Linear Plasma Waves

We start from the linearized set of fluid Maxwell equations:

$$\vec{\nabla}\times\vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \qquad \vec{\nabla}\times\vec{B}_1 = \mu_0\left[e n_0(\vec{V}_{i1}-\vec{V}_{e1})\right] + \mu_0\epsilon_0\frac{\partial \vec{E}_1}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}_1 = 0 \qquad \vec{\nabla}\cdot\vec{E}_1 = \frac{e(n_{1i}-n_{1e})}{\epsilon_0}$$

$$\begin{cases} \dfrac{\partial n_1}{\partial t} + n_0(\vec{\nabla}\cdot\vec{V}_1) = 0 \\ \dfrac{\partial}{\partial t}\vec{V}_1 = \dfrac{q}{m}(\vec{E}_1 + \vec{V}_1\times\vec{B}_0) - \dfrac{\gamma K T}{m n_0}\nabla n_1 \end{cases}$$

for each species

equation of state determines γ

To study the 1D linear wave, we will look for a solution in the form of $e^{i(\vec{K}\cdot\vec{r}-\omega t)}$. Here, $\vec{K}$ is the propagation vector

Derivatives:

$$\frac{\partial}{\partial t} \to -i\omega \qquad \vec{\nabla}\cdot(\,) \;\&\; \vec{\nabla}\times(\,) \to \vec{K}\cdot(\,) \;\&\; \vec{K}\times(\,)$$

We will consider the case of a "small-amplitude" plasma wave in a cold, unmagnetized plasma ($B_0 = 0$) where the electric field and propagation vector are perpendicular to each other.

$$\vec{\nabla}\cdot\vec{E}_1 = \vec{K}\cdot\vec{E}_1 = 0$$

$T_e = 0$ Cold plasma

$\vec{B}_0 = 0$ unmagnetized

This is a transverse electromagnetic wave (i.e. a Light wave) travelling through plasma as it includes both E_1 & B_1. Note that if $B_1 = 0$,

$$\vec{\nabla}\cdot\vec{E}_1 = 0 \;\&\; \vec{\nabla}\times\vec{E}_1 = 0 \xrightarrow[\text{boundary at } \infty]{\text{w/ } \vec{E}=0} \vec{E}_1 = 0 \Rightarrow \text{trivial solution}$$

$$\Rightarrow -\vec{\nabla}\times\vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t}$$

$$\vec{\nabla}\times\vec{B}_1 = \mu_0 e(\vec{V}_{i1} - \vec{V}_{e1}) n_0 + \mu_0\epsilon_0 \frac{\partial \vec{E}_1}{\partial t}$$

Linearized Euler's (momentum) equation:

$$\frac{\partial}{\partial t}\vec{V}_1 = \frac{q}{m}(\vec{E}_1 + \underbrace{\vec{V}_1\times\vec{B}_0}_{0}) - \underbrace{\frac{\gamma k T}{m n_0}}_{0} \nabla n_1$$

$$\boxed{\frac{\partial \vec{V}_1}{\partial t} = \frac{q\vec{E}_1}{m}} \quad \ldots (1)$$

for both ions & e^-

We can derive the electromagnetic wave equation by the usual method of taking $-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_1)$

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_1)$$

$$= -\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0}) + \nabla^2\vec{E}_1 = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}_1)$$

(transverse mode)

$$= \mu_0 e\left[\frac{\partial \vec{V}_{i1}}{\partial t} - \frac{\partial \vec{V}_{e1}}{\partial t}\right] n_0 + \mu_0\epsilon_0\frac{\partial^2\vec{E}_1}{\partial t^2}$$

Substitute $\frac{\partial V}{\partial t}$ for each species from Eqn. 1:

$q = -e$ for an electron

$$\nabla^2\vec{E}_1 - \frac{1}{c^2}\frac{\partial^2\vec{E}_1}{\partial t^2} = \mu_0 e n_0\left[\frac{e\vec{E}_1}{M} + \frac{e\vec{E}_1}{m}\right]$$

ion mass $\qquad$ e^- mass

$$= \mu_0\epsilon_0\left[\frac{e^2 n_0}{M\epsilon_0} + \frac{e^2 n_0}{m\epsilon_0}\right]\vec{E}_1$$

$$= \frac{1}{c^2}\left[\Omega_p^2 + \omega_p^2\right]\vec{E}_1$$

$$\therefore \boxed{\nabla^2\vec{E}_1 - \frac{1}{c^2}\frac{\partial^2\vec{E}_1}{\partial t^2} - \frac{1}{c^2}\left[\Omega_p^2 + \omega_p^2\right]\vec{E}_1 = 0} \quad \ldots (2)$$

Note that because of the discrepancy between the masses of electrons and ions,

$$\frac{\Omega_p^2}{\omega_p^2} = \frac{m}{M} \ll 1 \quad \left(\text{for hydrogen, it's } \frac{1}{1836}\right)$$

$$\Rightarrow -\Omega_p^2 + \omega_p^2 \approx \omega_p^2$$

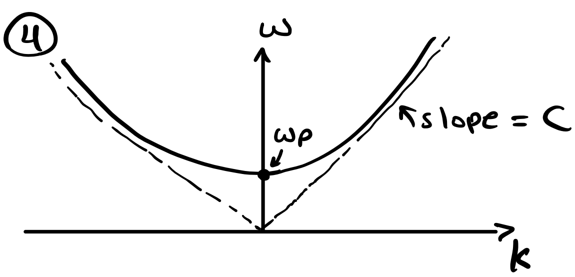
The implicit assumption in ignoring the ion plasma frequency is that ions are so heavy that we can essentially consider them as a uniform immobile background. In plasma physics jargon, we call this a high frequency plasma wave, since the wave oscillations occurs so fast that the ions don't have time to respond, and can be considered infinitely mobile.

Substituting the wave term, $\vec{E}_1 = \tilde{E}_1 e^{i(\vec{K}\cdot\vec{r} - \omega t)}$ in Eqn 2:

$$\Rightarrow \left[-K^2 + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}\right] \vec{E}_1 = 0 \quad \ldots (3)$$

Eqn 3 reveals the dispersion relation, $\omega(k)$ in a cold, unmagnetized plasma:

$$\boxed{\omega^2 = \omega_p^2 + c^2 K^2} \quad \ldots (4)$$



The dispersion relation allows us to calculate the phase & group velocity:

$$\begin{cases} V_\phi = \omega / k \\ V_g = \dfrac{d\omega}{dk} \end{cases}$$

∴ The group velocity is the slope of this curve, so

at $k = 0$, $V_g = 0$

$k \to \infty$, $V_g \simeq c$

We can derive exact expression for the phase & group velocity from the dispersion relation:

$$(4) \Rightarrow 2\omega d\omega = c^2 (2k\,dk) \Rightarrow \frac{d\omega}{dk} = \frac{c^2}{\omega/k} \Rightarrow \boxed{V_g V_\phi = c^2} \ldots (5)$$

Also, $(4) \Rightarrow \omega^2 = \omega_p^2 + c^2 K^2 \Rightarrow \eta = \dfrac{c}{V_\phi} = \dfrac{cK}{\omega} = \left(1 - \dfrac{\omega_p^2}{\omega^2}\right)^{1/2}$

index of refraction

$$\therefore \frac{\omega}{K} = V_\phi = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}}$$ always $> c$ ← c divided by a number smaller than 1

$$\therefore V_g = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$ always $< c$

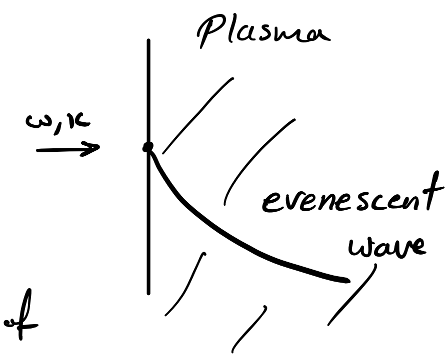
The group velocity is always less than *c*, as required by the special theory of relativity.

Cut off: from the dispersion relation, one can see that frequencies below ω_p are not supported by this wave:

$$\omega < \omega_p \Rightarrow cK = \sqrt{\omega^2-\omega_p^2} \text{ is imaginary.}$$

$$= i\sqrt{\omega_p^2-\omega^2}$$

$$\Rightarrow E \propto e^{iKz} = e^{-(\omega_p^2-\omega^2)z/c}$$



So, the wave is reflected at the boundary of plasma & an evenescent wave with skin depth δ extends into plasma:

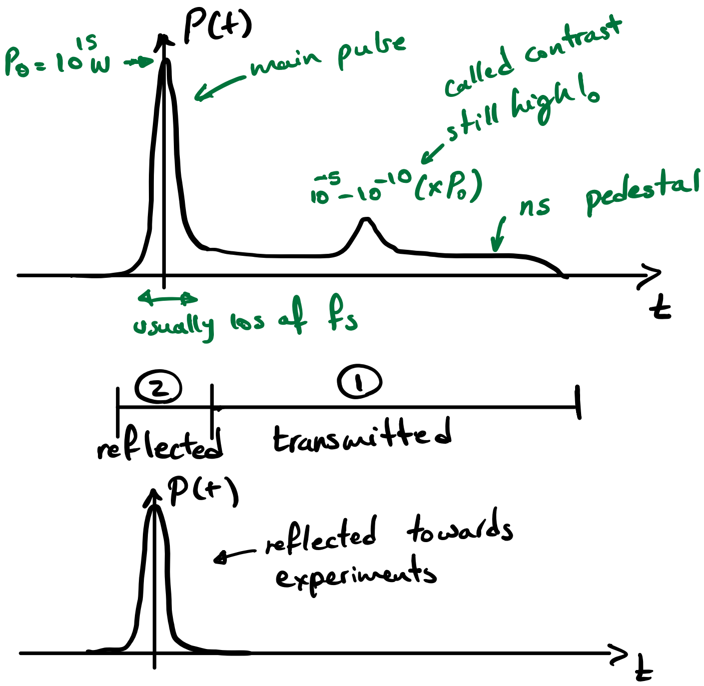
$$e^{iKx} = e^{-x/\delta} \Rightarrow \delta = \frac{c}{(\omega_p^2-\omega^2)^{1/2}}$$

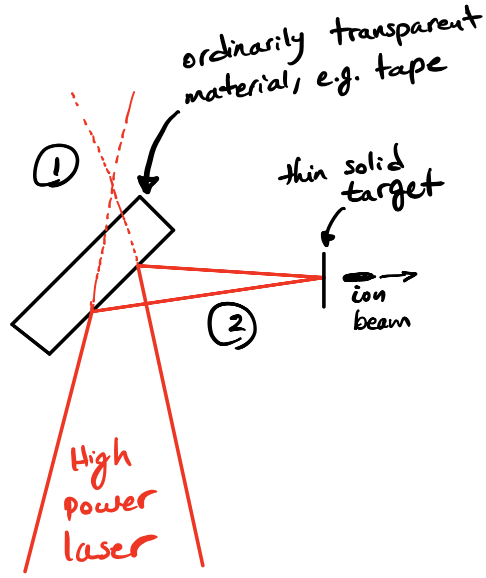
→ The density at which $\omega = \omega_p$ is called the critical density for that frequency:

$$\omega = \omega_p \Rightarrow \boxed{n_{crit} = m\epsilon_0\omega^2/e^2} \quad \dots (6)$$

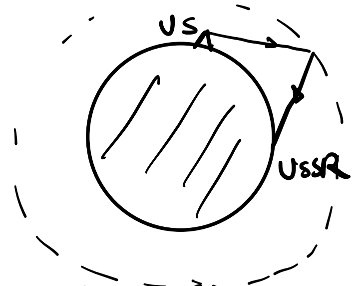
This is a very important conclusions. With some exceptions, such as very thin plasma and very high intensity lasers, a laser pulse impinging on a plasma with higher than the critical frequency will be completely reflected. There are several applications that rely on this property such as density measurement. One particular application used in petawatt laser experiments is the use of “plasma mirrors”, which remove ns or ps “pedestals” in the pulse which can create a pre plasma and are detrimental to physics under study, particularly in laser-solid interactions [see e.g. Thaury, et al, Nature Physics, 3, 424

(2007)].





Other examples of the reflection of waves at the cut of frequency include the reflection of short wave radios off of the ionosphere where plasma density is sufficiently high. This effect enabled the communication across continents



→ space shuttle reentry: the intense heat of friction on reentry creates plasma, resulting in communication blackout during reentry

Note: the index of refraction is density dependent:

$$\eta = \sqrt{1-\frac{\omega_p^2}{\omega^2}} = \left(1-\frac{n}{n_{crit}}\right)^{1/2}$$

Therefore density tailoring may be done to dictate the behavior of light (e.g. laser) as desired. Plasma density channels have been created to extend the region of laser-plasma interaction to tens of centimeter (many times Rayleigh length) using a method called ignitor-heater concept:

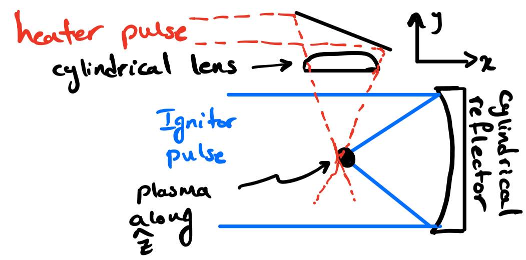
1- Ignitor creates a uniform plasma column

2- Heater laser is focused to the middle & heats plasma

3- Heated electrons move away from the axis, creating a parablic density profile, ideal for laser guiding. see e.g. P. Volfbeyn, "Guiding of laser pulses in plasma channels created by ignitor heater technique" Physics of plasmas, 6, 2269 (1999)

Ideal channel e^- density profile: $n(r) = \begin{cases} n_0 + \Delta n \dfrac{r^2}{r_0^2} & 0<r<r_{ch} \\ 0 & r>r_{ch} \end{cases}$

channel radius



Note: $v_\phi > c$, so no particles can keep up w/ & surf the wave.

Finally, if T_e were included, nothing changes w/ this analysis as you will show in a homework.

Nonlinear Plasma Waves in 1D

Generation of nonlinear plasma waves was investigated in a Russian paper in 1956 (A.I. Akhiezer, R.V. Polovin, “Theory of wave motion of an electron plasma”, Sov.Phys.JETP 3 (1956) 696). John Dawson at UCLA started looking at the same problem for acceleration of electrons in the 80’s.

To look at the for of the plasma waves, we start from the 1D cold fluid equation. We make the assumption that electrons move in 1D while ions stay in place. In this way, the sources for charge and current density are the electrons. Physically, this means that the driver and the accelerating wave have nearly uniform transverse profiles and are much larger than c/ω_p

Gauss's law: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0} \Rightarrow \dfrac{\partial}{\partial z} E_z = \dfrac{e}{\epsilon_0}(n_0 - n)$

background ion density e^- density

Faraday's law: $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Ampere's law: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$= \mu_0 en\vec{v} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

note, $J = \rho v$ & only e^- contribute to current density since ions are assumed immobile.

& $\vec{\nabla}\cdot\vec{B} = 0$ as always

Finally, the nonlinear fluid momentum equation for e^- is

$$\frac{\partial \vec{P}}{\partial t} + (v\cdot\vec{\nabla})\vec{P} = -e\vec{E} - e(\vec{v}\times\vec{B})$$

Note: cold plasma means $T=0 \Rightarrow \underline{\mathbb{P}} = 0$

In the words of Akhiezer and Polovin, “our problem consists of the general investigation of the [1D] wave motion in the plasma; I.e. of such electron motions for which <u>all</u> the variables entering [equations above] are functions not of ‘r’ and ‘t’ separately, but only of the combination $\zeta = v_\phi t - z$.
This combination identifies a wave that travels in the ‘z’ direction with a phase velocity v_ϕ

Thus all variables are a function of a single variable, which itself is a function of two variables. Using the chain rule, we convert these equations to those in terms of ζ

$$\zeta = V_\phi t - z = V_\phi t - \hat{z}\cdot\vec{r}$$

$$\frac{\partial}{\partial t} f(\zeta) = \frac{d}{d\zeta} f \; \frac{\partial \zeta}{\partial t} = V_\phi \frac{d}{d\zeta}$$

$$\frac{\partial}{\partial z} f(\zeta) = \frac{d}{d\zeta} f \; \frac{\partial \zeta}{\partial z} = - \frac{d}{d\zeta}$$

Spatial derivatives including Curl & divergence are expressed as

$$\begin{cases} \vec{\nabla} \times f = -\hat{z} \times \dfrac{df}{d\zeta} \\ \vec{\nabla} \cdot f = -\hat{z} \cdot \dfrac{df}{d\zeta} \end{cases}$$

Maxwell & fluid equations can then be rewritten as a function of ζ. For example

Faraday's Law: $-\hat{z} \times \frac{d\vec{E}}{d\zeta} = V_\phi \frac{d\vec{B}}{d\zeta} \Rightarrow \vec{B}$ is $\perp$r to $\hat{z} \Rightarrow \boxed{B_z = 0}$

Since this is a 1D wave motion, the motion of e^- as well as the wave propagation direction are in $\hat{z}$ direction. We are therefore interested in z component of the momentum eqn & related quantities:

$$\frac{\partial \vec{P}}{\partial t} + (v \cdot \vec{\nabla}) \vec{P} = -e\vec{E} - e(\vec{v} \times \vec{B})$$

z component = 0 since $\vec{v}$ is along z

z component $\Rightarrow$ $V_\phi \frac{dP_z}{d\zeta} - v \frac{dP_z}{d\zeta} = -e E_z$

E_z is described by Gauss's Law & Ampere's Law:

$$-\hat{z} \times \vec{B} = \mu_0 e n \vec{v} + V_\phi \mu_0 \epsilon_0 \frac{d\vec{E}}{d\zeta} \xRightarrow{z \text{ comp.}}$$

z component $\Rightarrow$ $0 = \mu_0 e n v + \frac{V_\phi}{c^2} \frac{dE_z}{d\zeta}$

Gauss's law $\rightarrow$ $-\frac{dE_z}{d\zeta} = \frac{e}{\epsilon_0}(n_0 - n)$

Normalize the equations using ω_p as normalizing frequency

& drop the z subscript

e.g. $\left(\frac{V_\phi}{c} - \frac{V}{c}\right) \frac{c}{\omega_p} \frac{dP_z}{dz} \frac{1}{mc} = -e E_z / mc\omega_p$

$\frac{V}{c} \rightarrow V \qquad \frac{eE}{mc\omega_p} \rightarrow E \qquad \frac{\omega_p}{c} z \rightarrow z$

$\frac{P}{mc} \rightarrow P \qquad \frac{n}{n_0} \rightarrow n \qquad \frac{1}{\epsilon_0}, \mu_0 \rightarrow 1$

$$\Rightarrow \begin{cases} (V_\phi - V) \frac{d}{dz} P = -E & \dots (1) \\ V_\phi \frac{d}{dz} E = nV & \dots (2) \\ -\frac{dE}{dz} = 1 - n & \dots (3) \end{cases}$$

$$\frac{(2)}{V_\phi} + (3) \Rightarrow 0 = \frac{nV}{V_\phi} + 1 - n$$

$$\Rightarrow \boxed{n = \frac{1}{1 - \frac{V}{V_\phi}}} \quad \dots (4)$$

In unnormalized units, $n = \dfrac{n_0}{1 - \frac{V}{V_\phi}}$

i.e. For there to be a physical solution, $V < V_\phi$

Note that (4) is a fully non-linear relationship. If the perturbation was small, we would Taylor expand the right hand side & would get a linear relationship between $\delta n = n - n_0$ & $\frac{V}{V_\phi}$.

Incidentally, equation (4) is consistent with the continuity equation:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{V}) = 0$$

$$\text{in 1D} \Rightarrow \frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(n\vec{V}) = 0$$

$$V_\phi \frac{d}{dz} n - \frac{d}{dz}(nV) = 0 \Rightarrow \frac{d}{dz}\left[n\left(1 + \frac{V}{V_\phi}\right)\right] = 0$$

Now, we want to solve for a relationship between the fields & density (or equivalently fluid velocity) to get the properties of this nonlinear wave.

2 $\Longrightarrow$ $$\frac{dE}{d\zeta} = \frac{nv}{v_\phi} \quad \cdots (5)$$

multiply (5) into (1) in reverse order

$$n \frac{v}{v_\phi}(v_\phi - v) \frac{d}{d\zeta} p = - E \frac{d}{d\zeta} E$$

$$v \underbrace{n\left(1 - \frac{v}{v_\phi}\right)}_{\equiv 1 \text{ from } (4)} \frac{dp}{d\zeta} = - \frac{d}{d\zeta}(E^2/2)$$

$$\frac{p}{\gamma}\frac{dp}{d\zeta} = - \frac{d}{d\zeta}(E^2/2) \quad \cdots (6)$$

Note, $\vec{p} = \gamma m \vec{v} \Rightarrow p = \gamma v$ in normalized units

$$\gamma = (1+p^2)^{1/2} \Rightarrow \frac{p}{\gamma}\frac{dp}{d\zeta} = \frac{d\gamma}{d\zeta}$$

$$\Rightarrow (6) \Rightarrow \frac{d}{d\zeta}\left(\gamma + \frac{E^2}{2}\right) = 0$$

$$\Rightarrow \boxed{\gamma + \frac{E^2}{2} = \text{constant} = C_0} \quad \cdots (7)$$

Note: Because of the complexity of particle motion in plasma, we are almost always on the lookout for them!

$$(7) \Rightarrow \boxed{E(\zeta) = \pm\sqrt{2}\left[C_0 - \sqrt{1-p^2(\zeta)}\right]^{1/2}} \quad \cdots (8)$$

From (1), $E=0$ at extremums of the fluid momentum, p, & vice versa. Let p oscillate between a maximum & minimum

$$p_- < p < p_+$$

We know that $E=0$ at both p_+ & p_-. The only way that can happen from equation 8, given that C_0 is a constant, is that $P_- = -P_+$.

$$-P_0 < P < +P_0 \quad \ldots \quad (9)$$

Let $P = P_0 \Rightarrow E = 0 \Rightarrow C_0 = \sqrt{1+P_0^2}$

$$\Rightarrow E(\zeta) = \sqrt{2}\left[\sqrt{1+P_0^2} - \sqrt{1+P(\zeta)^2}\right] \ldots \quad (10)$$

$\Rightarrow$ Peak value of E occurs at $P=0$

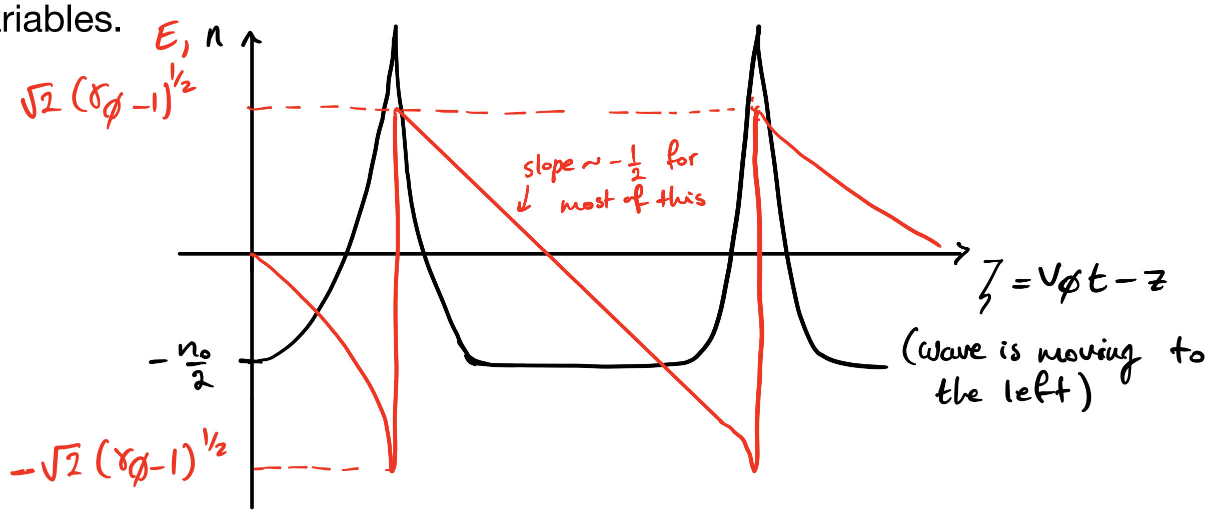
$$\boxed{E_0 = \sqrt{2}\left[\sqrt{1+P_0^2} - 1\right]^{1/2}} \quad \ldots \quad (11)$$

What is the maximum wave amplitude E_0 that the plasma can support? Max field amplitude happens when momentum amplitude has its max value:

$V_{max} = V_\phi \Rightarrow E_0^{max}$ occurs if $\sqrt{1+P_0^2} = \gamma_\phi$

$$\Rightarrow \boxed{E_{max} = E_{WB} = \sqrt{2}\left(\gamma_\phi - 1\right)^{1/2}} \quad (12)$$

This result was published first by the Akheizer paper referenced above. John Dawson (UCLA) realized in 1958 that physically this process is corresponds to wave breaking, which means that wave is steepening to the point that the top of the wave tilts over and crashes forward. This limit is strictly valid in the cold plasma limit. Using the relationships above we can plot the different variables.



→ slope of electric field is proportional to density; i.e. for almost the entire length of the wake, slope $\sim -\frac{1}{2}$, given by

$$n = \frac{1}{1 - v/v_\phi} \sim \frac{1}{2} \text{ for } v \sim -v_\phi$$

$$\frac{dE}{d\zeta} = n - 1 = -\frac{1}{2}$$

Finally, what is the wavelength?

in a linear wave,

$$\omega = \omega_p \Rightarrow \lambda_L = \frac{2\pi c}{\omega_p}$$

But this is a non-linear wave;

since the slope of the $E(\zeta)$ field is $\frac{1}{2}$ & the peak & valley of E are separated by $2\sqrt{2}(\gamma_\phi - 1)^{1/2}$,

$$\lambda_{NL} = 4\sqrt{2}\,(\gamma_\phi - 1)^{1/2} \quad \dots (13)$$

1D wakefield was one of the earliest attempts at developing a plasma wakefield theory & some of the conclusions (such as the slope of electric field) will remain valid even in 3D. The 1D theory however is of limited application. A 1D laser would have to be very intense and very wide compared to its wavelength. This has become possible only recently by the introduction of petawatt lasers. For the particle beam driver, we also typically operate in a regime where beam waist is small compared to the plasma wavelength. To get results that are closer to real physics of experiments, we will need analyze this physics in multi-dimensions and in particular in nonlinear regime.