PHY 554. Homework 3.

(a) The closed orbit of a three-bump system is

$$y(s) = \frac{\sqrt{\beta}}{2\sin\pi\nu}\sum_{i=1}^{3}\sqrt{\beta_i}\theta_i\cos(\pi\nu - |\psi - \psi_i|).$$

Using the condition $y(s_3) = y'(s_3) = 0$, Then

$$\begin{cases} \sqrt{\beta_1}\theta_1\cos(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\cos(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\cos\pi\nu = 0 \\ \sqrt{\beta_1}\theta_1\sin(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\sin(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\sin\pi\nu = 0 \end{cases}$$

where $\psi_{13} = \psi_1 - \psi_3$ and $\psi_{23} = \psi_2 - \psi_3$, we find

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}, \qquad \theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\frac{\sin\psi_{12}}{\sin\psi_{23}}.$$

(b) When $\psi_{31} = n\pi$, we find $\theta_2 = 0$, i.e. only two steering dipoles are needed for a local bump. Since $\psi_{32} = \psi_{31} - \psi_{21} = n\pi - \psi_{21}$, we have $\sin\psi_{32} = (-1)^{n-1}\sin\psi_{31}$, and $\theta_3 = (-1)^{n-1}\sqrt{\beta_1/\beta_2}\,\theta_1$.

1. Problem 2

Given a distribution $\rho(X, X')$ with $\int \rho(X, X')dXdX' = 1$, we have

$$
\begin{aligned}
< X > &= \int X\rho(X, X')dXdX' \\
< X' > &= \int X'\rho(X, X')dXdX' \\
\sigma_X^2 &= \int (X- < X >)^2\rho(X, X')dXdX' \\
\sigma_{X'}^2 &= \int (X'- < X' >)^2\rho(X, X')dXdX' \\
\sigma_{XX'} &= \int (X- < X >)(X'- < X' >)\rho(X, X')dXdX'
\end{aligned}
$$

And we can simplify $\sigma_X^2, \sigma_{X'}^2, \sigma_{XX'}$ as

$$
\begin{aligned}
\sigma_X^2 &= < X^2 > - < X >^2 \\
\sigma_{X'}^2 &= < X'^2 > - < X' >^2 \\
\sigma_{XX'} &= < XX' > - < X >< X' >
\end{aligned}
$$

If we take derivative over s, we have

$$
\begin{aligned}
\frac{d\sigma_X^2}{ds} &= 2 < XX' > -2 < X >< X' > \\
\frac{d\sigma_{X'}^2}{ds} &= 2 < X'X'' > -2 < X' >< X'' > \\
\frac{d\sigma_{XX'}}{ds} &= < X'^2 > - < X' >^2 - < X >< X'' > + < XX'' >
\end{aligned}
$$

Note that X satisfies the function

$$X'' + KX = 0$$

So we can replace X'' by $-KX$.
And emittance is defined as

$$\varepsilon^2 = \sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2$$

Therefore

$$\begin{aligned}
\frac{d\varepsilon^2}{ds} &= \sigma_X^2 \frac{d\sigma_{X'}^2}{ds} + \sigma_{X'}^2 \frac{d\sigma_X^2}{ds} - 2\sigma_{XX'} \frac{d\sigma_{XX'}}{ds} \\
&= \left(< X^2 > - < X >^2\right)(2 < X'X'' > -2 < X' >< X'' >) \\
&\quad + \left(< X'^2 > - < X' >^2\right)(2 < XX' > -2 < X >< X' >) \\
&\quad -2\,(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2 - < X >< X'' > + < XX'' >\right) \\
&= \left(< X^2 > - < X >^2\right)(-2K < XX' > +2K < X >< X' >) \\
&\quad + \left(< X'^2 > - < X' >^2\right)(2 < XX' > -2 < X >< X' >) \\
&\quad -2\,(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2 +K < X >^2 -K < X^2 >\right) \\
&= 2K\left(< X^2 > - < X >^2\right)(< X >< X' > - < XX' >) \\
&\quad +2\left(< X'^2 > - < X' >^2\right)(< XX' > - < X >< X' >) \\
&\quad -2\,(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2\right) \\
&\quad -2K\,(< XX' > - < X >< X' >)\left(< X >^2 - < X^2 >\right) \\
&= 0
\end{aligned}$$

Note that the two terms with K are canceled and the two terms without K are canceled. So we have

$$\frac{d\varepsilon^2}{ds} = 0$$