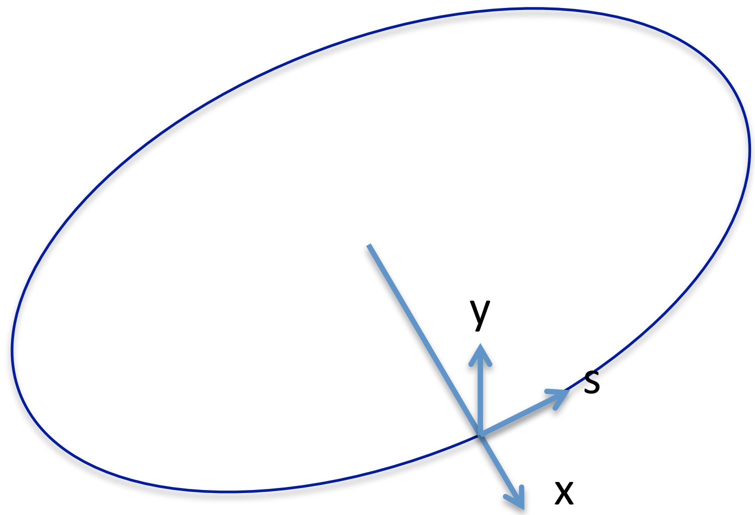
Problem 1. 5 points. Plane symmetry and plane trajectories.



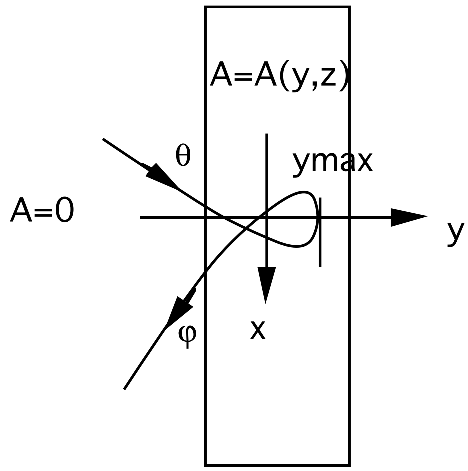
(a) Plane reference orbit (torsion κ=0) requires that total out-of plane force is equal zero. Find ratio between radial (x, horizontal) magnetic field and of-plane (vertical, y) electric field to satisfy this condition. What's happens when both of them are equal zero?

(b) Define full set of condition on EM field providing that all in-plane trajectries (e.g. all trajectroies with y=0 and y'=0, but otherwise arbitrary) to stay in-plane, i.e. y=0 is a solutions. Consider that particles have different energies.

Hint: use Lorentz force

Problem 2. 10 points. Magnetic Mirror: An electron propagates through a magnetic field with vector potential $\vec{A} = \vec{A}(y,z)$. Find an additional invariant of motion caused by independence of vector potential on x. Write explicit expression for p_x using this invariant. Consider a magnet with mid-plane symmetry ($\vec{H} = \hat{e}_z H(y)$ *at* $z = 0$; z is perpendicular to the plane of figure) shown below with $\vec{A} = \vec{A}(y,z)$ inside the magnet and $\vec{A} = 0$ outside the magnet. Let's consider an electron entering the magnet in the middle plane $z = 0$ with mechanical momentum $\vec{p} = \hat{e}_x p_x + \hat{e}_y p_y = p(\hat{e}_x \cos\theta + \hat{e}_y \sin\theta)$ ($\vec{A} = 0$) laying in the x-y plane, making turn in the magnet and coming out.

1. Show that trajectory of electron remains in the plane $z = 0$;
2. Find angle φ of out-coming trajectory of the electron (reflected angle).
3. Find equation defining depth of penetration of electron inside the magnet $y_{\max}$ using $A(y, z = 0)$.



Clues: use Lorentz force to find (1), Use canonical momentum to connect mechanical momentum with $\vec{A} = \vec{A}(y, z = 0)$ for (2,3)

Homework 3 with solutions:

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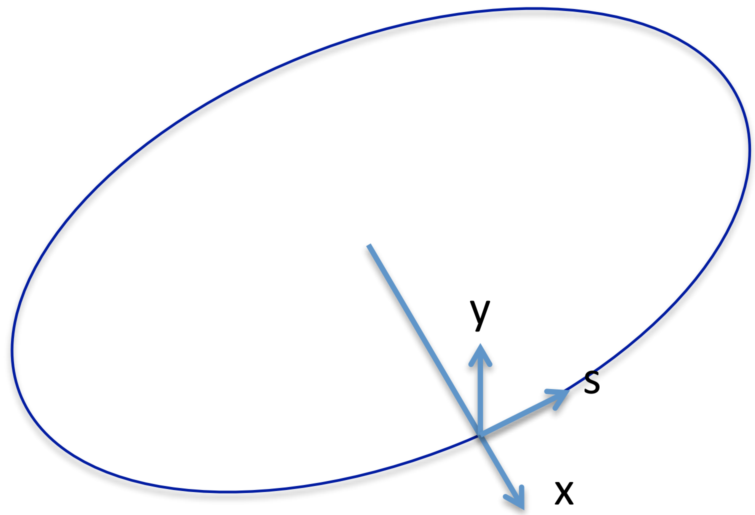

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Hint: use Lorentz force

Solution:
(a) Let's use Lorentz force for a particle on designed trajectory

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\left[\vec{\mathrm{v}} \times \vec{B}\right];$$

$$\vec{\mathrm{v}} = \hat{\tau}\mathrm{v}; \quad \frac{dp_y}{dt} = eE_y - \frac{e}{c}\mathrm{v}B_x = 0$$

to determine when vertical force is zero:

$$E_y = \frac{\mathrm{v}}{c}B_x$$

Then $E_y = B_x = 0$ is a specific case of plane trajectory. Note that there is no condition on E_s, B_s, E_x, B_y.

(b) We have to extend previous condition $E_y = \frac{\mathrm{v}}{c}B_x$ to the case of various energies, e.g. various velocities. Hence, we will arrive to $E_y = B_x = 0$ in all plane *y=0*. It means that all derivatives about *x* and *s* are also zero:

$$E_y(x,s,y=0)=0;\;\; B_x(x,s,y=0)=0$$

$$\partial_x^n\partial_s^m E_y=0\,@\,y=0;\;\; \partial_x^n\partial_s^m B_x=0\,@\,y=0;$$

For an arbitrary velocity direction in y=0 plane:

$$\vec{\mathrm{v}}=\hat{\tau}\mathrm{v}_s+\hat{n}\mathrm{v}_x;$$

$$\frac{dp_y}{dt}=eE_y-\frac{e}{c}\mathrm{v}_s B_x+-\frac{e}{c}\mathrm{v}_s B_s=0$$

we also have to conclude that longitudinal (along the reference trajectory) component of the field is also has to be zero:

$$B_s(x,s,y=0)=0;\;\; \partial_x^n\partial_s^m B_s=0\,@\,y=0;$$

In short, for particle's trajectories to be able to stay in one plane, the perpendicular component of the electric field has to be zero in all point of the plane (e.g. only components of electric field parallel to the surface y=0 are allowed), and magnetic field has to be strictly perpendicular to the plane y=0 (e.g. only B_y is allowed in the plane y=0).

Problem 2. 10 points.

Magnetic Mirror: An electron propagates through a magnetic field with vector potential $\vec{A}=\vec{A}(y,z)$. Find an additional invariant of motion caused by independence of vector potential on x. Write explicit expression for p_x using this invariant. Consider a magnet with mid-plane symmetry ($\vec{H}=\hat{e}_z H(y)$ *at* $z=0$; z is perpendicular to the plane of figure) shown below with $\vec{A}=\vec{A}(y,z)$ inside the magnet and $\vec{A}=0$ outside the magnet. Let's consider an electron entering the magnet in the middle plane $z=0$ with mechanical momentum $\vec{p}=\hat{e}_x p_x+\hat{e}_y p_y=p(\hat{e}_x\cos\theta+\hat{e}_y\sin\theta)$ ($\vec{A}=0$) laying in the x-y plane, making turn in the magnet and coming out.

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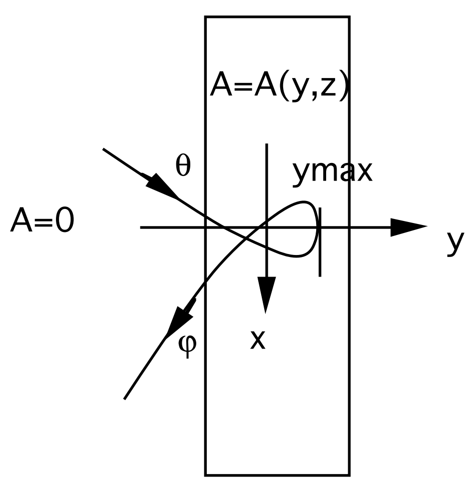

Clues: use Lorentz force to find (1), Use canonical momentum to connect mechanical momentum with $\vec{A} = \vec{A}(y, z = 0)$ for (2,3)

Solution:

1. Lorentz force in the middle plane z=0

$$\vec{f} = \frac{e}{c}\left[\vec{v} \times \vec{H}\right] = \frac{e}{c} H(y)\left[\vec{v} \times \hat{e}_z\right]$$

lays in the x-y plane:

$$f_z = \vec{f}\hat{e}_z = \frac{e}{c} H(y)\hat{e}_z\left[\vec{v} \times \hat{e}_z\right] \equiv 0.$$

i.e. nothing can move particle out of plane trajectory if $v_{z0} = 0$. $p_z = 0$ #1

2. Hamiltonian of the particle does not depend on x because of $\vec{A} = \vec{A}(y,z)$. It means that x-component of canonical momentum is constant

$$P_x = p_x + \frac{e}{c} A_x(y) = const = p_0 \cos\theta; \frac{dP_x}{dt} = -\frac{\partial H}{\partial x} \equiv 0;$$

which means that

$$p_x = const - \frac{e}{c} A_x(y)$$

depends only on y. In magnetic field the energy of the particle and total momentum are constants: (For the plane trajectory $p_z = 0$)

$$\vec{p}^2 = p_o^2 = p_x^{\,2} + p_y^{\,2} = E^2 / c^2 - m^2c^2 = const; p_y = \pm\sqrt{p_o^2 - p_x^{\,2}(y)}\,.$$

For particle to leave system, p_y should be negative. Therefore:

$$\tan\varphi = -\frac{p_y(y)}{p_x(y)}(out) = f(y) = \tan\theta = \frac{p_y(y)}{p_x(y)}(in);$$

is function of y only and

$$\varphi(y) = \theta(y)$$

is generalization of the answer. #2

3. Maximum depth $y_{\max}$ is achieved when $p_y = 0$, i.e. $p_x = p_0$:

$$p_o = p_o \cos\theta - \frac{e}{c} A_x(y_{\max});$$

$$y_{\max} = A_x^{-1}\left(\frac{c}{e} p_o(\cos\theta - 1)\right) \text{ \#3}$$

Additional information: conservation of P_x and total mechanical momentum (energy) does not depend on the plane of trajectory but only on fact that particle move in the magnetic field with no dependence on x! For arbitrary trajectory we can state that:

$$p_x = p_o \cos\theta - \frac{e}{c} A_x(y)$$

$$p_y^{\,2} + p_z^{\,2} = p_o^2 - \left(p_o \cos\theta - \frac{e}{c} A_x(y)\right)^2;$$

$$\frac{\sqrt{dy^2 + dz^2}}{dx} = \sqrt{\left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} = \sqrt{1 - \left(\cos\theta - \frac{e}{p_o c} A_x(y)\right)^2};$$

i.e. we have correlation's between vertical (z) and horizontal angles at the exit.

Magnetic mirrors are very useful devices allowing to forget about details of magnetic field dependencies. Providing appropriate dependence of $\vec{H} = \hat{e}_z H(y)$, one can make design trajectory to cross at the entrance or to focus beam.