Problem 1. Chromaticity compensation: total 50 points

1a. 10 points. Consider a weak-focusing storage ring with bending radius ρ and equations of motion

$$x'' + K_x \cdot x = 0;\;\; K_x = \frac{1-n}{\rho^2};\; n = -\frac{\rho}{B_y}\frac{\partial B_y}{\partial x};$$

$$y'' + K_y \cdot y = 0;\;\; K_y = \frac{n}{\rho^2};\;\; 0 < n = < 1.$$

Derive expressions for horizontal (x) and vertical (y) β and D functions:

$$D'' + \frac{1-n}{\rho^2}\cdot D = \frac{1}{\rho},$$

and show that β and D functions are simply zeros. Calculate chromaticity of vertical and horizontal oscillation in such a ring

1b. 15 points. Show that adding sextuple component of the magnetic field

$$\frac{B_2}{B_0} = \frac{b}{2}\cdot\left(\hat{x}\cdot\left(x^2 - y^2\right) - 2\hat{y}\cdot xy\right)$$

cannot compensate both chromaticity, i.e. that depending on sign of S it can reduce one chromaticity while increasing the other. In other words you should prove that chromaticity can be compensated only in lattice where β and D function are not constant, for example in strong focusing FODO lattice.

1c. 25 points. Consider a FODO lattice with thing quadrupole lenses which are combined with thin sextuples with integrated strength of (note sign change in definition – you would need "defocusing" sextuple to compensate vertical chromaticity)

$$S_F = \int_{QF} K_2 ds;\;\; S_D = -\int_{QD} K_2 ds;$$

As you may remember from the lectures, horizontal β-function as well as (horizontal) dispersion D reach their maxima in focusing quadrupole (QF) and minima in the defocusing quadrupoles (QD). Similarly, value of vertical β-function is minimal in QF and maximal in QD. Let's assume that – <u>you do not need to derive this</u>! – that

$$\beta_{xF}, \beta_{xD}, \beta_{yF}, \beta_{yD}, D_F, D_D$$

are the optics functions in F and D quadrupoles. Furthermore, let's assume – <u>you do not need to derive this</u>! – that C_x and C_y are horizontal and vertical chromaticities of the FODO cell, which you need to compensate. Find necessary strength of S_F and S_D and identify conditions – i.e. combination of $\beta_{xF}, \beta_{xD}, \beta_{yF}, \beta_{yD}$ - when such compensation is impossible. Also show that stable FODO lattice provides condition for chromaticity compensations.

Solution:

1.a There are multiple way to determine β-function but the easiest is to remember that β-function is wavelength of the betatron oscillations:

$$x''+K_x\cdot x=0;\ K_x=\frac{1-n}{\rho^2}\Leftrightarrow x=a_x\cdot\cos\left(s\cdot\sqrt{K_x}+\varphi_x\right);\psi_x=s\cdot\sqrt{K_x}+\varphi_x;\quad \beta_x=\left(\frac{d\psi_x}{ds}\right)^{-1}=\frac{1}{\sqrt{K_x}}=\frac{\rho}{\sqrt{1-n}}$$

$$y''+K_y\cdot y=0;\ K_y=\frac{n}{\rho^2};\quad \Leftrightarrow y=a_y\cdot\cos\left(s\cdot\sqrt{K_y}+\varphi_y\right);\psi_y=s\cdot\sqrt{K_y}+\varphi_y;\quad \beta_y=\left(\frac{d\psi_y}{ds}\right)^{-1}=\frac{1}{\sqrt{K_y}}=\frac{\rho}{\sqrt{n}}$$

Phase advances per turn and tunes are therefore:

$$\mu_x=2\pi\rho\cdot\sqrt{K_x}=2\pi\sqrt{1-n};\ \nu_x=\sqrt{1-n};\ \mu_y=2\pi\rho\cdot\sqrt{K_y}=2\pi\sqrt{n};\ \nu_y=\sqrt{n};$$

Since β-functions are constants, their derivatives and therefore α-functions are zero.
Dispersions is also constant:

$$D''+\frac{1-n}{\rho^2}\cdot D=\frac{1}{\rho}\Rightarrow D=\frac{\rho}{1-n}$$

1.b To calculate chromaticity, we need to see how rigidity of oscillations and the length of trajectory chance with beam momentum. Accurate formula for weak focusing can be found in the bottom of p.155 of S.Y. Lee's "Accelerator Physics" book. Taking into account that this ring has uniform field as function of azimuth, D' =0 the expressions for the change in the focusing strengths are:

$$K=\frac{B_1}{B\rho}=-\frac{1}{B\rho}\frac{\partial B_y}{\partial x}=\frac{n}{\rho^2};\gamma_x=\frac{1+\alpha_x^2}{\beta_x}=\frac{1}{\beta_x};\quad \frac{\gamma_x D}{\rho\beta_x{}^2}=\frac{1}{\rho^2};\quad \delta=\frac{\delta p}{p_o}$$

$$\Delta K_x=\left(\frac{2}{\rho^2}\cdot\left(\frac{D}{\rho}-1\right)+\frac{n}{\rho^2}\left(\frac{2D}{\rho}-1\right)+1\right)\cdot\delta=\frac{\delta}{\rho^2}\cdot\frac{(1+n)^2}{1-n};\ \Delta K_y=\left(\frac{n}{\rho^2}\left(\frac{D}{\rho}-1\right)+1\right)\cdot\delta=\frac{\delta}{\rho^2}\cdot\frac{1-n+n^2}{1-n}$$

$$C_x=\frac{2\pi\rho}{4\pi}\cdot\beta_x\cdot\frac{\Delta K_x}{\delta}=\frac{1}{2}\cdot\frac{(1+n)^2}{\left(\sqrt{1-n}\right)^3};\ C_y=\frac{2\pi\rho}{4\pi}\cdot\beta_y\cdot\frac{\Delta K_y}{\delta}=\frac{1}{2}\cdot\frac{1-n+n^2}{(1-n)\sqrt{n}}$$

Any of you, who did this rather detailed calculation and found that both chromaticities are positive, i.e. betatron tunes increase with increase of the beam energy, will have extra 50 "STAR" points for this part of the problem.

Those of you who resort to standard expression typical for strong focusing or isomagnetic lattices from http://case.physics.stonybrook.edu/images/4/4a/PHY554_Lecture16_F2024.pdf will finish with negative chromaticities in both direction:

$$K=\frac{B_1}{B\rho}=-\frac{1}{B\rho}\frac{\partial B_y}{\partial x}=\frac{n}{\rho^2};\quad \delta=\frac{\delta p}{p_o}$$

$$\Delta K_x=\left(-\frac{2}{\rho^2}+\frac{n}{\rho^2}\right)\cdot\delta=\frac{\delta}{\rho^2}\cdot(n-2);\ \Delta K_y=-\frac{n}{\rho^2}\cdot\delta;$$

$$C_x=\frac{2\pi\rho}{4\pi}\cdot\beta_x\cdot\frac{\Delta K_x}{\delta}=\frac{1}{2}\cdot\frac{n-2}{\sqrt{1-n}}<0;\ C_y=\frac{2\pi\rho}{4\pi}\cdot\beta_y\cdot\frac{\Delta K_y}{\delta}=-\frac{n}{2\sqrt{n}}<0$$

and will get regular points for this part of the program.
Significant difference in the answers is result of the significant increase of the ring circumference

$$C = 2\pi \cdot (\rho + D \cdot \delta) = 2\pi \cdot \rho \cdot \left(1 + \frac{\delta}{1-n}\right)$$

which is neglected in eq. (2.306), p. 155 in S.Y. Lee book.
In any case, both calculations show that chromaticities in the weak focusing ring have the same sign. Attempt of compensating them using sextupoles would result in changing chromaticities in opposite directions:

$$\Delta C_x = \frac{1}{4\pi} \oint K_2(s)\, \beta_x(s)\, D(s)\, ds = \frac{\beta_x D}{4\pi} \cdot \oint K_2(s)\, ds$$

$$\Delta C_y = -\frac{1}{4\pi} \oint K_2(s)\, \beta_y(s)\, D(s)\, ds = -\frac{\beta_y D}{4\pi} \cdot \oint K_2(s)\, ds$$

$$\Delta C_y = -\frac{\beta_y}{\beta_x} \cdot \Delta C_x = -\sqrt{\frac{n}{1-n}} \cdot \Delta C_x$$

which means that they cannot be compensated because both optics functions are constant can be taken out of the integrals.
1c. 25 points. Idea here is to compensate chromaticities in each FODO cell of the accelerator. Let's assume that we need to compensate chromaticity accumulated in the FODO cells

$$C_x = -\frac{1}{4\pi} \int_{cell} K_x(s)\, \beta_x(s)\, D(s)\, ds;\; C_y = -\frac{1}{4\pi} \int_{cell} K_y(s)\, \beta_x(s)\, D(s)\, ds;$$

which usually have negative values, by introducing thin sextupoles in QF and QDL

$$K_2(s) = S_F \cdot \delta\left(s - s_F\right) - S_D \cdot \delta\left(s - s_F\right)$$

$$\Delta C_x = \frac{1}{4\pi} \int_{cell} K_2(s)\, \beta_x(s)\, D(s)\, ds = \frac{1}{4\pi}\left(S_F \cdot \beta_x\left(s_F\right) \cdot D\left(s_F\right) - S_{FD} \cdot \beta_x\left(s_D\right) \cdot D\left(s_D\right)\right)$$

$$\Delta C_y = -\frac{1}{4\pi} \int_{cell} K_2(s)\, \beta_y(s)\, D(s)\, ds = -\frac{1}{4\pi}\left(S_F \cdot \beta_y\left(s_F\right) \cdot D\left(s_F\right) - S_{FD} \cdot \beta_y\left(s_D\right) \cdot D\left(s_D\right)\right)$$

$$\begin{bmatrix} \Delta C_x \\ \Delta C_y \end{bmatrix} = \begin{bmatrix} \beta_x(s_F) & -\beta_x(s_D) \\ -\beta_y(s_F) & \beta_y(s_D) \end{bmatrix} \cdot \begin{bmatrix} S_F \cdot D_F \\ S_D \cdot D_D \end{bmatrix} = -\begin{bmatrix} C_x \\ C_y \end{bmatrix}$$

Solving this pair of question is equivalent to inversion if the matrix, which is possible when its determinant is not equal zero:

$$det\begin{bmatrix} \beta_x(s_F) & -\beta_x(s_D) \\ -\beta_y(s_F) & \beta_y(s_D) \end{bmatrix} = \beta_x(s_F)\cdot\beta_y(s_D) - \beta_x(s_D)\cdot\beta_y(s_F) \neq 0$$

$$\begin{bmatrix} S_F\cdot D_F \\ S_D\cdot D_D \end{bmatrix} = -\frac{4\pi}{\beta_{xF}\cdot\beta_{yD}-\beta_{xD}\cdot\beta_{yD}}\begin{bmatrix} \beta_y(s_D) & \beta_x(s_D) \\ \beta_y(s_F) & \beta_x(s_F) \end{bmatrix}\cdot\begin{bmatrix} C_x \\ C_y \end{bmatrix};$$

$$S_F = -\frac{4\pi\left(\beta_{yD}\cdot C_x + \beta_{xD}\cdot C_y\right)}{D_F\left(\beta_{xF}\cdot\beta_{yD}-\beta_{xD}\cdot\beta_{yF}\right)};\ S_D = -\frac{4\pi\left(\beta_{yF}\cdot C_x + \beta_{xF}\cdot C_y\right)}{D_D\left(\beta_{xF}\cdot\beta_{yD}-\beta_{xD}\cdot\beta_{yF}\right)};$$

It also clarifies again that when $\beta_{xF}\cdot\beta_{yD} = \beta_{xD}\cdot\beta_{yF}$, chromaticity compensation is impossible – for example in weak focusing ring. In contrast, in FODO cell we always have:

$$\beta_{xF}\cdot\beta_{yD} > \beta_{xD}\cdot\beta_{yF}\ .$$

Problem 2. 20 points. CERN is considering building 100 TeV proton collider. They plan to build circular storage ring with circumference of 100 km colliding 50 TeV proton beams. Assuming 70% filling factors by bending dipoles – total length of the bending magnets is 70 km! -, find the following:

i. Bending radius and necessary magnetic filed in the dipoles
ii. Energy loss of protons for synchrotron radiation and critical wavelength of radiation
iii. Assuming iso-magnetic lattice, calculate damping times for synchrotron (energy) and betatron oscillations

Sollution:

i. Radius of curvature is nothing else than length of magnets dived by 2π, which also gives us necessary magnetic field:

$$\rho[km] = \frac{70}{2\pi} = 11.141\ km;\ B\rho[T\cdot km] = \frac{pc\{TeV\}}{0.29979..} \approx 166.78;\ B = 14.97\ T$$

ii. Synchrotron radiation loss of of a charge particles was identified in Lectures 14-15: http://case.physics.stonybrook.edu/images/f/fc/PHY554_Lecture14_15_F2024.pdf

$$U_o[GeV] = C_\gamma\frac{E^4[GeV]}{\rho[m]};\ C_\gamma = \frac{4\pi\cdot r_c}{(mc^2)^3} = 7.783\cdot 10^{-18}\ \frac{m}{GeV^3}\ for\ protons$$

Multiplying numbers give us losses per turn of 0.00437 GeV (4.37 MeV).

Calculating critical wavelength of synchrotron radiation requires knowledge of bending radius (11,142 m) and relativistic factor γ=50 TeV/m_pc^2=53,289:

$$\omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho} \Rightarrow \lambda_c = 2\pi\frac{c}{\omega_c} = \frac{4\pi}{3}\frac{\rho}{\gamma^3} = 3.084\cdot 10^{-10}\ m = 3.08\ \mathring{A}$$

iii. In isomagnetic lattice J_s=2, J_x=J_y=1.

Revolution frequency in 100 km ring is f_o*=3 kHz*, and damping time for synchrotron (energy oscillations) is

$$\tau_s = \frac{E[GeV]}{f_o \cdot U_o[GeV]} = 3,820\ sec$$

i.e. it is just a bit longer than 1 hour. Naturally damping time for betatron oscillations is twice longer.

Problem 3 30 points. Design diffraction-limited FODO light source for hard-X-rays. The diffraction limited X-ray source requires transverse beam emittance to satisfy

$$\varepsilon_{x,y} \leqslant \frac{\lambda}{4\pi}$$

where $\varepsilon_{x,y}$ are geometric (not normalized) beam emittances. Assume that λ=1 angstrom (0.1 nm) and that coupling provides for equal splitting (κ=1) of natural emittance induced by quantum fluctuation of synchrotron radiation. Assume that dipoles (part of FODO cells) have the uniform magnetic field and occupy 2/3 of the circumference ring circumference *C*.

i. Derive equation for necessary number of the FODO cells and bending angle and length of each dipole magnet as function of the beam energy and ring circumference
ii. Calculate bending angle and length of each dipole magnet for two cases:
   a. APS storage ring: E=6 GeV, C=1,104 m
   b. NSLS II storage ring E=3 GeV, C=792 m

Solution:

i. First, let's define what emittances we need in the storage ring for coherence at 1 Å:

$$\varepsilon_{x,y} \leqslant \frac{100\ pm}{4\pi} = 7.95\ pm \cdot rad \Rightarrow \varepsilon_{x,y} = 7.5\ pm \cdot rad$$

Since emittance is shared between two direction, natural emittance of the lattice should be 15 pm rad. Using formula from http://case.physics.stonybrook.edu/images/f/fc/PHY554_Lecture14_15_F2024.pdf we get:

ii. As you learned in the class, natural emittance does not depend on radius of curvature, only of lattice parameters and beam energy:

$$\varepsilon_{nat} = F_{FODO} \cdot C_\gamma \cdot \gamma^2 \cdot \theta^3 = F_{FODO} \cdot C_\gamma \cdot \gamma^2 \cdot \left(\frac{2\pi}{N}\right)^3 \Rightarrow N = 2\pi \cdot \sqrt[3]{\frac{F_{FODO} \cdot C_\gamma \cdot \gamma^2}{\varepsilon_{nat}}}$$

were N is number of dipoles. Since there are two dipoles in each FODO cell, number of FODO cell is N/2 – it means that we should round answer to upper even integer.

Putting numbers in we get:

$$\varepsilon_{nat} = 15\cdot 10^{-12} m,\ C_\gamma = 3.83\cdot 10^{-13} \Rightarrow N \cong 1.85\sqrt[3]{F_{FODO}}\cdot \gamma^{\frac{2}{3}};$$

$$l_{dipole} = \rho\cdot\frac{2\pi}{N} = 3.4\cdot\frac{\rho}{\sqrt[3]{F_{FODO}}\cdot\gamma^{\frac{2}{3}}};$$

With F_{FODO} minimum reaching ~1.3 we can specify that

$$N \geqslant 2\cdot\gamma^{\frac{2}{3}};\ l_{dipole} = \rho\cdot\frac{2\pi}{N} < 3.14\cdot\rho\cdot\gamma^{-\frac{2}{3}}$$

Putting number for 6 GeV we get N ≥ 1,034 for APS ring tunnel and length of magnets less than 0.712 m (i.e. 2/3 C/N). Similarly, for NSLS energy of 3 GeV we get N ≥ 650 and length of dipole magnets less than 0.81 m.

Naturally, number of magnets would increase as cube root of F_{FODO} when FODO is not tuned for minimum emittance.