# USPAS'23: Hadron Beam Cooling in Particle Accelerators Solutions: Stochastic Cooling Final 

## Damping rates of Optical Stochastic Cooling (Total: 25 points)

Let us consider an Optical Stochastic Cooler (OSC) and find the horizontal and longitudinal damping rates of OSC. Assume a particle motion that is only coupled between longitudinal and horizontal planes such that the vertical motion is uncoupled and can be safely omitted.

1. (5 points) The relative longitudinal momentum change of a particle can be expressed as $\delta p / p=$ $\kappa \sin (k s)$ with $k=2 \pi / \lambda$. Show that the linearized longitudinal kick takes the following form:

$$
\begin{equation*}
\delta p / p=\kappa k\left(M_{51}^{\mathrm{PK}} x+M_{52}^{\mathrm{PK}} \theta_{x}+M_{56}^{\mathrm{PK}} \Delta p / p\right) \tag{1}
\end{equation*}
$$

where $\mathbf{M}^{\mathrm{PK}}$ is a pick-up to kicker transfer matrix.

## Solution:

For the case of linearized longitudinal kick:

$$
\delta p / p=\kappa \sin (k s) \approx \kappa k s
$$

Using the P-to-K transfer matrix:

$$
\mathbf{M}=\left[\begin{array}{cccc}
M_{11} & M_{12} & 0 & M_{16} \\
M_{21} & M_{22} & 0 & M_{26} \\
M_{51} & M_{52} & 1 & M_{56} \\
0 & 0 & 0 & 1
\end{array}\right], \mathbf{x}=\left[\begin{array}{c}
x \\
\theta_{x} \\
s \\
\Delta p / p
\end{array}\right]
$$

One obtains

$$
\delta p / p=\kappa k\left(M_{51}^{\mathrm{PK}} x+M_{52}^{\mathrm{PK}} \theta_{x}+M_{56}^{\mathrm{PK}} \Delta p / p\right)
$$

2. (10 points) We will use the result obtained in part 1 to define the matrix $\mathbf{M}^{\mathrm{C}}$ such as:

$$
\mathbf{M}^{\mathrm{C}}=\kappa k\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{2}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{51}^{\mathrm{PK}} & M_{52}^{\mathrm{PK}} & 0 & M_{56}^{\mathrm{PK}}
\end{array}\right] .
$$

The perturbation theory for the case of symplectic unperturbed motion yields that the tune shifts are:

$$
\delta Q_{k}=\frac{1}{4 \pi} \mathbf{v}_{k}^{\dagger} \mathbf{U} \mathbf{M}^{C} \mathbf{U}\left(\mathbf{M}^{\mathrm{PK}}\right)^{T} \mathbf{U} \mathbf{v}_{k}=\frac{k \kappa}{4 \pi} \mathbf{v}_{k}^{\dagger}\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{3}\\
0 & 0 & 0 & 0 \\
M_{26}^{\mathrm{PK}} & -M_{16}^{\mathrm{PK}} & 0 & M_{56}^{\mathrm{PK}} \\
0 & 0 & 0 & 0
\end{array}\right] \mathbf{v}_{k}
$$

Find the damping rate of the betatron motion $\lambda_{x}=-2 \pi \operatorname{Im} \delta Q_{1}$ if the eigenvector equals to:

$$
\mathbf{v}_{1}=\left[\begin{array}{c}
\sqrt{\beta_{2}}  \tag{4}\\
-\left(i+\alpha_{2}\right) / \sqrt{\beta_{2}} \\
-\left(i D_{2}\left(1-i \alpha_{2}\right)+D_{2}^{\prime} \beta_{2}\right) / \sqrt{\beta_{2}} \\
0
\end{array}\right]
$$

## Solution:

First, let's obtain the tune shift using the given eigenvector:

$$
\left.\begin{array}{rl}
\delta Q_{1} & =\frac{k \kappa}{4 \pi}\left[\sqrt{\beta_{2}}\right. \\
& \left(i-\alpha_{2}\right) / \sqrt{\beta_{2}} \quad\left(i D_{2}\left(1+i \alpha_{2}\right)-D_{2}^{\prime} \beta_{2}\right) / \sqrt{\beta_{2}}
\end{array} 0\right] \times x+\begin{array}{ccc}
0 & 0 & 0 \\
0 \\
0 & 0 & 0 \\
0 \\
M_{26}^{\mathrm{PK}} & -M_{16}^{\mathrm{PK}} & 0
\end{array} M_{56}^{\mathrm{PK}}\left[\begin{array}{c}
-\left(i+\alpha_{2}\right) / \sqrt{\beta_{2}} \\
0
\end{array}\right.
$$

After the matrix multiplication, one obtains the following:

$$
\delta Q_{1}=\frac{k \kappa}{4 \pi}\left(\left(i D_{2}\left(1+i \alpha_{2}\right)-D_{2}^{\prime} \beta_{2}\right) / \sqrt{\beta_{2}}\right) \times\left(\sqrt{\beta_{2}} M_{26}^{(\mathrm{PK})}+M_{16}^{(\mathrm{PK})}\left(i+\alpha_{2}\right) / \sqrt{\beta_{2}}\right)
$$

After taking the imaginary part of the resulting expression, we can find the damping rate for betatron motion:

$$
\lambda_{x}=-2 \pi \operatorname{Im} \delta Q_{1}=-\frac{k \kappa}{2}\left(D_{2} M_{26}^{(\mathrm{PK})}-D_{2}^{\prime} M_{16}^{(\mathrm{PK})}\right)
$$

3. (10 points) Find the damping rate of the synchrotron motion $\lambda_{s}=-2 \pi \operatorname{Im} \delta Q_{2}$ if the eigenvector is:

$$
\mathbf{v}_{2}=\left[\begin{array}{c}
-i D_{2} / \sqrt{\beta_{s}}  \tag{5}\\
-i D_{2}^{\prime} / \sqrt{\beta_{s}} \\
\sqrt{\beta_{s}} \\
-i / \sqrt{\beta_{s}}
\end{array}\right], \text { with } \beta_{s}=R \eta / \nu_{s}
$$

Solution: Similarly to part 2 :

$$
\begin{gathered}
\delta Q_{2}=\frac{k \kappa}{4 \pi}\left[\begin{array}{llll}
i D_{2} / \sqrt{\beta_{s}} & i D_{2}^{\prime} / \sqrt{\beta_{s}} & \sqrt{\beta_{s}} & i / \sqrt{\beta_{s}}
\end{array}\right]\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{26}^{\mathrm{PK}} & -M_{16}^{\mathrm{PK}} & 0 & M_{56}^{\mathrm{PK}} \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
-i D_{2} / \sqrt{\beta_{s}} \\
-i D_{2}^{\prime} / \sqrt{\beta_{s}} \\
\sqrt{\beta_{s}} \\
-i / \sqrt{\beta_{s}}
\end{array}\right] \\
\delta Q_{2}=\frac{k \kappa}{4 \pi}\left(\sqrt{\beta_{s}}\right)\left(-i M_{26}^{(\mathrm{PK})} D_{2} / \sqrt{\beta_{s}}+i M_{16}^{(\mathrm{PK})} D_{2}^{\prime} / \sqrt{\beta_{s}}-i M_{56}^{(\mathrm{PK})} / \sqrt{\beta_{s}}\right) \\
\lambda_{s}=-2 \pi \operatorname{Im} \delta Q_{2}=\frac{k \kappa}{2}\left(D_{2} M_{26}^{(\mathrm{PK})}-D_{2}^{\prime} M_{16}^{(\mathrm{PK})}+M_{56}^{(\mathrm{PK})}\right)
\end{gathered}
$$

