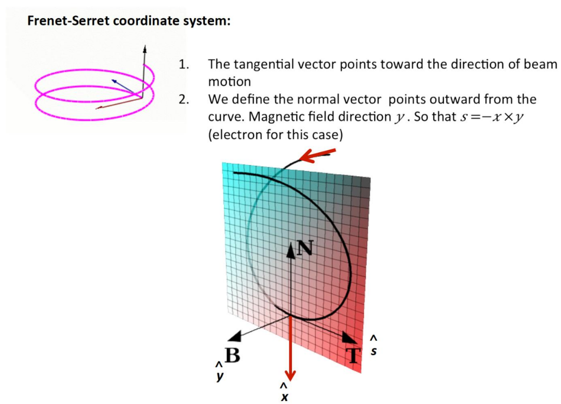
PHY 554

Fundamentals of Accelerator Physics

Lecture 4: Linear Betatron Motion

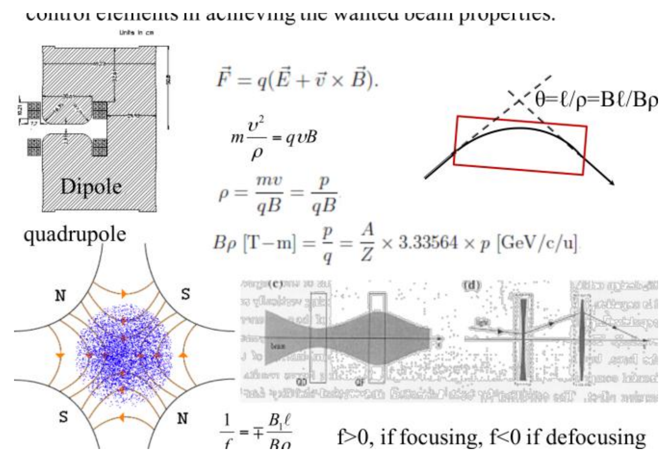


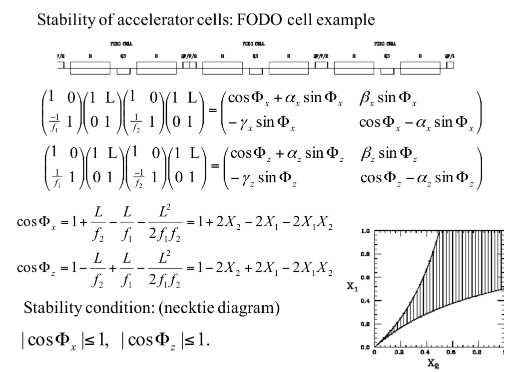
New coordinate system

From coordinates and magnets to Hill's equation, transfer matrices, and FODO lattice stability.

Today's route: from forces to lattice design

- Define local accelerator coordinates around a reference orbit.
- Use Lorentz force to identify bending and focusing terms.
- Introduce dipoles, quadrupoles, sextupoles and multipole language.
- Write linear motion as Hill's equation and transfer matrices.
- Use Courant-Snyder parameters to interpret stability and beam size.
- Apply the ideas to a FODO cell and real synchrotron lattices.

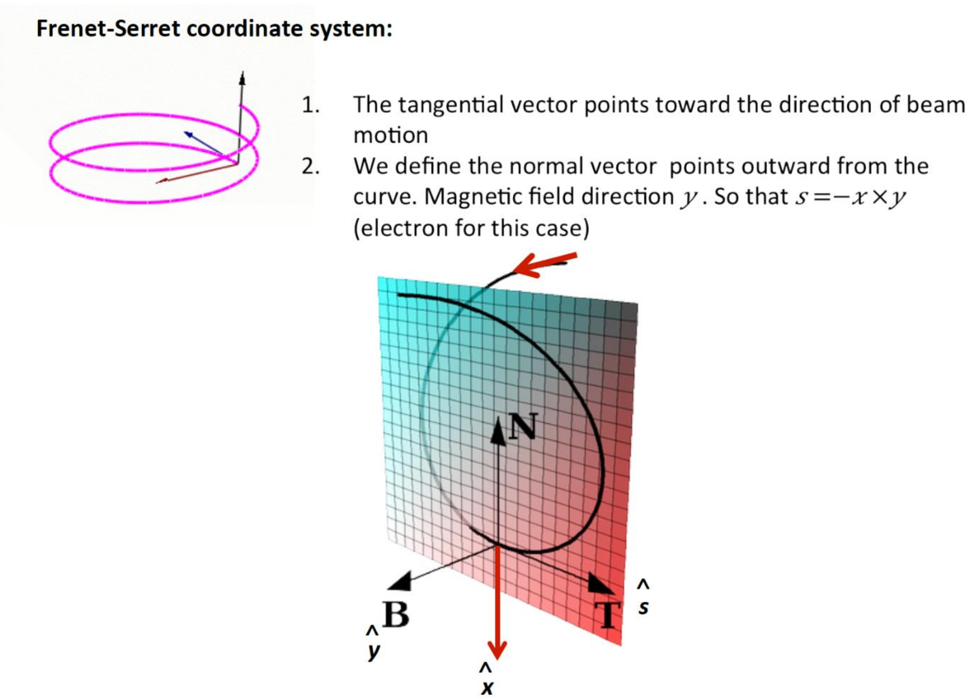




$$\text{fields} \rightarrow \text{forces} \rightarrow \text{matrices} \rightarrow \text{optics} \rightarrow \text{lattice}$$

A lattice is a sequence of simple linear maps.

New coordinates follow the reference orbit



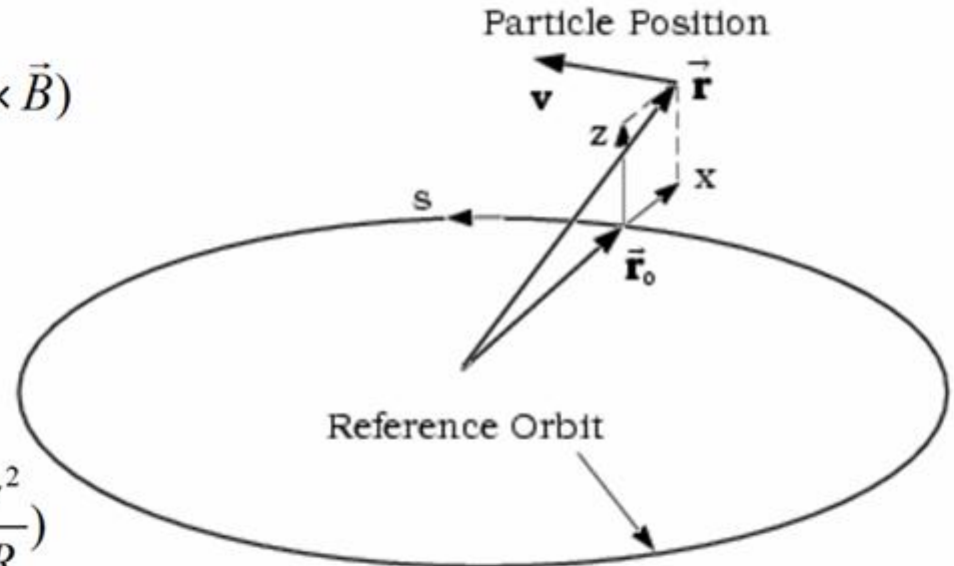
- Use path length s along the design orbit as the independent variable.
- x is the horizontal displacement from the reference orbit.
- y is the vertical displacement; some older notes use z for this coordinate.
- Small deviations allow linearized transverse equations.
- The reference orbit is set by dipole bending and beam rigidity.

$$\theta = s/\rho, \quad x, y \ll \rho$$

We do not solve motion in a global Cartesian frame; we expand around the orbit the accelerator is designed to follow.

New term: Frenet-Serret, a localized curvilinear coordinate system

Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$



$$F_x = \frac{d}{dt}(\gamma m \dot{x}) = \frac{\gamma m \beta^2 c^2}{R} = -q\beta c B_y$$

$$F_y = \frac{d}{dt}(\gamma m \dot{y}) = q\beta c B_x$$

$$\frac{d}{dt}(\gamma m \dot{x}) \approx \gamma m(\ddot{x} - \frac{v^2}{R}) = \gamma m(\frac{\beta^2 c^2}{R^2}\frac{d^2 x}{d\theta^2} - \frac{v^2}{R})$$

How to transform from the original coordinate system into the Frenet-Serret coordinate system?

$$\theta = \frac{s}{R} = \frac{\beta c t}{R}$$

$$n = -\frac{\rho}{B_0}(\frac{\partial B_y}{\partial x})_{x=0}$$

Constant when $B_y \sim x^0$ or x^1

$$\frac{d^2 x}{d\theta^2} + (1 + \frac{\rho}{B_0}\frac{\partial B_y}{\partial x})x = 0 \qquad \frac{d^2 y}{d\theta^2} + (-\frac{\rho}{B_0}\frac{\partial B_y}{\partial x})y(1 + 2\frac{x}{\rho}) = 0$$

$$\frac{d^2 x}{d\theta^2} + (1-n)x = 0 \qquad \frac{d^2 y}{d\theta^2} + ny = 0$$

$$\frac{dx}{d\theta} = \sqrt{1-n}(-A\sin\sqrt{1-n}\theta + B\cos\sqrt{1-n}\theta) \qquad \frac{dy}{d\theta} = \sqrt{n}(-C\sin\sqrt{n}\theta + D\cos\sqrt{n}\theta)$$

Number of transverse oscillation in one beam revolution ~ $\sqrt{1-n} / \sqrt{n}$, weak focusing!

For weak focusing, stable solution requires 0<n<1, which makes transverse (betatron) oscillation having a tune (number of circles per revolution) $Qx, Qy = \sqrt{1-n}, \sqrt{n}$ less than 1. The beam sizes in such machines scales with $C^{1/2}/(1-n)^{1/4}$, $C^{1/2}/n^{1/4}$, where C is the circumference.

For a modern machine, especially light sources where smaller beam sizes are required to achieve higher beam brightness, strong focusing is required => external focusing magnets (quadrupoles) are often used.

Hill's equation

$$x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \quad K_y(s) = \pm\frac{B_1}{B\rho}$$

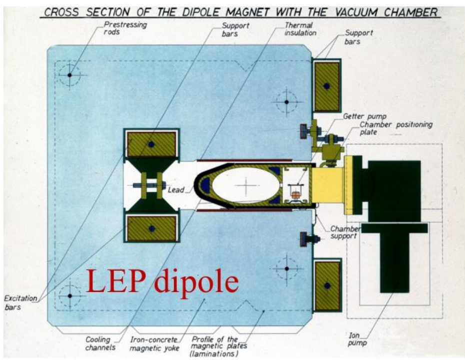
Higher order magnet, usually field errors

Natural focusing from dipoles

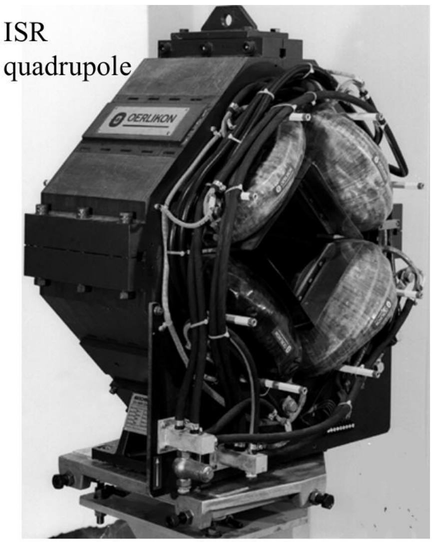
Focusing from quadrupoles

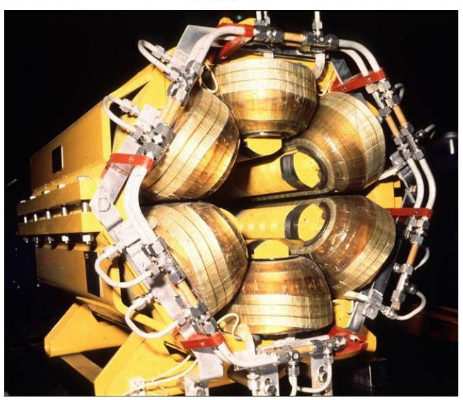
Message: weak focusing works, but it scales poorly.

The basic knobs: dipoles, quadrupoles, sextupoles



ISR quadrupole





LEP Sextupole

- Dipole: bends the reference orbit.
- Quadrupole: focuses one transverse plane and defocuses the other.
- Sextupole: for nonlinear dynamics.
- Real magnets have multipole errors and alignment errors.
- A lattice combines these elements to achieve desired beam properties.

$$\theta = \ell/\rho = B\ell/(B\rho)$$

$$1/f = \pm B_1\ell/(B\rho)$$

$$B_1 = \partial B_\gamma/\partial x$$

The sign depends on plane and magnet polarity.

Rigidity translates momentum into magnet strength

Magnets bend according to momentum per charge, not energy by itself.

$$B\rho = p/q$$

$$B\rho\ [\mathrm{T \cdot m}] = p\ [\mathrm{GeV}/c]\ /\ 0.2998$$

- Higher momentum beams need larger B or larger bending radius.
- Integrated gradient gives quadrupole focusing strength.
- Thin-lens language is useful for quick lattice estimates.

$$k = (1/B\rho)\ \partial B_\gamma/\partial x$$

$$1/f = k\ell$$

A short magnet can be approximated by an instantaneous angular kick.

Field expansion: a common language for magnets and errors

$$B_y + jB_x = B_0 \sum_n (b_n + ja_n)(x+jy)^n, \qquad A_s = \mathrm{Re}\left\{ B_0 \sum_n \frac{1}{n+1} (b_n + ja_n)(x+jy)^{n+1} \right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_y = B_0 b_0, \quad B_x = B_0 a_0,$

b_1 : quad, a_1 : skew quad; $B_y = B_0 b_1 x, \quad B_x = B_0 b_1 y, \; B_y = -B_0 a_1 y, \quad B_x = B_0 a_1 x,$

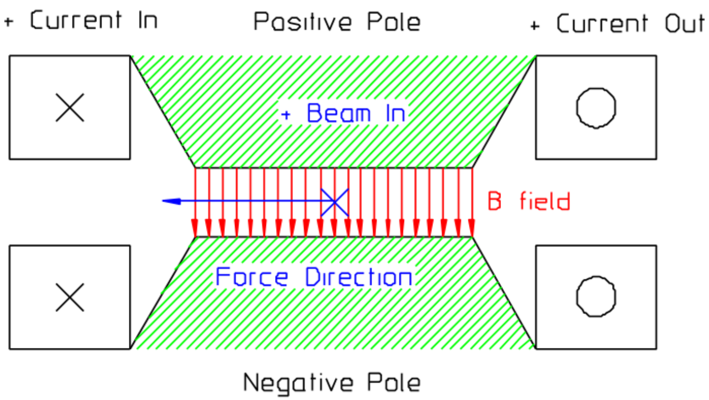
b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_y + jB_x) = \mp \frac{1}{\rho} \sum_n (b_n + ja_n)(x+jy)^n$$

- In a current-free 2D aperture, magnetic fields can be expanded as multipoles.
- Normal components b_n and skew components a_n describe field content.
- Design fields and imperfections use the same language.
- This becomes important for orbit errors, chromaticity and nonlinear dynamics.

Dipole and quadrupoles are the focus for now. Higher order components will be introduced later in the course.

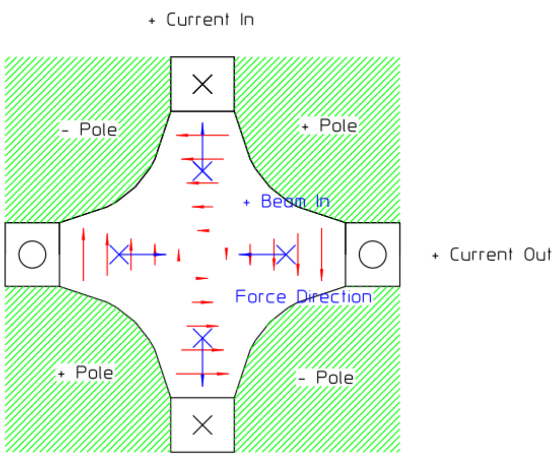
Real magnets turn equations into hardware



$$B_0 = \frac{\mu_0 NI}{g}, \quad L = \frac{\mu_0 N^2 \ell w}{g}$$

$$A_s = B_0 x$$

Formula only applies to good field region, real magnet design requires fringe field consideration



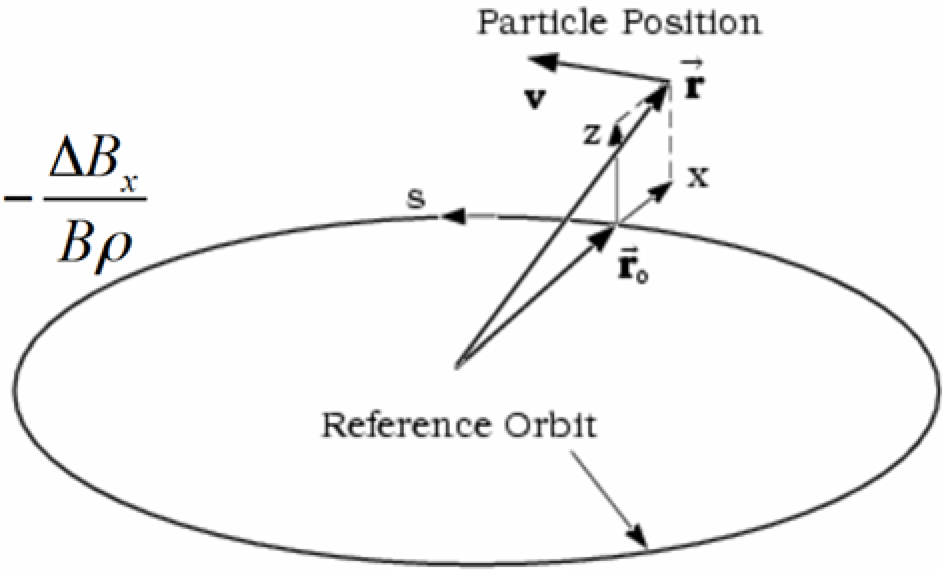
$$B_1 = \frac{2\mu_0 NI}{a^2}, \quad L = \frac{8\mu_0 N^2 \ell x_c^2}{a^2}$$

$$A_s = \frac{B_1}{2}\left(x^2 - z^2\right)$$

- Pole shape, coil current and iron geometry determine field quality.
- Dipole fields are approximately uniform in the good-field region.
- Quadrupole fields grow linearly with transverse displacement.
- A superconducting magnet package also includes cryostat, bus, collaring and alignment constraints.

How to solve the Hill's equation?

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$



Ideal Linear accelerator:

$$x'' + K_x(s)x = 0, \quad y'' + K_y(s)y = 0$$

Let X represent x or y:

$$X'' + K(s)X = 0,$$

$$X(s) = \begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

- K(s) is set by dipoles and quadrupoles.
- Field errors appear as inhomogeneous driving terms.
- M(s,s0) is the transfer matrix from one location to another.
- Hamiltonian motion preserves phase-space area: det M = 1.

$$\det \mathrm{M} = 1$$

The focusing function is piecewise constant!

$$K(s)=\begin{cases}+K\\-K\\0\end{cases}\quad y=\begin{cases}A\sin\sqrt{K}s+B\cos\sqrt{K}s\\A\sinh\sqrt{K}s+B\cosh\sqrt{K}s\\A+Bs\end{cases}$$

$$M(s,s_0)=\begin{pmatrix}\cos\sqrt{K}(s-s_0) & \frac{1}{\sqrt{K}}\sin\sqrt{K}(s-s_0)\\-\sqrt{K}\sin\sqrt{K}(s-s_0) & \cos\sqrt{K}(s-s_0)\end{pmatrix},\quad \dots$$

$$\ell = s - s_0.$$

$$M(s|s_0)=\begin{cases}\begin{pmatrix}\cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell\\-\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell\end{pmatrix} & K>0\text{: focusing quad.}\\ \begin{pmatrix}1 & \ell\\0 & 1\end{pmatrix} & K=0\text{: drift space}\\ \begin{pmatrix}\cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell\\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell\end{pmatrix} & K<0\text{: defocussing quad.}\end{cases}$$

In thin-lens approximation with $\ell \to 0$, the transfer matrix for a quadrupole reduces to

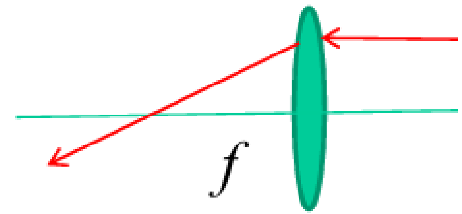
$$M_{\text{focusing}}=\begin{pmatrix}1 & 0\\-1/f & 1\end{pmatrix},\quad M_{\text{defocussing}}=\begin{pmatrix}1 & 0\\1/f & 1\end{pmatrix}\qquad f=\lim_{\ell\to 0}\frac{1}{|K|\ell}$$

$$y'' + K(s)y = 0, \qquad K_x(s) = \frac{1}{\rho^2} \mp \frac{1}{B\rho}\frac{\partial B_z}{\partial x}, \quad K_z(s) = \pm \frac{1}{B\rho}\frac{\partial B_z}{\partial x},$$

Thin lens approximation: Let $|K|\,\ell \to 1/f$ as $\ell \to 0$.

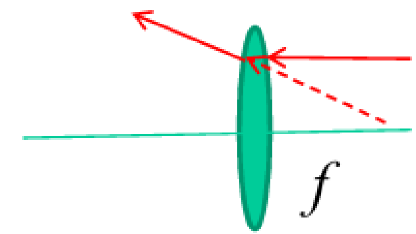
1. focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$



2. de-focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$



3. Dipole: $K_x(s)=1/\rho^2$. $\quad M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix} \to \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$

4. Drift space: K=0 $\quad M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$

Thin lenses are not exact magnets; they are a compact way to see stability and phase advance.

Lattice is multiplication of matrices

- A lattice is modeled as a sequence of elements with simple maps.
- Drifts transport an angle into a position offset.
- Focusing quadrupoles rotate phase space.
- Defocusing quadrupoles give hyperbolic maps in that plane.
- The full lattice map is an ordered matrix product.

$$M_{total} = M_n \ldots M_3 \, M_2 \, M_1$$

Order matters: the first element encountered by the beam is the rightmost matrix.

The most general representation of the matrix **M**(s) with **unit modulus** is given by the Courant-Snyder parameterization.

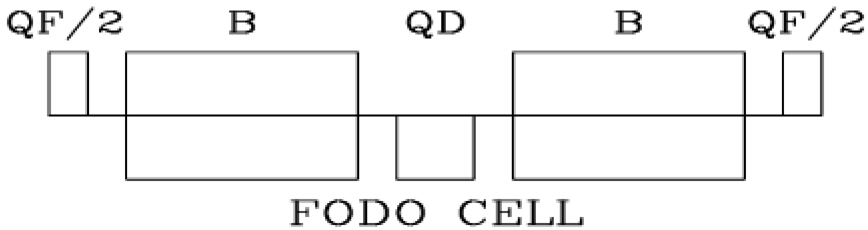
$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

The ambiguity in the sign of sin can be resolved by requiring β to be a positive definite number if |Trace(M)| ≤ 2, and by requiring Im(sinΦ)>0 if |Trace(M)| > 2. The definition of the phase factor is still ambiguous up to an integral multiple of 2π. This ambiguity will be resolved when the matrix is tracked along the accelerator elements. Using the property of matrix J, we obtain the De Moivere's theorem:

$$\mathbf{M}^k = (\mathbf{I}\cos\Phi + \mathbf{J}\sin\Phi)^k = \mathbf{I}\cos k\Phi + \mathbf{J}\sin k\Phi,$$
$$\mathbf{M}^{-1} = \mathbf{I}\cos\Phi - \mathbf{J}\sin\Phi.$$

Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$
\begin{aligned}
\mathrm{M} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}
\end{aligned}
$$

$$
\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}
$$

The sequence can be stable in both planes over one full period by using QF and QD.

$$
\cos\Phi = \tfrac{1}{2}\mathrm{Tr}(\mathrm{M})
$$

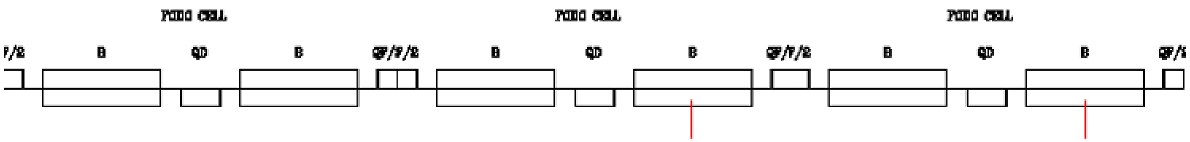
$$
\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}
$$

$$
\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi}
$$

$$
\alpha = 0
$$

Same cell, different starting point: what changes?

Example: FODO cell



$$\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

Answer:

$$\begin{pmatrix}1 & \frac{L}{4}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{3L}{4}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

Questions:

1) Will Φ of these above matrix identical?

2) Will α and β of these matrices identical?

3) What is the meaning of these parameters?

Answers:

- The phase advance over one complete periodic cell is invariant under a cyclic shift.
- The local Twiss parameters alpha and beta depend on where the cell is started.
- We will learn this in following classes.

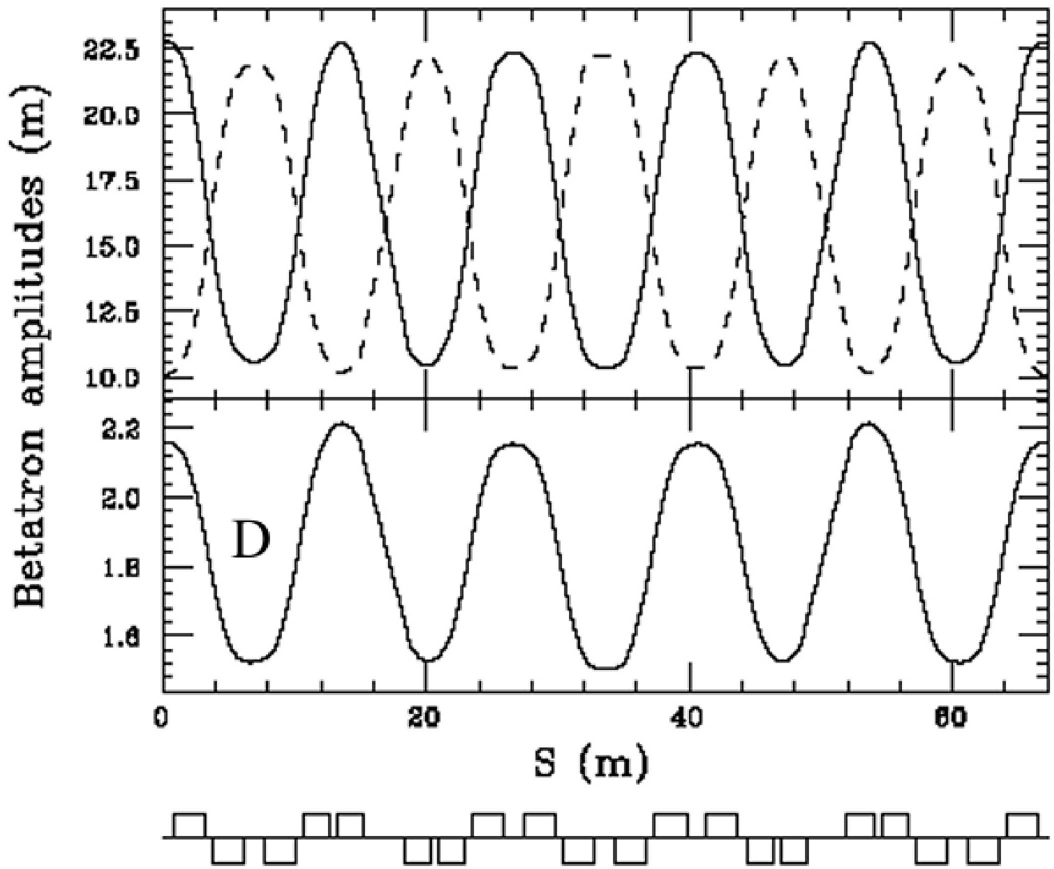
Real optics for Alternating Gradient Synchrotron

AGS

$$\beta = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$

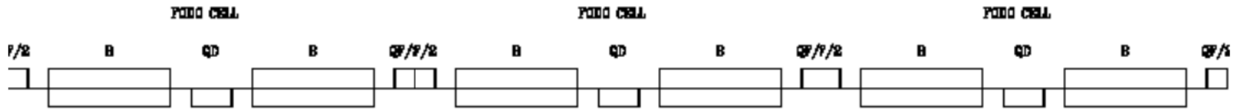
$$\alpha = 0$$

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f}$$



- Beta functions oscillate through the lattice.
- Focusing and defocusing elements exchange large and small beta between planes.
- Real machines use repeated cells, insertions, matching sections and correction systems.

Stability of accelerator cells: FODO cell example



$$\begin{pmatrix}1 & 0\\ \frac{-1}{f_1} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f_2} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\Phi_x + \alpha_x \sin\Phi_x & \beta_x \sin\Phi_x\\ -\gamma_x \sin\Phi_x & \cos\Phi_x - \alpha_x \sin\Phi_x\end{pmatrix}$$

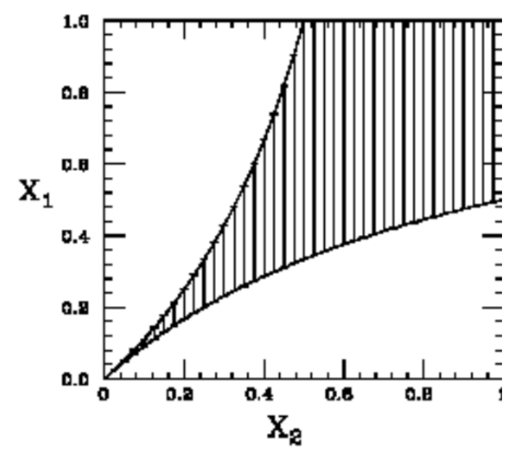
$$\begin{pmatrix}1 & 0\\ \frac{1}{f_1} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f_2} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\Phi_z + \alpha_z \sin\Phi_z & \beta_z \sin\Phi_z\\ -\gamma_z \sin\Phi_z & \cos\Phi_z - \alpha_z \sin\Phi_z\end{pmatrix}$$

$$\cos\Phi_x = 1 + \frac{L}{f_2} - \frac{L}{f_1} - \frac{L^2}{2 f_1 f_2} = 1 + 2X_2 - 2X_1 - 2X_1 X_2$$

$$\cos\Phi_z = 1 - \frac{L}{f_2} + \frac{L}{f_1} - \frac{L^2}{2 f_1 f_2} = 1 - 2X_2 + 2X_1 - 2X_1 X_2$$

Stability condition: (necktie diagram)

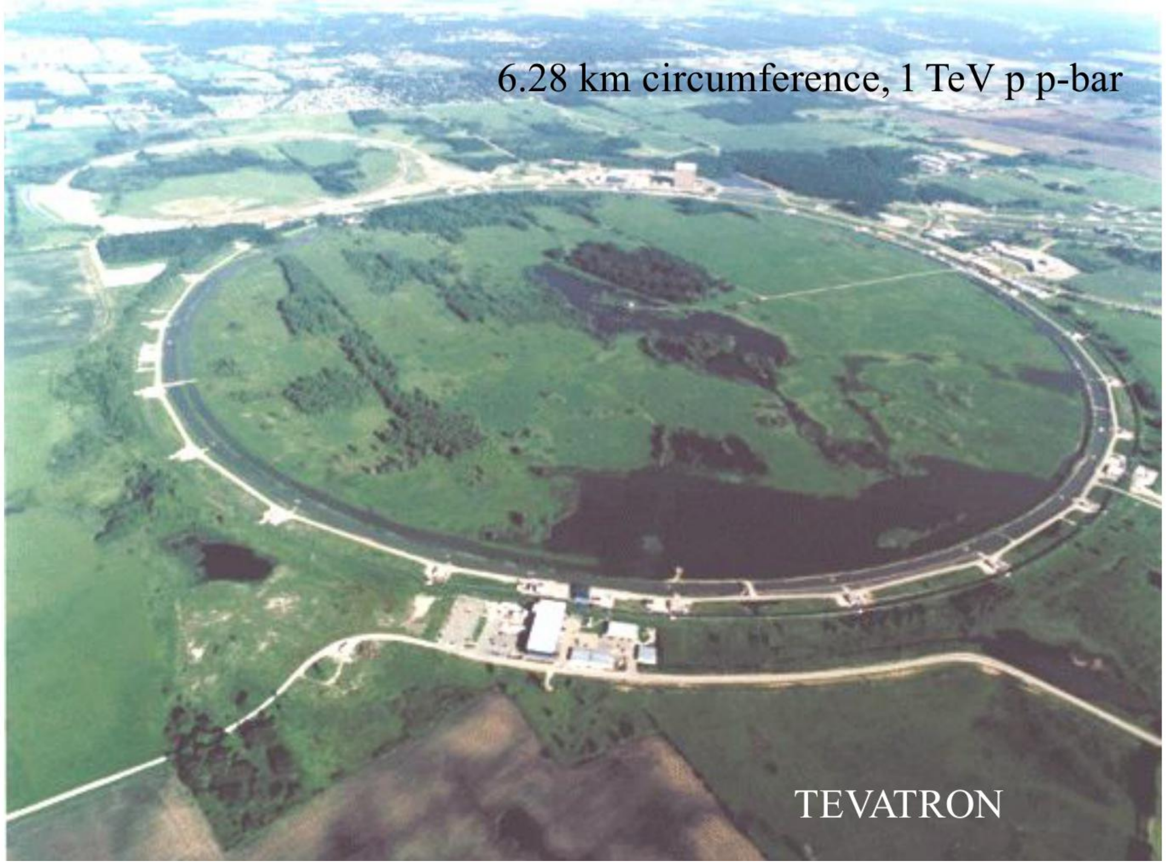
$$|\cos\Phi_x| \le 1, \quad |\cos\Phi_z| \le 1.$$



- The x and y planes impose separate stability conditions.
- A pair of focusing strengths must lie in the stable region for both planes.
- In practice, tune choice also avoids resonances and accommodates errors.

MACHINE EXAMPLE

Tevatron



- 6.28 km circumference proton-antiproton collider.
- Cells combine quadrupoles, dipoles and correction regions.
- Each sector bends the beam by 60 degrees in the example layout.
- Stored magnetic energy was enormous: about 350 MJ at 1 TeV.

$$E\_mag = (1/2)\, L\, I^2$$

A lattice is not only beam dynamics: it is also magnet technology, cryogenics, protection and operations.

What to remember from Lecture 4

- Accelerator coordinates are local coordinates around a reference orbit.
- Dipoles bend; quadrupoles focus; sextupoles and higher multipoles introduce nonlinear effects and corrections.
- Linear transverse motion is described by Hill's equation.
- Transfer matrices map (x, x') through elements; stable maps have bounded repeated motion.
- Courant-Snyder parameters turn matrix motion into optics language: beta, alpha, phase advance and tune.
- FODO cells show how alternating-gradient focusing creates stable transport in both planes.

Selected references

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