Homework 20

1. A point particle with charge Q moves with velocity \( \vec{v} = (0, 0, v_z) \) in the longitudinal direction. The particle has a horizontal offset, \( x = a \). Show that, in the cylindrical system, the charge distribution due to the particle can be written as:

\[
\rho(r, \theta, z, t) = \sum_{m=0}^{\infty} \frac{Q_m \cos(m\theta)}{\pi a^{m+1}(1 + \delta_{m,0})} \delta(z - v_z t) \delta(r - a)
\]

with \( Q_m = Q a^m \) and \( \delta_{m,0} = \left\{ \begin{array}{ll} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{array} \right. \)

Hint 1. prove that the delta function in cylindrical coordinate is \( \rho(x, y, z) = \frac{1}{r} \delta(r - a) \delta_\theta(\theta) \delta(z) \)

with \( \delta_\theta(\theta) = \sum_{2\pi k} \delta(\theta - 2\pi k) \) (Jackson page 120 footnote)

Hint 2. perform Fourier transformation to \( \delta_\theta(\theta) \)
Homeworks
(Taken from exercise 2.16 from [4] by A. Chao)

2. Perform a contour integral of $\frac{Z_{//}(\omega')}{\omega' - \omega}$ in the complex $\omega'$-plane over the upper half plane along the contour shown in the figure. Show that if $Z_{//}(\omega')$ converges sufficiently fast as $|\omega| \to \infty$

$$Z_{//}(\omega) = -\frac{i}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega' - \omega} d\omega' \quad (1)$$

Show that eq. (1) leads to Kramers-Kronig relations

$$\text{Re}[Z_{//}(\omega)] = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Im}[Z_{//}(\omega')]}{\omega' - \omega} d\omega'$$

$$\text{Im}[Z_{//}(\omega)] = -\frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Re}[Z_{//}(\omega')]}{\omega' - \omega} d\omega'$$

About Principal Value Integral:

The trick of P.V. is to utilize the property that the divergences on the side $\omega' < \omega$ and the side $\omega' > \omega$ are of opposite signs and, if the integration is taken symmetrically about the singularity so that the divergences on the two sides cancel each other, the integral is actually well defined. Algebraically, this leads to

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{f(x)}{x - a} = \int_{0}^{\infty} du \frac{f(a + u) - f(a - u)}{u},$$

where the expression on the right is well behaved at $u = 0$. 