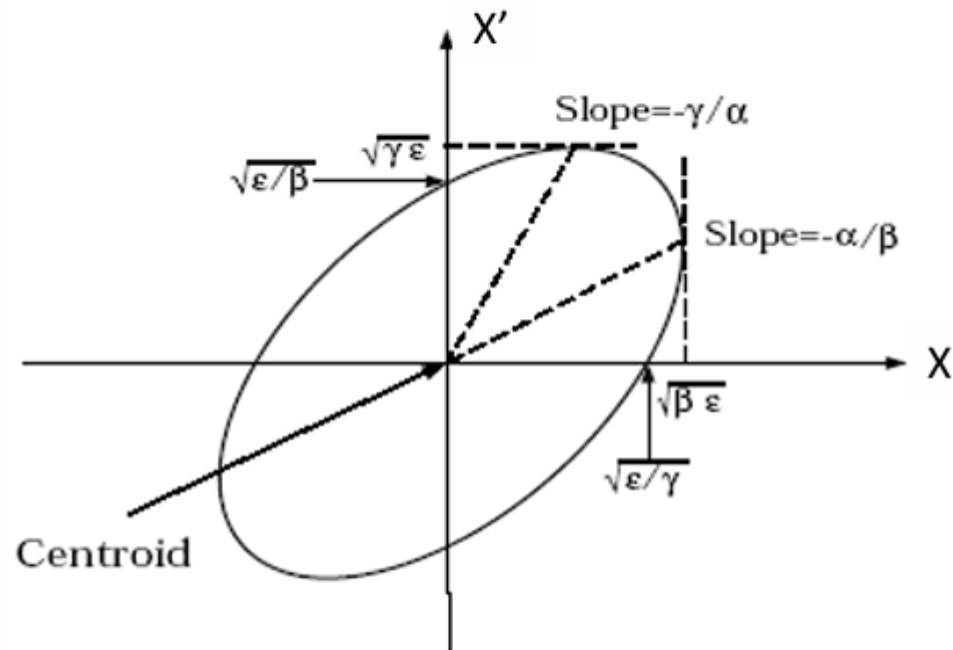
$$X = \sqrt{2\beta J} \cos \psi, \quad X' = -\sqrt{\frac{2J}{\beta}} (\sin \psi + \alpha \cos \psi)$$

Define: $P_X = \beta X' + \alpha X = -\sqrt{2\beta J} \sin \psi$

(X, P_X) form a **normalized phase space coordinates** with $X^2 + P_X^2 = 2\beta J$, here J is called **action**.

Courant-Snyder Invariant

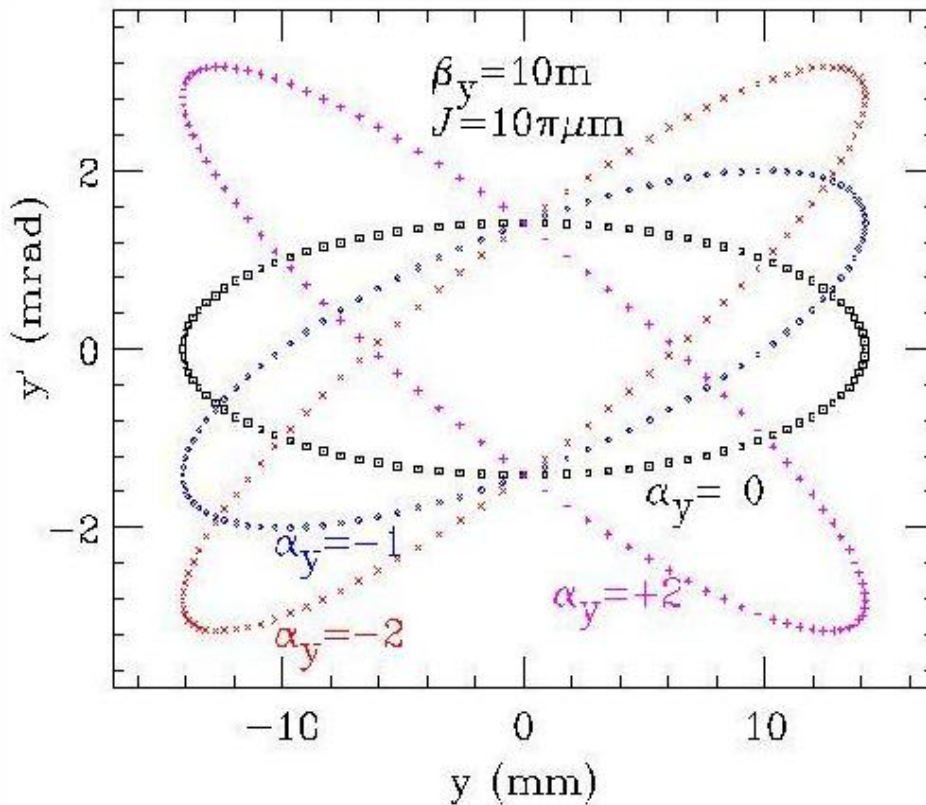
$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta} [X^2 + (\alpha X + \beta X')^2] = 2J \equiv \varepsilon$$



Questions:

- 1) When we have two particles with different action J_1 and $J_2 = 2J_1$, what will their ellipses look like?
- 2) Will the ellipses intersect with each other?

What beta and alpha do to the ellipse



- Beta controls the maximum coordinate excursion: X_{max} scales as $\sqrt{\beta \epsilon}$.
- Gamma controls the maximum angular excursion: X'_{max} scales as $\sqrt{\gamma \epsilon}$.
- Alpha tilts the ellipse by coupling coordinate and angle.
- Area is linearly proportional to the action J ($A = 2 \beta J$)

Key take-away: beta is not a particle property. It is an optical property of the lattice location. While action J is the particle property and stays invariant over lattice location.

What to remember from Lecture 5

- Periodic focusing systems allow Floquet solutions: periodic amplitude plus advancing phase.
- The amplitude function $w(s)$ is $\sqrt{\beta(s)}$, and α is $-\beta' / 2$.
- Tune is the number of betatron oscillations per revolution: $Q = (1/2\pi) \int ds / \beta$.
- Transfer matrices can be factored into local Twiss scaling, a phase rotation, and inverse scaling.
- Normalized coordinates turn betatron motion into a simple harmonic oscillator.
- The Courant-Snyder invariant is the phase-space ellipse area scale for linear motion.

Questions for students

- What is local about β and α , but global about tune?
- Why does a small β imply faster phase advance?
- What changes when a particle has twice the action?

Selected references and cleanup notes

- E. D. Courant and H. S. Snyder, Theory of the Alternating-Gradient Synchrotron, Annals of Physics, 1958.
- S. Y. Lee, Accelerator Physics, World Scientific.
- H. Wiedemann, Particle Accelerator Physics, Springer.
- M. Sands, The Physics of Electron Storage Rings: An Introduction.