One of our tasks in this section is to study transverse stability, I.e. Do particles of the same energy, but with slightly different transverse coordinates, either in position or direction, remain near each other in the course of their motion in the accelerator.

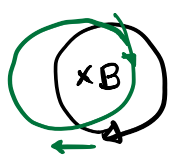
We will find the criterion for such stability and discuss the solution to the associated equations of motion. We will show that in general, stability consists of bounded oscillatory motion about the design trajectory. This motion is called *betatron oscillations*.

We will see that the transverse oscillation frequencies are much higher than the typical phase oscillations, thus allowing us to treat the longitudinal degree of freedom independently.

Note that to create the restoring force for a relatively slow moving particle, we can use electric fields, whereas for a particle moving near the speed of light, we need magnetic fields. This is because for such a fast moving particle, 1 T (typical) and 300 MV/m (outside reach!) create the same focusing force.

Stability of transverse oscillations

Why is focusing required? Consider the simple case of a particle rotating in a constant magnetic field:



If this particle is deflected ⊥r to B, it would continue to rotate in the same plane & does not go out of control (i.e it's stable)

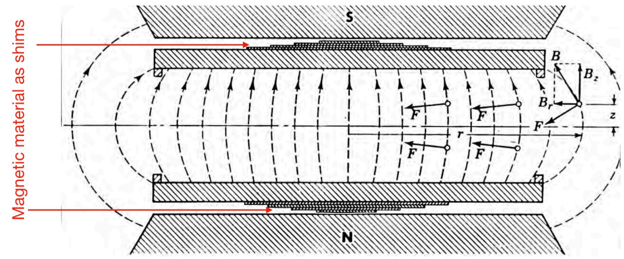
But if it recieves a kick in the same direction as B, it will go out of control. (Leaves accelerator, unstable)

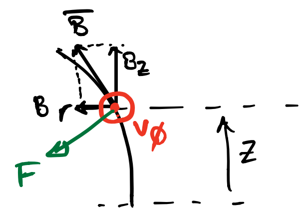
So, we need to design a focusing system to keep these particles in the accelerator.

Weak focusing

The exploration of weak focusing was initially worked on by Edwin

McMillan in the context of a cyclotron. This technique involves bending the magnetic fields using various permanent magnet configurations and was in use until early 1950's when it was replaced by strong focusing. This magnetic configuration provides focusing stability for particles with transverse velocity components. Consider for instance a particle at a distance 'z' above the center line in this image executing cyclotron oscillations:





$$\bar{V} = v_\phi \hat{\phi}$$

$$\bar{F} = q(\bar{v} \times \bar{B})$$

$$= q\left(v_\phi \hat{\phi} \times (B_z \hat{z} - B_r \hat{r})\right)$$

$$= q\, v_\phi B_z(-\hat{r}) - B_r v_\phi \hat{z}$$

Force focuses particles towards the center of cyclotron

Note that at higher radius, the field lines become farther apart (I.e. the strength of field decreases), and therefore the focusing is achieved at the expense of the cyclotron force and frequency.

Recall

$\omega_c = \frac{qB}{\gamma m} \Rightarrow$ Lower B results is smaller cyclotron freq

$r_L = \frac{P}{eB} \Rightarrow$ Lower B results in higher cyclotron radius

Suppose vertical component of the field along the midplane is given by

$$B_y = \frac{B_0}{r^n}$$

So, if $n=0$, we get the uniform field back.

If $n>1$, it can be shown that the field could not provide the necessary centripital force to keep particles moving in a circular path in constant radius

So for stability, $0<n<1$

The disadvantage is that as the energy and the circumference of the orbit increases, so does the required aperture to contain a given angular deflection. Because focusing is weak, the radial oscillations are only slightly affected.
The weak focusing is not tenable high energy particles because the size and cost of constant magnetic fields at the required radius became unreasonable. This situation led to the invention of alternating gradient focusing, otherwise known as strong focusing, which will be the focus of the remainder of class.

Strong focusing

We want the restoring force on the particle displaced from design trajectory to be as strong as possible.

In the weak focusing, in the absence of current density, field gradients that provide restoring forces in both transverse degrees of freedom simultaneously is not possible. This is because

$$\bar{\nabla}\times\bar{B}=0 \Rightarrow \frac{\partial B_y}{\partial x}=\frac{\partial B_x}{\partial y}$$ (This is the $\hat{z}$ component)

Note that in this equation, $\frac{\partial \bar{E}}{\partial t}$ is implicitely assumed to be zero. This is because RF accelerating and magnetic focusing section operate in distinct region of an RF accelerator.

For small displacements, (x,y)

$$\bar{B} = B_x\hat{x} + B_y\hat{y}$$

$$\bar{B} = \left(B_x(0,0) + \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x\right)\hat{x} + \left(B_y(0,0) + \frac{\partial B_y}{\partial x} x + \frac{\partial B_y}{\partial y} y\right)\hat{y}$$

potential restoring force

force $\perp r$ to displacement

$\hat{z}\begin{pmatrix} x \\ y \end{pmatrix}$

direction of force, $\bar{v} \times \bar{B}$ =

$$\hat{z} \times \bar{B} = \left(B_x(0,0) + \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x\right)\hat{y} + \left(-B_y(0,0) - \frac{\partial B_y}{\partial x} x - \frac{\partial B_y}{\partial y} y\right)\hat{x}$$

because $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$ (curl condition)

So if $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} > 0$ for example, an x displacement generates a force in $-$ve x direction (focusing), while a y displacement generates a force in $+$ve x direction, i.e.

only one of the 2nd terms is focusing & the other defocusing

The standard magnet that produces this focusing character is the quadrupole

<u>Focal length of a thin quadrupole</u>

Imagine a charged particle moving through the quadrupole at a distance x from the magnets axis of symmetry.

Use <u>thin lens approximation:</u> the length of magnet ‘l’ is short enough that the displacement x is unaltered as the particle passes through the magnet

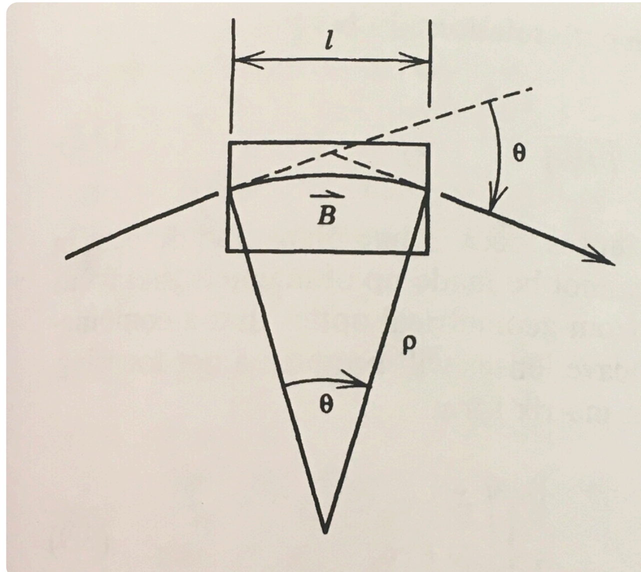
i.e. magnetic field $B_y = \left(\frac{\partial B_y}{\partial x}\right) x$ is constant along the particle trajectory

angle $= x' \equiv \frac{dx}{ds}$ (ds: distance measured along the ideal trajectory)

Slope change $= \Delta x' = -\frac{L}{\rho}$

$$= -l \cdot \left(\frac{eB_y}{p}\right)$$

$$\frac{1}{\rho} = \frac{eB}{p}$$



In a quadrupole $B(0,0) = 0$ &

we assumed only x displacement

$$\therefore \Delta x' = -\left(\frac{eB'l}{p}\right)x \quad (3.5)$$

$B' = \frac{\partial B_y}{\partial x}$ is the gradient of the quadrupole magnet

As a lens, $\Delta x' = -\frac{x}{f}$, where f is the focal length of the quarupoles

$$\therefore \frac{1}{f} = \frac{eB'l}{p}$$

→ The ratio of momentum to charge, $\frac{p}{e}$ is often called the magnetic rigidity, and written as $B\rho$ (single symbol)

In MKS units, $(B\rho) = \frac{10}{2.9979} p(\mathrm{GeV}/c)$ in T.m

$$\therefore \frac{1}{f} = \frac{B'l}{(B\rho)}$$

Since the quadrupole is focusing in one direction and defocusing in the other, we will need multiple quadrupoles!

Let's recast (3.5) in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in} \quad \leftarrow \text{for convex lens } ()$$

in free space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

$\therefore$ matrix transport corresponding to concave, drift, convex is

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{f} & L \\ -\frac{L}{f^2} & 1-\frac{L}{f} \end{pmatrix}$$

It's clear that for $\frac{L}{f} \ll 1$, the total is focusing

If the two lenses are interchanged, the total result would still be focusing. In fact, the focal length need not be large compared with the lens spacing for this to occur

This discussion suggests that one can focus in two degrees of freedom simultaneously using a system of magnetic elements whose gradients alternate in sign. But this discussion was based on tracing particles through a single pass of a system of two lenses. In modern accelerators, where particles must be transport d through great distances, the stability of particle motion through repetitive encounters of such system must be studied. This is what we'll do next.

We will see that *strong focusing* leads to beam sizes which are dependent on the spacing of the lenses and their strength and independent of the scale of the accelerator.

Stability criterion

We want to know the relationship between lens strengths and spacing that lead to stable oscillations as opposed to oscillations that grow in amplitude in time.

Lattice: the detailed description of the way in which magnets and intervening spaces are placed to form the accelerator.

If a particle traverses a series of elements $M_1, M_2, \ldots, M_n$, input & output conditions are related by

$$M = M_n M_{n-1} \cdots M_2 M_1$$

If these sequence of elementary matrices are encountered repeatedly, as in the case of a circular accelerator, then we can use this matrix to inquire about the stability of transverse oscillations:

For stable oscillations, $M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$ has to be bounded for $n \to \infty$

Let V_1 & V_2 be the two eigenvectors of M, corresponding to eigenvalues λ_1, λ_2

Any initial conditions can be represented as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{in} = \underbrace{A V_1 + B V_2}_{\text{constants}}$$

$\therefore$ propagations through n periods: $M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{in} = A\lambda_1^n V_1 + B\lambda_2^n V_2$

$\Rightarrow$ stability criteria: λ_1 & λ_2 do not grow w/ n or

$$|\lambda_1|, |\lambda_2| < 1$$

Note: $\det(M_i) = 1$ for all matrices of concern

$\therefore \quad \det(M) = 1$

$\therefore$ Eigenvalues of M are reciprical of each other

$\lambda_2 = \frac{1}{\lambda_1}$

In general, $\lambda_1 = e^{i\mu}$, μ complex ; Actually, it has to be real for stability

$\lambda_2 = e^{-i\mu}$

Now, we can solve the eigenvalue equation for M:

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Eigenvalue equation: $\det(M - \lambda I) = 0$

$$\therefore \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\underbrace{(ad-bc)}_{\equiv \det(M) = 1} - (a+d)\lambda + \lambda^2 = 0$$

$$\therefore 1 = (a+d)\lambda - \lambda^2 \Rightarrow (a+d)\lambda = 1 + \lambda^2$$

$$\therefore (a+d) = \lambda^{-1} + \lambda$$

$\uparrow$ Trace of M

In terms of μ (where $\lambda_{1,2} = e^{\pm i\mu}$)

$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = \mathrm{Tr}\,M$$

↑symbol for trace of M

$|\cos\mu| < 1$, so the stability condition is

$$-1 \leq \frac{1}{2}\mathrm{Tr}(M) \leq +1$$

<u>FODO lattice:</u> As an example, consider a FODO lattice, I.e. A lattice characterized by equally spaced focusing and defocusing lenses, each of which we will assume to be thin. If the order is focusing, drift of length L, defocusing lens, the matrix is

$$M = \begin{bmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ -\frac{L}{f} & 1 + \frac{L}{f} \end{bmatrix}$$

$\therefore$ stability condition is $\quad -1 \leq 1 - \frac{1}{2}\left(\frac{L}{f}\right)^2 \leq 1$

$$\therefore -2 \leq -\frac{1}{2}\left(\frac{L}{f}\right)^2 \leq 0$$

$$0 \leq \left(\frac{L}{f}\right)^2 \leq 4$$

$$\left|\frac{L}{2f}\right| \leq 1$$

This means that in order for a FODO lattice to be stable, the focal length has to be greater than half the lens spacing.

Critical point: The wavelength of an oscillation is just four times the lens spacing and therefore is unrelated to the size of the accelerator. This alternating gradient principle enables us to decouple the transverse aperture requirement from the size and hence the energy of an accelerator.

This is demonstrated here for the case of the FODO lattice with focal length exactly half the lens spacing: