PHY 564
Advanced Accelerator Physics
Lecture 7
Matrices of accelerator elements

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Sylvester formulae

Standard case of distinct eigen values, 1… 2n

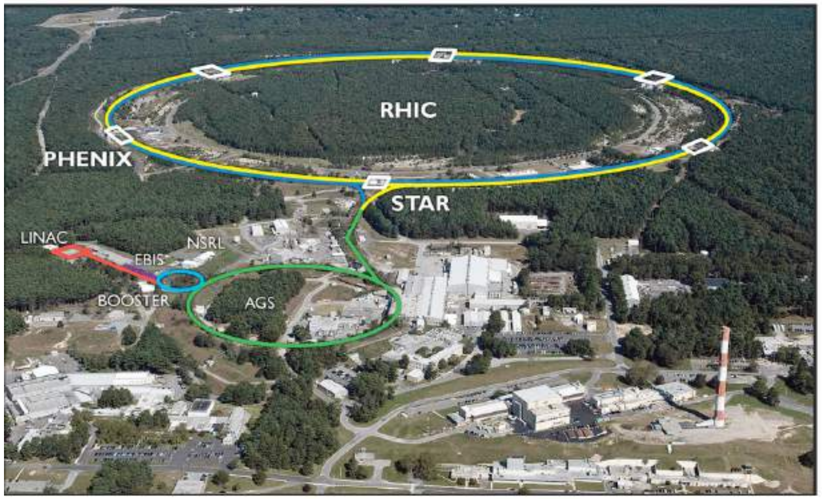
$$\exp\left[\mathbf{D}s\right] = \sum_{k=1}^{2n} e^{\lambda_k s} \prod_{j \neq k} \frac{\mathbf{D} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j}$$

Degenerated case of with m < 2n distinct eigen values, n_i is the height (index) of the eigen value

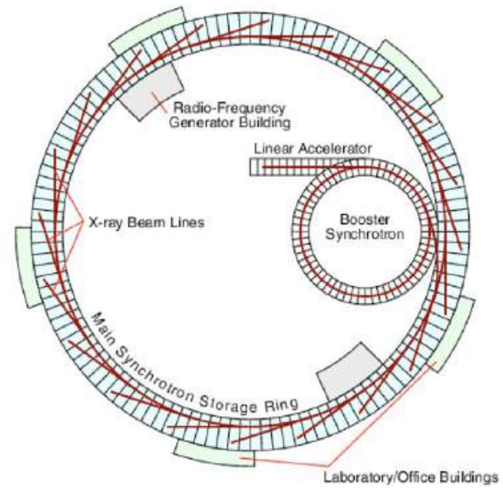
$$\exp\left[\mathbf{D}s\right] = \sum_{i=1}^{m} \left[\left(\sum_{j=0}^{l_i - 1} \frac{\phi_i^{(j)}\left(\lambda_i, s\right)}{j!} \left(\mathbf{D} - \lambda_i I\right)^j \right) \prod_{j \neq i} \left(\mathbf{D} - \lambda_j I\right)^{l_j} \right];$$

$$\phi_i\left(\lambda, s\right) = \frac{e^{\lambda s}}{\prod_{j \neq i} \left(\lambda - \lambda_j\right)^{l_j}}; \phi_i^{(j)}\left(\lambda\right) = \frac{\partial^j \phi_i}{\partial \lambda^j}.$$

Each modern accelerator has a large number of elements serving various purposes: guns (or sources, including targets producing positrons, muons, antiprotons…) generating charged particles, which are both accelerated and transported for their intended use. Accelerated particles either dumped into a target or circulated in synchrotron or storage rings. Modern accelerator complexes build for high energy colliders or generating synchrotron radiation are comprised of a multiple dedicated accelerators and transport channels connecting the later.



(a)



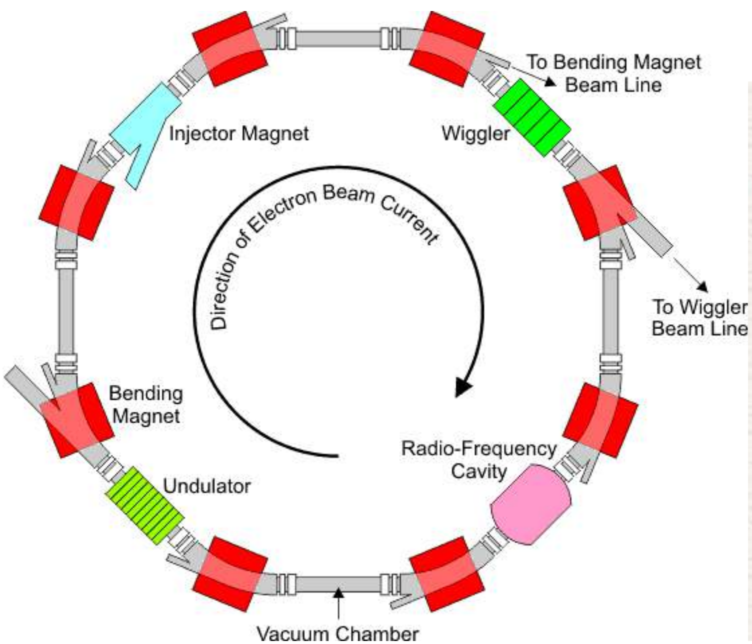
(b)

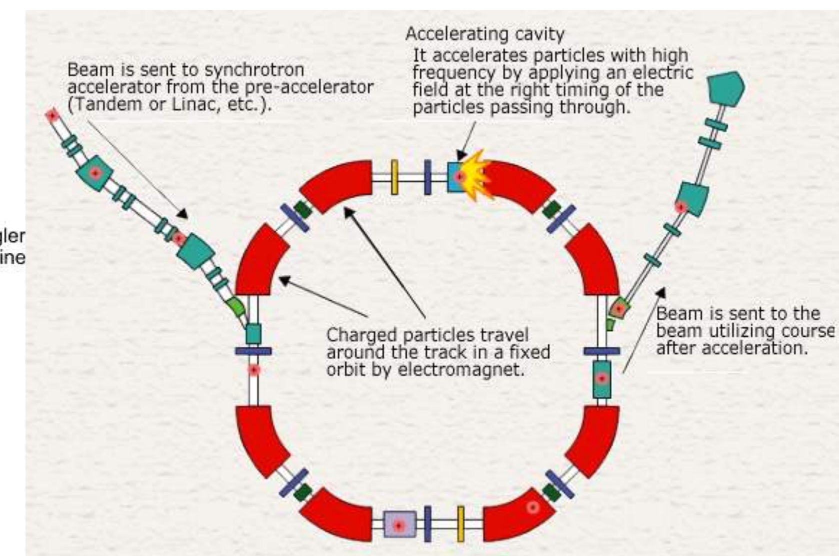
(a) RHIC (Relativistic Heavy Ion Collider) has sources of polarized protons and various ions. A linear accelerator, a booster and AGS (Alternating gradient Synchrotron) and two super-conducting (blue and yellow) RHIC rings– all serving for staged accelerating protons to 250 GeV and heavy ions to 100 GeV/u. Accelerated beams are circulating and colliding in RHIC for many hours. The chain of accelerator/rings are connected to each other by transport channels.

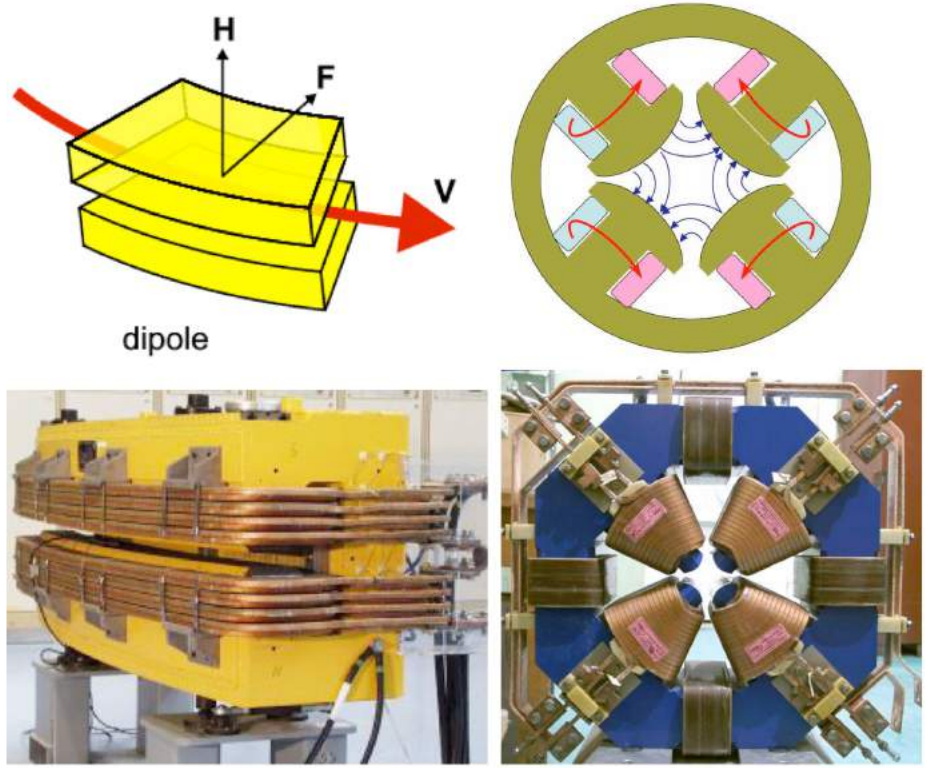
(b) A typical synchrotron radiation source has a booster synchrotron for acceleration to the final energy (typically 3-8 GeV). Beam at the top energy is injected into the storage ring where it circulates for many hours.

We will discuss the sources of particles later in this course, while dedicating some of next classes for discussion of linear accelerator and RF accelerating cavities for storage rings and boosters. In general, time dependent elements of accelerators may require computer simulation when analytical solutions are not available.

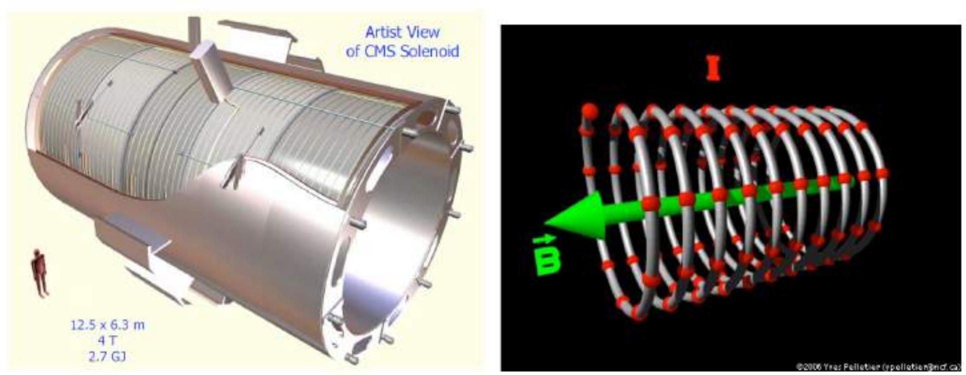
In this class we will focus on majority of elements, which are used in accelerators. These elements either have time independent EM field (DC) or it is varies very slow when compared with the time required for particles to pass through the elements. These elements are used for bending and focusing beams of particles in accelerators and transport channels or as elements of detectors. They include dipoles, quadrupoles, Sq-quadrupoles, wigglers, undulators, solenoids, etc.







Dipole and quadrupole magnets.



Superconducting solenoid for CMS detector at LHC.

Matrix of a general DC accelerator element (including twisted quads or helical wiggler) is the focus of this lecture. With all diversity of possible elements on accelerators, DC (or almost DC) magnets play the most prominent role. Even in linear RF accelerators bending and focusing elements are placed between accelerator sections, e.g. in the locations where energy is constant. In this case we can use reduced variables. Furthermore, large number of terms is the Hamiltonian simply disappear and from the previous lectures we have:

$$\tilde{h}_n = \frac{\pi_1^2 + \pi_3^2}{2} + f\frac{x^2}{2} + n \cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o^2}{2} \cdot \frac{m^2c^2}{p_o^2} + g_x x\pi_o + g_y y\pi_o; \qquad (7\text{-}1)$$

It is also true that for high energy particles we are using magnetic elements and electrostatic elements are exceptionally rare. The reason for this is three fold:

(a) for particles moving with speed close to the speed of light magnetic element are much stronger and much more effective. A typical magnetic field of 20 kGs (2 T) for room temperature steel-dominated magnets corresponds to 6 MV/cm electric field, which would very hard (most likely impossible because of arcing) to achieve with typical gap of few cm. Needles to say that superconducting magnets are edging towards 100 kGs (10T) field, and 30 MV/cm E-field is beyond reach.

(b) Electric fields do not penetrate through metal vacuum chamber, which is used for high vacuum systems. It means that there should be penetrations and internal electrostatic structures – both limiting the available voltages/E-fields and creating beam-unfriendly environment: we will discuss wakefields and collective effects later in the course.

(c) Electrostatic system are much more complicated and much more hazardous for humans – you can be killed by just 100 V, not mentioning 10 MV. In contrast, magnetic field does not arc – the only danger is that it attract magnetic materials.

Even though it is tempting to remove electric fields, it does not either helps or hurts our consideration for matrix of a generic DC accelerator element. As you can see from next equation, it adds only a single non-relativistic term g_y.

For fields in vacuum we have

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}, \ \frac{\partial E_x}{\partial x} + KE_x + \frac{\partial E_y}{\partial y} = 0$$

resulting in

$$\tilde{h}_n = \frac{\pi_1{}^2 + \pi_3{}^2}{2} + f\frac{x^2}{2} + n \cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o{}^2}{2} \cdot \frac{m^2c^2}{p_o{}^2} + g_x x\pi_o + g_y y\pi_o;$$

$$\pi_1 = \frac{P_1}{p_o}; \ \pi_3 = \frac{P_3}{p_o}; \pi_o = \frac{\delta}{p_o} = \frac{E - E_o}{cp_o}$$

$$f = K^2 - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_x}{p_o{}^2}\right)^2;$$

$$g = \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_z}{p_o{}^2}\right)^2; \qquad ; \ (7\text{-}2)$$

$$2n = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K \cdot \frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o{}^2}\right)\left(\frac{meE_x}{p_o{}^2}\right)$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad g_x = \frac{(mc)^2 \cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \ g_y = \frac{(mc)^2 \cdot eE_y}{p_o^3};$$

In the absence of longitudinal electric field, the momentum P_2 is constant as well $\pi_o = const$, δ=const. The fact that particle's energy does not change in such element is rather obvious (It is completely correct for magnetic elements. Presence of electric field makes it less obvious, but it comes from the fact that Hamiltonian does not depend on time!): $\pi_o{}' = -\frac{\partial h}{\partial \tau} = 0$, e.g. absence of the accelerating/decelerating electric field component.

Equations of motion become specific:

$$\mathbf{X}^T = [x, \pi_1, y, \pi_3, \tau, \pi_o] = [X^T, \tau, \pi_o];\ X^T = [x, \pi_1, y, \pi_3], \tag{7-3}$$

$$\frac{d\mathbf{X}}{ds} = \mathbf{D}(s)\cdot\mathbf{X}; \mathbf{D} = \mathbf{S}\cdot\mathbf{H}(s) = \begin{bmatrix} 0 & 1 & -L & 0 & 0 & 0 \\ -f & 0 & -n & -L & 0 & -g_x \\ L & 0 & 0 & 1 & 0 & 0 \\ -n & L & -g & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & m^2c^2/p_o^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \tag{7-4}$$

and can be rewritten in a slightly different (just deceivingly looking better) way:

$$\frac{dX}{ds} = D(s)\cdot X + \pi_o \cdot C(s);$$

$$\frac{d\tau}{ds} = g_x x + g_y y + \pi_o \cdot m^2c^2/p_o{}^2; D(s) = \begin{bmatrix} 0 & 1 & -L & 0 \\ -f & 0 & -n & -L \\ L & 0 & 0 & 1 \\ -n & L & -g & 0 \end{bmatrix}; C(s) = \begin{bmatrix} 0 \\ -g_x \\ 0 \\ -g_y \end{bmatrix}. \tag{7-5}$$

Hence, solution for transverse motion (4-vector) in such an element can be written as combination general solution of homogeneous equation plus specific solution of inhomogeneous one:

$$X = M(s)\cdot X_o + \pi_o \cdot R(s);\ \frac{dM}{ds} = D(s)\cdot M;\ M\left(s_o\right) = I;$$
$$\frac{dR}{ds} = D(s)\cdot R + C(s);\ R\left(s_o\right) = 0. \qquad (7\text{-}6)$$

It worth noting that C=0 as soon as there is no EM field on the orbit – $E_{ro}=0,\ B_{ro}=0$. In this case $R=0$.

Before finding 4x4 matrixes M and vector R, let's see what we will know about the 6x6 matrix after that. First, the obvious:

$$\mathbf{M}_{6x6} = \begin{bmatrix} & \mathbf{M}_{4x4} & & & 0 & R \\ R_{51} & R_{52} & R_{53} & R_{54} & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7\text{-}7)$$

with a natural question of what are non-trivial R_{5k} elements? Usually these elements, with exception of R_{56} are not even mentioned in most of textbooks. Fortunately for us, Mr. Hamiltonian gives us a hand in the form of symplecticity of transport matrixes. Using (18) and (18-1) we can find that:

$$\mathbf{M}_{6x6}{}^T\mathbf{S}\mathbf{M}_{6x6} == \begin{bmatrix} \mathbf{M}^T{}_{4x4} & Q^T & 0 \\ 0 & 1 & 0 \\ R^T & R_{56} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{S}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{M}_{4x4} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}^T{}_{4x4}\mathbf{S}_{4x4} & 0 & Q^T \\ 0 & 0 & 1 \\ R^T\mathbf{S}_{4x4} & -1 & R_{56} \end{bmatrix}\begin{bmatrix} \mathbf{M}_{4x4} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}^T{}_{4x4}\mathbf{S}_{4x4}\mathbf{M}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ R^T\mathbf{S}_{4x4}\mathbf{M}_{4x4} - Q & -1 & R^T\mathbf{S}_{4x4}R \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

where we used $Q = [R_{51}, R_{52}, R_{53}, R_{54}]$.

We should note what $\mathbf{X^T S X = 0}$ for any vector, $\mathbf{M}^T{}_{4x4}\mathbf{S}_{4x4}\mathbf{M}_{4x4} = \mathbf{S}_{4x4}$ and only non-trivial condition from the equation above is:

$$R^T \mathbf{S}_{4x4}\mathbf{M}_{4x4} - Q = 0$$

which gives us very valuable dependence of the arrival time on the transverse motions:

$$Q = R^T \mathbf{S}_{4x4}\mathbf{M}_{4x4} \quad or \quad Q^T = -\mathbf{M}^T{}_{4x4}\mathbf{S}_{4x4} R \ . \tag{7-8}$$

Element R_{56} is decoupled form the symplectic condition in this case and should be determined by direct integration - no magic here:

$$\begin{gathered} \boldsymbol{\tau}(s) = \boldsymbol{\tau}(s_o) + \pi_o \cdot \left\{ m^2c^2 / p_o^2 (s - s_o) + \int_{s_o}^{s} \left(g_x R(s)_{16} + g_y R_{36}(s) \right) ds \right\} \\ R_{56} = m^2c^2 / p_o^2 (s - s_o) + \int_{s_o}^{s} \left(g_x R(s)_{16} + g_y R_{36}(s) \right) ds \end{gathered} \tag{7-9}$$

It is important to notice that all relation derived above were done without any assumptions about *s*-dependence of the magnetic and electric field, except assumptions that energy for the beam (reference particle) is constant and fields are time-independent. It means that all conclusions about structure of the 6x6 transport matrix, 4-vector *R* and eqs. (52-54) is correct for any beam-line combined of such elements.

Since there is no general solution for s-depend linear equations, the matrix of the is

$$\frac{dM}{ds} = D(s)\cdot M \Rightarrow M(s) = \lim_{N\to\infty}\prod_{i=0}^{N} e^{D(s^*_i)\Delta s_i};\; s^*_i \in \{s_i, s_{i+1}\}\;, s_N = s,\; \Delta s_i = s_{i+1} - s_i;$$

$$\frac{dR}{ds} = D(s)\cdot R + C(s) \Rightarrow R(s) = \lim_{N\to\infty} R_N;\; \left\{ R_0 = 0;\; R_{i+1} = e^{D(s^*_i)\Delta s_i}\left(R_i + \Delta R_i\right), i = 0,1..,N\right\}$$

$$\Delta R_i = \int_{s_i}^{s_{i+1}} e^{-D(s^*_i)(s-s_i)} C(s)\,ds; \qquad (7\text{-}10)$$

A standard analytical approach to accelerator design is to assume that the coefficients in the Hamiltonian (7-4) are step-constant functions in intervals $\{s_j, s_{j+1}\}$, or so called hard-edge element approach:

$$D = \{D_{i,k}(s)\};\; D_{i,k}(s) = const;\; s \in \{s_j, s_{j+1}\}; \qquad (7\text{-}11)$$

This approach is frequently justified by design of the magnetic elements having constant field components along a patch of the reference orbit, whose length is much longer that the extend of the edge-field (e.g. transition area). In this case edge-field are typically included as a “local kick”.

From pure theoretical point of view, we always can chop the beam-line into segment where $D_{i,k}(s)$ are constant or have negligibly small variations* and write the step-wise solutions

$$\frac{dM}{ds} = D(s)\cdot M \Rightarrow M_k(s) = e^{D_k(s-s_k)}\prod_{i=0}^{k-1} e^{D_i\Delta s_i};\; s \in \{s_{k-1}, s_k\}\;;$$

$$\frac{dR}{ds} = D(s)\cdot R + C(s) \Rightarrow R_k(s) = e^{D_k(s-s_k)} R_{k-1} + \Delta R_k(s) \qquad (7\text{-}12)$$

with our goal today to find exact expression for $e^{D_k(s-s_k)}$ and $\Delta R_k(s)$. Let’s find the solutions for 4x4 matrixes of arbitrary element and corresponding R-vectors.

**The later has to be carefully checked*

As we discussed in previous class, characteristic equation for any linear Hamiltonian

$$\mathbf{X}' = \mathbf{D}\cdot\mathbf{X} = \mathbf{SH}\cdot\mathbf{X} \rightarrow d(\lambda) = \det[\mathbf{D} - \lambda\mathbf{I}] = \prod_{i=1}^{2n}(\lambda_i - \lambda) = 0 \qquad (7\text{-}13)$$

system is bi-quadratic. It comes from the fact that if λ_i is an eigen value of $\mathbf{D}$ (solution of $d(\lambda)=0$), than $-\lambda_i$ is also an eigen value of $\mathbf{D}$:

$$\det \mathbf{A}^T = \det \mathbf{A};\ \det \mathbf{AB} = \det \mathbf{BA};\ (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T;\ \det[-\mathbf{A}_{m\times m}] = (-1)^m \det \mathbf{A}$$

$$\mathbf{S}^T = -\mathbf{S};\ \mathbf{H}^T = \mathbf{H};\ \mathbf{S}^2 = -\mathbf{I};\ \det \mathbf{S} = 1;\ \Rightarrow$$

$$d(\lambda) = \det[\mathbf{D} - \lambda\mathbf{I}] = \det[\mathbf{D} - \lambda\mathbf{I}]^T = \det\left[(\mathbf{SH})^T - \lambda\mathbf{I}\right] = \det[-(\mathbf{HS} + \lambda\mathbf{I})] \qquad (7\text{-}14)$$

$$-\mathbf{S}(\mathbf{HS} + \lambda\mathbf{I})\mathbf{S} = (\mathbf{SH} + \lambda\mathbf{I});\ \det[\mathbf{S}(\mathbf{HS} + \lambda\mathbf{I})\mathbf{S}] = \det[\mathbf{HS} + \lambda\mathbf{I}]\det^2\mathbf{S} = \det[\mathbf{HS} + \lambda\mathbf{I}] \Rightarrow$$

$$d(\lambda) = \det[-(\mathbf{HS} + \lambda\mathbf{I})] = (-1)^{2n}\det[\mathbf{HS} + \lambda\mathbf{I}] = d(-\lambda)$$

e.g. $d(\lambda)$ contains only even powers of λ, e.g. $d(\lambda) \equiv d_1(\lambda^2)$. This fact has dramatic consequences for accelerators: it reduces power of the eigen value equation by a factor of two and allows to find analytical expressions for all possible cases. In 3D case it reduces equation to a cubic equation on λ^2, which has analytical solution. In general (non-Hamiltonian case) we would face finding roots of a 6-th order polynomial, which does not have known analytical expressions. Again, this is demonstration of power of Hamiltonian approach and its symplectic metrics.

Next step

$$\det\left[D-\lambda I\right]=\det\begin{bmatrix} -\lambda & 1 & -L & 0 \\ -f & -\lambda & -n & -L \\ L & 0 & -\lambda & 1 \\ -n & L & -g & -\lambda \end{bmatrix};$$

Matrix of "a hard-edge" accelerator element. Here we are interested in finding solution for 4x4 matrices, e.g. Hamiltonian nature of the problem reduces eigen value problem to quadratic equation on λ^2, which is easy to solve. Indeed, characteristic equation for D in eq. (7-5) is biquadratic:

$$\det[D-\lambda I] = \lambda^4 + \lambda^2\left(f+g+2L^2\right) + fg + L^4 - L^2(f+g) - n^2 = 0 \tag{7-15}$$

with easy roots (eigen values):

$$\lambda^2 = a \pm b;\ a = -\frac{f+g+2L^2}{2};\ b^2 = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2 \tag{7-16}$$

allowing a simple classification of various cases. Before starting classification of the cases, let's note that

$$f+g = K^2 + 2\left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_x}{p_o^{\ 2}}\right)^2 + \left(\frac{meE_z}{p_o^{\ 2}}\right)^2 \geq 0$$

i.e. $a \leq 0;\ b^2 \geq 0;\ \operatorname{Im}(b) = 0$.

Analytical solutions with the full set of eigen values $\{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2\}$ can be classified as following:

I. $\lambda_1 = \lambda_2 = 0;\ \ a = 0;\ b = 0;$

II. $\lambda_1 = \lambda_2 = i\omega;\ \ a = -\omega^2;\ b = 0;$

III. $\lambda_1 = 0;\ \ \lambda_2 = i\omega;\ \ a + b = 0;\ \ 2b = \omega^2$

IV. $\lambda_1 = i\omega_1;\ \ \lambda_2 = i\omega_2;\ \ \omega_1^{\ 2} = -a - b;\ \omega_2^{\ 2} = -a + b;\ |a| > b$

V. $\lambda_1 = i\omega_1;\ \ \lambda_2 = \omega_2;\ \omega_1^{\ 2} = -a - b;\ \omega_2^{\ 2} = b - a;\ b > |a|$

Before going to case-by-case calculations, lets use Sylvester's formulae and try to find solution of inhomogeneous equation:

$$\frac{d\mathbf{R}}{ds} = \mathbf{D}\cdot \mathbf{R} + \mathbf{C};\ \ \mathbf{R}(0) = 0\cdot \tag{7-17}$$

When matrix det **D**≠0, (7-17) can be inversed using a $\mathbf{R} = A + e^{Ds}\cdot B$ as a guess and the boundary condition $\mathbf{R}(0) = 0$:

$$\mathbf{R} = \left(\mathbf{M_{4x4}(s)} - \mathbf{I}\right)\cdot \mathbf{D}^{-1}\cdot \mathbf{C} \tag{7-18}$$

is the easiest solution. Prove is just straight forward:

$$\mathbf{R}' = \mathbf{D}\cdot\mathbf{M_{4x4}}\cdot\mathbf{D^{-1}}\cdot\mathbf{C};$$

$$\mathbf{D}\cdot\left(\mathbf{M_{4x4}} - \mathbf{I}\right)\cdot\mathbf{D^{-1}}\cdot\mathbf{C} + \mathbf{C} = \mathbf{D}\cdot\mathbf{M_{4x4}}\cdot\mathbf{D^{-1}}\cdot\mathbf{C}\,\#$$

In all cases we can use method of variable constants to find it:

$$\frac{dR}{ds} = R' = \mathbf{D}\cdot R + \mathbf{C};\quad \mathbf{M}' = \mathbf{DM};$$

$$R = \mathbf{M}(s)A(s) \Rightarrow \mathbf{M}'A + \mathbf{M}A' = \mathbf{DM}A + C;\quad R(0) = 0 \Rightarrow A_o = 0 \tag{7-19}$$

$$A' = \mathbf{M}^{-1}(s)C \Rightarrow A = \int_0^s \mathbf{M}^{-1}(z)C\,dz = \left(\int_0^s e^{-\mathbf{D}z}dz\right)\cdot C;\quad R = e^{\mathbf{D}s}\left(\int_0^s e^{-\mathbf{D}z}dz\right)\cdot C$$

It is important to remember that $\mathbf{M^{-1}(s)}$ is just the $\mathbf{M(-s)} = \mathbf{e^{-Ds}}$. Hence in all our formulae for matrixes from previous lectures we need to replace s by –s to get $\mathbf{M^{-1}(s)}$**.**

Other vice, we have to use general formula (33) for the homogeneous solution and use method of variable constants (see Appendix F in last class) to find it:

$$R(s) = \sum_{k=1}^{m}\left\{\prod_{i\neq k}\left[\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k-\lambda_i}\right]\sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j\right\}\sum_{n=0}^{n_k-1}(\mathbf{D}-\lambda_k\mathbf{I})^n\frac{s^n}{n!}\cdot\sum_{p=0}^{n_k-1}(-1)^{p+1}(\mathbf{D}-\lambda_k\mathbf{I})^p\cdot\mathbf{C}\cdot\left[\sum_{q=0}^{p1}\frac{s^{p-q}}{(p-q)!\,\lambda_k^{\,q+1}} - \frac{e^{\lambda_k}}{\lambda_k^{\,p+1}}\right] \tag{7-20}$$

In all specific cases I, II, III, IV and V, integrating (7-19) directly is usually easier that using general form of (7-20).

Now we are ready to write down exact solutions for all five cases:

$$\frac{f+g+2L^2}{2}=0;\ \frac{(f-g)^2}{4}+2L^2(f+g)+n^2=0;$$

Case I.

$$f+g=pos^2\geq 0\Rightarrow (f-g)^2=0;\ L^2(f+g)=0;\ n^2=0$$

$$f+g+2L^2=pos^2+2L^2=0\Rightarrow L=0;\ f+g=0\Rightarrow$$

$$f-g=0\Rightarrow f=g=L=n=0!!!$$

means that there is nothing in the Hamiltonian but p^2– is this the drift section matrix of which we already know. Hence, there is not curvature as well and R=0.

$$\mathbf{M}_{\mathbf{4x4}}=\begin{bmatrix}1 & s & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & s\\ 0 & 0 & 0 & 0\end{bmatrix};\ \mathbf{R}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \qquad \text{(I-1)}$$

The only not trivial (ha-ha – it is also as trivial as it can be) is R_{56}:

$$R_{56}=\frac{m^2c^2}{p_o^2}s \qquad \text{(I-2)}$$

we already had seen it when studied nilpotent case…

Case II: $b = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2 = 0;$

$$f = g;\ n = 0\ \ and\ \ L^2(f+g) = L^2\left(K^2 + \Omega^2 + El^2\right) = 0; \Omega = eB_s/2p_o c; E_\perp = 0.$$

i.e. there are two cases: L=0 or $f + g = 0$.

If both are equal zero, i.e. $f + g = 0;\ \ L = 0$, this is equivalent to the case I above.

Case II a: $f + g = 0$, K=0, Bs=0 -> L=κ. Thus, this is just a drift (straight section) with rotation, whose matrix is trivial: Drift + rotation. There is not transverse force – hence R=0.

$$\mathbf{M_{4x4}} = \begin{bmatrix} M_d \cdot \cos\kappa s & -M_d \cdot \sin\kappa s \\ M_d \cdot \sin\kappa s & M_d \cdot \cos\kappa s \end{bmatrix};\ M_d = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix};\ \ \mathbf{R} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(IIa-1)}$$

R_{56} is as for a drift:

$$R_{56} = \frac{m^2 c^2}{p_o^2} s \qquad \text{(IIa-2)}$$

Case II b: L=0; $f = g = \left(K^2+\Omega^2\right)/2; \kappa = -\Omega$; i.e. the motion is uncoupled:

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -f & 0 \end{bmatrix};\ C = \begin{bmatrix} 0 \\ g_x \\ 0 \\ g_y \end{bmatrix}.$$

$$\mathbf{M}_{4x4} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix};\ M = \begin{bmatrix} \cos\omega s & \sin\omega s/\omega \\ -\omega\sin\omega s & \cos\omega s \end{bmatrix} \qquad \text{(IIb-1)}$$

Here we may have non-zero R: yes, it may be! It is simple integrals to be taken care of:

$$C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix};\ M^{-1}(z) = \begin{bmatrix} \cos\omega z & -\sin\omega z/\omega \\ \omega\sin\omega z & \cos\omega z \end{bmatrix} C_{x,y} = g_{x,y}\begin{bmatrix} \sin\omega z/\omega \\ -\cos\omega s \end{bmatrix};$$

$$\int_0^s \mathbf{M}^{-1}(z)C_{x,y}dz = g_{x,y}\begin{bmatrix} \int_0^s \sin(\omega z)dz/\omega \\ -\int_0^s \cos(\omega z)dz \end{bmatrix} = \frac{g_{x,y}}{\omega}\begin{bmatrix} (1-\cos\omega s)/\omega \\ -\sin(\omega s) \end{bmatrix}$$

$$C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \int_0^s \mathbf{M}^{-1}(z)dz \cdot C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$M(s)\int_0^s \mathbf{M}^{-1}(z)C_{x,y}dz = \frac{g_{x,y}}{\omega}\begin{bmatrix} \cos\omega s & \sin\omega s/\omega \\ -\omega\sin\omega s & \cos\omega s \end{bmatrix}\cdot\begin{bmatrix} (1-\cos\omega s)/\omega \\ -\sin(\omega z) \end{bmatrix} = \frac{g_{x,y}}{\omega^2}\begin{bmatrix} \cos\omega s - 1 \\ -\omega\sin\omega s \end{bmatrix}$$

Case II b cont…

$$R_{56} = s \cdot m^2c^2 / p_o + \int_0^s \left(g_x R(z)_{16} + g_y R_{36}(z) \right) dz =$$

$$\int_0^s \left(g_x R(z)_{16} + g_y R_{36}(z) \right) dz = \frac{g_x{}^2 + g_y{}^2}{\omega^2} \int_0^s (\cos \omega z - 1) dz = \frac{g_x{}^2 + g_y{}^2}{\omega^2} \left(\frac{\sin \omega s}{\omega} - s \right)$$

with the result:

$$R = \begin{bmatrix} \frac{g_x}{\omega^2}(\cos \omega s - 1) \\ -\frac{g_x}{\omega} \sin \omega s \\ \frac{g_y}{\omega^2}(\cos \omega s - 1) \\ -\frac{g_x}{\omega} \sin \omega s \end{bmatrix}; \quad R_{56} = \frac{m^2c^2}{p_o{}^2} s + \frac{g_x{}^2 + g_y{}^2}{\omega^2} \left(\frac{\sin \omega s}{\omega} - s \right) \quad \text{(IIb-2)}$$

Case III: $a+b=0$; $\det D=0$; $\omega^2=2b$; $\lambda_{1,2}=\pm i\omega;\lambda_3=0$; $m=3$.

We have to use degenerated case formula, but the maximum height of the eigen vector is 2 and only for 3-rd eigen value. Since it is not scary at all: $n_1=1;n_2=1;n_3=2$. Let's do it step by step:

$$\exp\left[\mathbf{D}s\right]=e^{\lambda_1 s}\frac{\left(\mathbf{D}-\lambda_2\mathbf{I}\right)}{\left(\lambda_1-\lambda_2\right)}\frac{\left(\mathbf{D}-\lambda_3\mathbf{I}\right)^2}{\left(\lambda_1-\lambda_3\right)^2}+e^{\lambda_2 s}\frac{\left(\mathbf{D}-\lambda_1\mathbf{I}\right)}{\left(\lambda_2-\lambda_1\right)}\frac{\left(\mathbf{D}-\lambda_3\mathbf{I}\right)^2}{\left(\lambda_2-\lambda_3\right)^2}$$

$$\left(\phi_3\left(\lambda_3,s\right)+\phi_3'\left(\lambda_3,s\right)\left(\mathbf{D}-\lambda_3\mathbf{I}\right)\right)\left(\mathbf{D}-\lambda_1\mathbf{I}\right)\left(\mathbf{D}-\lambda_2\mathbf{I}\right)$$

$$\phi_3\left(\lambda,s\right)=\frac{e^{\lambda s}}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)};\ \lambda_{1,2}=\pm i\omega;\ \lambda_3=0$$

$$\exp\left[\mathbf{D}s\right]=e^{i\omega s}\frac{\left(\mathbf{D}+i\omega\mathbf{I}\right)}{2i\omega}\frac{\mathbf{D}^2}{\left(i\omega\right)^2}-e^{-i\omega s}\frac{\left(\mathbf{D}-i\omega\mathbf{I}\right)}{2i\omega}\frac{\mathbf{D}^2}{\left(i\omega\right)^2}+\frac{\mathbf{I}+\mathbf{D}\cdot s}{\omega^2}\left(\mathbf{D}^2+\omega^2\mathbf{I}\right)$$

$$\phi_3\left(\lambda,s\right)=\frac{e^{\lambda s}}{\left(\lambda^2+\omega^2\right)};\phi_3'\left(\lambda,s\right)=\frac{s\cdot e^{\lambda s}}{\left(\lambda^2+\omega^2\right)}+\frac{2\lambda\cdot e^{\lambda s}}{\left(\lambda^2+\omega^2\right)^2};$$

$$\phi_3\left(\lambda=0,s\right)=\frac{1}{\omega^2};\phi_3'\left(\lambda=0,s\right)=\frac{s}{\omega^2}$$

$$\exp\left[\mathbf{D}s\right]=\left(\mathbf{I}+\mathbf{D}\cdot s\right)\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)-\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s+\mathbf{D}\sin\omega s\right)$$

$$\mathbf{M}_{4x4}\left(s\right)=\left(\mathbf{I}+\mathbf{D}\cdot s\right)\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)-\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s+\mathbf{D}\sin\omega s\right)\qquad\text{(III-1)}$$

Case III continued..

Similarly

$$R = \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}s + \mathbf{D}\frac{s^2}{2} \right) + \frac{\mathbf{D}^2}{\omega^4} \big(\mathbf{D}(\cos\omega s - 1) - \mathbf{I}\omega \sin\omega s \big) \right\} C \qquad \text{(III-2)}$$

Next is just

$$\int_o^s C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}z + \mathbf{D}\frac{z^2}{2} \right) + \frac{\mathbf{D}^2}{\omega^4} \big(\mathbf{D}(\cos\omega z - 1) - \mathbf{I}\omega \sin\omega z \big) \right\} C dz =$$

$$C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}\frac{s^2}{2} + \mathbf{D}\frac{s^3}{6} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D} \left(\frac{\sin\omega s}{\omega} - s \right) \mathbf{I}(\cos\omega z - 1) \right) \right\} C$$

with result of:

$$R_{56} = m^2 c^2 / p_o s + C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}\frac{s^2}{2} + \mathbf{D}\frac{s^3}{6} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D} \left(\frac{\sin\omega s}{\omega} - s \right) \mathbf{I}(\cos\omega z - 1) \right) \right\} C \qquad \text{(III-3)}$$

Case IV: all roots are different, no degeneration. Use formula (36)

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2}\left(\frac{e^{\lambda_k s}+e^{-\lambda_k s}}{2}\mathbf{I}+\frac{e^{\lambda_k s}-e^{-\lambda_k s}}{2\lambda_k}\mathbf{D}\right)\prod\left(\frac{\mathbf{D}^2-\lambda_j{}^2\mathbf{I}}{\lambda_k{}^2-\lambda_j{}^2}\right)$$

with only one term in the product:

$$\mathbf{M}_{\mathbf{4x4}} = \frac{1}{\omega_1{}^2-\omega_2{}^2}\left\{\left(\mathbf{I}\cos\omega_1 s+\mathbf{D}\frac{\sin\omega_1 s}{\omega_1}\right)\left(\mathbf{D}^2+\omega_2{}^2\mathbf{I}\right)-\left(\mathbf{I}\cos\omega_2 s+\mathbf{D}\frac{\sin\omega_2 s}{\omega_2}\right)\left(\mathbf{D}^2+\omega_1{}^2\mathbf{I}\right)\right\} \qquad \text{(IV-1)}$$

For R we invoke a simplest formula:

$$\mathbf{R} = (\mathbf{M}_{\mathbf{4x4}}(\mathbf{s})-\mathbf{I})\mathbf{D}^{-1}\cdot\mathbf{C} \qquad \text{(IV-2)}$$

For R56 it is tedious but easy:

$$R_{56} = m^2c^2/p_o s + C^T \mathsf{M}\mathbf{D}^{-1}C;$$

$$\mathsf{M} = \frac{1}{\omega_1{}^2-\omega_2{}^2}\left\{\begin{array}{l}\left(\mathbf{I}\frac{\sin\omega_1 s}{\omega_1}+\mathbf{D}\frac{1-\cos\omega_1 s}{\omega_1{}^2}\right)\left(\mathbf{D}^2+\omega_2{}^2\mathbf{I}\right)- \\ \left(\mathbf{I}\frac{\sin\omega_2 s}{\omega_2}+\mathbf{D}\frac{1-\cos\omega_2 s}{\omega_2{}^2}\right)\left(\mathbf{D}^2+\omega_1{}^2\mathbf{I}\right)-\mathbf{I}\cdot s\end{array}\right\} \qquad \text{(IV-3)}$$

Case V: all roots are different, no degeneration. Use formula (36) again

$$\mathbf{M}_{\mathbf{4x4}} = \frac{1}{\omega_1^{\ 2} + \omega_2^{\ 2}} \left\{ \left(\mathbf{I} \cos\omega_1 s + \mathbf{D} \frac{\sin\omega_1 s}{\omega_1} \right) \left(\mathbf{D}^2 - \omega_2^{\ 2} \mathbf{I} \right) - \left(\mathbf{I} \cosh\omega_2 s + \mathbf{D} \frac{\sinh\omega_2 s}{\omega_2} \right) \left(\mathbf{D}^2 + \omega_1^{\ 2} \mathbf{I} \right) \right\} \quad \text{(V-1)}$$

$$\mathbf{R} = (\mathbf{M}_{\mathbf{4x4}}(\mathbf{s}) - \mathbf{I})\mathbf{D}^{-1} \cdot \mathbf{C} \quad \text{(V-2)}$$

$$R_{56} = m^2 c^2 / p_o s + C^T \mathsf{M} \mathbf{D}^{-1} C;$$

$$\mathsf{M} = \frac{1}{\omega_1^{\ 2} + \omega_2^{\ 2}} \left\{ \begin{array}{l} \left(\mathbf{I} \dfrac{\sin\omega_1 s}{\omega_1} + \mathbf{D} \dfrac{1 - \cos\omega_1 s}{\omega_1^{\ 2}} \right) \left(\mathbf{D}^2 - \omega_2^{\ 2} \mathbf{I} \right) - \\ \left(\mathbf{I} \dfrac{\sinh\omega_2 s}{\omega_2} + \mathbf{D} \dfrac{\cosh\omega_2 s - 1}{\omega_2^{\ 2}} \right) \left(\mathbf{D}^2 + \omega_1^{\ 2} \mathbf{I} \right) - \mathbf{I} \cdot s \end{array} \right\} \quad \text{(V-3)}$$

Now we are ready to calculate matrix of arbitrary DC element in accelerator. To finish discussion of few remaining topics for 6x6 matrix of an accelerator. First is multiplication of the 6x6 matrixes for purely magnetic elements:

$$\mathbf{M}_{k(6x6)} = \begin{bmatrix} \mathbf{M}_{k(4x4)} & 0 & R_k \\ L_k & 1 & R_{56_k} \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{M}_{2(6x6)}\mathbf{M}_{1(6x6)} = \begin{bmatrix} \mathbf{M}_{(4x4)} & 0 & R \\ L & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_2\mathbf{M}_1 & 0 & R_2 + \mathbf{M}_2 R_1 \\ L_2 + L_1\mathbf{M}_2 & 1 & R_{56_1} + R_{56_2} + L_2 R_1 \\ 0 & 0 & 1 \end{bmatrix} \quad (7\text{-}21)$$

i.e. having transformation rules for mixed size objects: a 4x4 matrix M, 4-elemetn column R, 4 element line L, and a number R_{56}. As you remember, L is dependent (L4-7) and expressed as $\mathbf{L} = \mathbf{R}^T\mathbf{S}\mathbf{M}$. Thus:

$$\mathbf{M}_{(4x4)} = \mathbf{M}_2\mathbf{M}_1;\;\; R = \mathbf{M}_2 R_1 + R_2;\;\; L = L_2\mathbf{M}_1 + L_1;\;\; R_{56} = R_{56_1} + R_{56_2} + L_2 R_1 \quad (7\text{-}22)$$

Just as an exercise, let's consider a transport matrix of quadrupole (CASE V) :

$$D_x = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}; D_y = \begin{bmatrix} 0 & 1 \\ K_1 & 0 \end{bmatrix}; K_1 = -\frac{e}{p_o c}\frac{\partial B_y}{\partial x};$$

$$\begin{pmatrix} x \\ \pi_1 \end{pmatrix} \equiv \begin{pmatrix} x \\ x' \end{pmatrix}; \begin{pmatrix} y \\ \pi_3 \end{pmatrix} \equiv \begin{pmatrix} y \\ y' \end{pmatrix};$$

$$\left\{ \begin{matrix} K_1 > 0; \ M_x = M_F; M_y = M_D \\ K_1 < 0; \ M_x = M_D; M_y = M_F \end{matrix} \right\}; \ \omega = \sqrt{|K|}; \varphi = \omega s; \qquad (7\text{-}23)$$

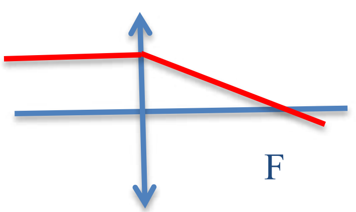
$$M_F = \begin{bmatrix} \cos\varphi & \dfrac{\sin\varphi}{\omega} \\ -\omega\sin\varphi & \cos\varphi \end{bmatrix}; M_D = \begin{bmatrix} \cosh\varphi & \dfrac{\sinh\varphi}{\omega} \\ \omega\sinh\varphi & \cosh\varphi \end{bmatrix};$$

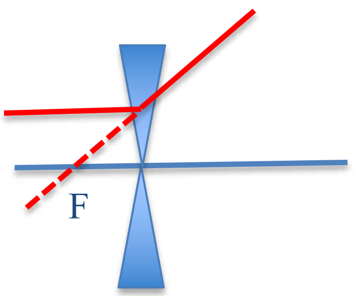
this is the most popular focusing/defocusing element in accelerators. In a case when length of the quadrupole is very short, but the strength is finite we can use a thin-lens approximations:

$$\varphi = l\sqrt{K_1} \to 0; K_1 l = const = \frac{1}{F}$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\dfrac{p_o}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \dfrac{p_o}{F} & 1 \end{bmatrix}; \qquad (7\text{-}24)$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \dfrac{1}{F} & 1 \end{bmatrix}$$





How we do it?

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2n} e^{\lambda_k s} \prod_{j \neq k} \frac{\mathbf{D} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j}$$

$$\mathbf{D}_{4x4} = \begin{bmatrix} \mathbf{D}_x & 0 \\ 0 & \mathbf{D}_y \end{bmatrix}; \exp[\mathbf{D}_{4x4} s] = \begin{bmatrix} \exp[\mathbf{D}_x s] & 0 \\ 0 & \exp[\mathbf{D}_y s] \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{bmatrix}; \det[\mathbf{D} - \lambda \mathbf{I}] = (\lambda - i\Omega)(\lambda + i\Omega); \lambda_{1,2} = \pm i\Omega;$$

$$\mathbf{M}_F = \exp[\mathbf{D}s] = \frac{\mathbf{D} + i\Omega \mathbf{I}}{2i\Omega} e^{i\Omega s} - \frac{\mathbf{D} - i\Omega \mathbf{I}}{2i\Omega} e^{-i\Omega s} = \begin{bmatrix} \cos\Omega s & \frac{\sin\Omega s}{\Omega} \\ -\Omega \cdot \sin\Omega s & \cos\Omega s \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}; \det[\mathbf{D} - \lambda \mathbf{I}] = (\lambda - \Omega)(\lambda + \Omega); \lambda_{1,2} = \pm \Omega;$$

$$\mathbf{M}_D = \exp[\mathbf{D}s] = \frac{\mathbf{D} + \Omega \mathbf{I}}{2\Omega} e^{\Omega s} - \frac{\mathbf{D} - \Omega \mathbf{I}}{2i\Omega} e^{-\Omega s} = \begin{bmatrix} \cosh f\Omega s & \frac{\sinh f\Omega s}{\Omega} \\ \Omega \cdot \sinh f\Omega s & \cosh f\Omega s \end{bmatrix};$$

One more - less trivial case, a solenoid

- It is interesting that it can be found in two ways
  - Directly without using torsion – case 3
  - Using torsion –case 2b

Matrix of solenoid –Case 3

$$\tilde{h}_n = \frac{\pi_1^{\,2}+\pi_3^{\,2}}{2} + \Omega^2\frac{x^2}{2} + \Omega^2\frac{y^2}{2} + \Omega\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o^{\,2}}{2}\cdot\frac{m^2c^2}{p_o^{\,2}};\; L = \Omega = \frac{eB_s}{2p_oc}; f = g = \Omega^2; n = 0..$$

$$a = -\frac{f+g+2L^2}{2} = -2\Omega^2; b = \sqrt{\frac{(f-g)^2}{4} + 2L^2(f+g) + n^2} = 2\Omega^2;$$

$$a+b=0;\; \det D = 0;\quad \omega^2 = 2b;\quad \lambda_{1,2} = \pm i\omega; \lambda_3 = 0;\; m = 3.$$

$$\mathbf{M}_{\mathbf{4x4}} = \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2}\right)\left(\mathbf{I} + s\mathbf{D}\right) - \frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s + \frac{\mathbf{D}}{\omega}\sin\omega s\right)$$

$$\omega = 2\Omega$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & -L & 0 \\ -f & 0 & -n & -L \\ L & 0 & 0 & 1 \\ -n & L & -g & 0 \end{bmatrix} == \begin{bmatrix} 0 & 1 & -\Omega & 0 \\ -\Omega^2 & 0 & 0 & -\Omega \\ \Omega & 0 & 0 & 1 \\ 0 & \Omega & -\Omega^2 & 0 \end{bmatrix}; \frac{\mathbf{D}^2}{\omega^2} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 & -\Omega^{-1} \\ 0 & -1 & \Omega & 0 \\ 0 & \Omega^{-1} & -1 & 0 \\ -\Omega & 0 & 0 & -1 \end{bmatrix};$$

$$\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -\Omega^{-1} \\ 0 & 1 & \Omega & 0 \\ 0 & \Omega^{-1} & 1 & 0 \\ -\Omega & 0 & 0 & 1 \end{bmatrix}; \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2}\right)\mathbf{D} = 0 \rightarrow \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2}\right)\mathbf{D}s\;\; \textit{vanishes}$$

Matrix of solenoid:
using case 3we get an unusual matrix

$$\mathbf{M}_{\mathbf{4x4}} = \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2}\right)\left(\mathbf{I} + s\mathbf{D}\right) - \frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s + \frac{\mathbf{D}}{\omega}\sin\omega s\right)$$

$$\omega = 2\Omega;\ \Omega = \frac{eB_s}{p_o c}$$

$$\mathbf{M}_{4x4} = \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2}\right) - \frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s + \frac{\mathbf{D}}{\omega}\sin\omega s\right) = \frac{1}{2}\begin{bmatrix} 1+\cos 2\Omega s & \frac{\sin 2\Omega s}{\Omega} & -\sin 2\Omega s & -\frac{1-\cos 2\Omega s}{\Omega} \\ -\Omega\sin 2\Omega s & 1+\cos 2\Omega s & \Omega(1-\cos 2\Omega s) & -\sin 2\Omega s \\ \sin 2\Omega s & \frac{1-\cos 2\Omega s}{\Omega} & 1+\cos 2\Omega s & \frac{\sin 2\Omega s}{\Omega} \\ -\Omega(1-\cos 2\Omega s) & \sin 2\Omega s & -\Omega\sin 2\Omega s & 1+\cos 2\Omega s \end{bmatrix} \quad \text{(s1)}$$

Actually this matrix has a very simple structure, which can be easily reviled if we use torsion

Matrix of solenoid
To bring it to Case 2, we can use torsion to eliminate coupling terms in the Hamiltonian

$$\tilde{h}_n = \frac{\pi_1^{\,2} + \pi_3^{\,2}}{2} + \Omega^2 \frac{x^2}{2} + \Omega^2 \frac{y^2}{2} + \frac{\pi_o^{\,2}}{2} \cdot \frac{m^2c^2}{p_o^{\,2}}; \ \Omega = \frac{eB_s}{2p_oc}; \kappa = -\Omega \Rightarrow L = \frac{eB_s}{2p_oc} + \kappa = 0;$$

$$a = -\frac{f+g}{2} = -\Omega^2; b = 0; \quad \mathbf{D}_{4\text{x}4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\Omega^2 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{bmatrix}; \exp\left[\mathbf{D}_{4x4}s\right] = \begin{bmatrix} \exp[\mathbf{D}s] & 0 \\ 0 & \exp[\mathbf{D}s] \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{bmatrix}; \det\left[\mathbf{D} - \lambda\mathbf{I}\right] = (\lambda - i\Omega)(\lambda + i\Omega); \lambda_{1,2} = \pm i\Omega;$$

$$\mathbf{M} = \exp[\mathbf{D}s] = \frac{\mathbf{D} + i\Omega\mathbf{I}}{2i\Omega} e^{i\Omega s} - \frac{\mathbf{D} - i\Omega\mathbf{I}}{2i\Omega} e^{-i\Omega s} = \begin{bmatrix} \cos\Omega s & \dfrac{\sin\Omega s}{\Omega} \\ -\Omega \cdot \sin\Omega s & \cos\Omega s \end{bmatrix}; \mathbf{M}_{4\text{x}4} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix}$$

It means that in rotating coordinate system x and y motions are decoupled and it is simple oscillation caused by uniform focusing in both directions. What remains is rotation by angle

$$\varphi = \kappa s = -\Omega s$$

Matrix of solenoid… continued

To bring it to the same coordinate frame we need to rotate the coordinate back by angle $-\varphi = -\kappa s = \Omega s$

$$R_{4x4}(\theta) = \begin{bmatrix} \mathbf{I}\cos\theta & -\mathbf{I}\sin\theta \\ \mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{bmatrix}; \bar{\mathbf{M}}_{4x4} = R_{4x4}(\Omega s)\cdot \mathbf{M}_{4x4} = \begin{bmatrix} \mathbf{M}\cos\Omega s & -\mathbf{M}\sin\Omega s \\ \mathbf{M}\sin\Omega s & \mathbf{M}\cos\Omega s \end{bmatrix} \quad \text{(s2)}$$

It means that solenoid focuses equally in all direction and rotates planes of oscillation by and angle $\theta = \frac{eB_s}{2p_o s} s$

It easy to show that 4x4 matrices (s2) and (s1) are identical using simple ratios like $\cos^2\theta = \frac{1+\cos 2\theta}{2}; \sin\theta\cos\theta = \frac{\sin 2\theta}{2}; \sin^2\theta = \frac{1-\cos 2\theta}{2}$

It is interesting that matrix of s solenoid with arbitrary dependence of magnetic field has the same form

$$\bar{\mathbf{M}}_{4x4} = \begin{bmatrix} \mathbf{M}(s)\cos\theta & -\mathbf{M}(s)\sin\theta \\ \mathbf{M}(s)\sin\theta & \mathbf{M}(s)\cos\theta \end{bmatrix}; \theta = \frac{1}{2p_o c}\int^{s} B_s(z)\,dz.$$

What we learned today?

- Majority of accelerator elements are either drifts or magnets, located in the places where energy of the beam is constant
- Many of them can be considered to be DC, e.g. time independent
- Typical approach of calculating a beamline transport matrix is to consider elements with step-wise constant "coefficients"
- Since energy if the beam is constant, the 6x6 matrix is reduced to 4x4 matrix, as special solution (4-vector) for particle with deviated energy and a slip-factor R_{56} accounting for dependence of the travel time on the particle's energy.
- We applied Sylvester formulae, derive during last class.
- There is only five distinct cases covering any possible DC hard-edge elements, or any shot slice of *s*-dependent magnet parameter (such as magnet edge field)
- Now you should be able to write matrix of any DC element you encounter in accelerator