Homework 5. Due October 2

Problem 1. 2x5 points. Field expansion and field integral.
Accelerator physicists often prefer to express magnetic fields in Cartesian coordinate system. A typical magnetic field (with higher order components) can be conveniently written in a complex form,

$$\vec{B} = B_y(x,y,z) + iB_x(x,y,z) = \sum_{n=0}^{\infty} a_n(z)\,(x+iy)^n$$

where $B_x(x,y,z)$ and $B_y(x,y,z)$ are both real.

a) Show that the fields expressed in this way satisfies relations

$$\left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$$

and

$$\left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$$

which effectively shows that the field divergence (transverse) and curl (longitudinal) are zero

$$\nabla_\perp \cdot \vec{B} = \left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0$$

$$\left(\nabla \times \vec{B}\right)_\parallel = \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = 0$$

i.e., no current in free space.

b) The scalar potential $\Phi(\mathrm{x},\mathrm{y},\mathrm{z})$ of this field ($\vec{B} = -\nabla\Phi$) should satisfy Laplace's equation:

$$\Delta\Phi \stackrel{\text{def}}{=} \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

using Laplace's equation and by defining a integrated (over z) scalar potential as $\overline{\Phi}(\mathrm{x},\mathrm{y}) \stackrel{\text{def}}{=} \int_{z_1}^{z_2} \Phi(\mathrm{x},\mathrm{y},\mathrm{z})\,dz$

show that we can write the transverse Laplace of the integrated scalar potential as

$$\Delta_\perp\overline{\Phi} \stackrel{\text{def}}{=} \frac{\partial^2\overline{\Phi}}{\partial x^2} + \frac{\partial^2\overline{\Phi}}{\partial y^2} = B_z(x,y,z_2) - B_z(x,y,z_1)$$

discuss under what condition does the integrated field satisfy a two dimensional Laplace's equation? Discuss what are the requirements to use this relation when measuring the integrated field of a magnet, i.e., should the magnet be long or short, should the coil be long or short?