Homework 2. PHY 564

Problem 1. 10 points Motion of non-radiating charged particle in constant uniform magnetic field is a well known spiral:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = \frac{e}{c}H\left[\hat{e}_x v_y - \hat{e}_y v_x\right]; \vec{H} = \hat{e}_z H$$

$$\mathrm{E} = c\sqrt{m^2c^2 + \vec{p}^2} = conts;\ \gamma = const; v = const;$$

$$p_z = const;\ z = v_{oz}t + z_o;$$

$$p_x^2 + p_y^2 = const;\ p_x + ip_y = p_\perp e^{i\varphi(t)} = m\gamma v_\perp e^{i\varphi(t)}$$

simple substitution gives:

$$m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = -i\frac{e}{c}Hv_\perp e^{i\varphi(t)}$$

$$r_\perp = x + iy = i\omega m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt}$$

$$\varphi(t) = \omega t + \varphi_o;\ \omega = -\frac{eH}{m\gamma c}$$

and trajectory: $z = v_{oz}t + z_o; x + iy = v_\perp / \omega \cdot e^{i\omega t}$. Do not forget to apply Re or Im to all necessary formulae. Use analytical extension of the Lorentz transformation to complex values by going into a reference frame with x-velocity going approaching infinity $\beta \Rightarrow \infty; \chi \to 0; \chi\beta \to 1$. Show that transverse electric field becomes a magnetic field (with an imaginary value) and visa versa. Follow this path and transfer 4-coordinates to that frame. Use analytical extension of *exp, sin, cos* to complex values and transform the solution above in that for motion in constant magnetic field. Compare it with known solution is your favorite EM book .

Problem 2. 4 points

Find maximum energy of a charged particle (with unit charge *e*!) which can be circulating in Earth's larges possible storage ring: the one going around Earth equator with radius of 6,384 km.

First, find it for storage ring using average bending magnetic field of a super-conducting magnet with strength of 10 T (100 kGs).

Second, find it for a very strong DC electric dipole fields of 10 MV/m.

Compare these energies with current largest (27 km in circumference) circular collider, LHC, circulating 6.5 TeV (1 TeV = 10^{12} eV).

Hint: assume that particles move with speed of the light. Check the final result for protons having rest mass of 938.27 MeV/c^2

With solutions:

Problem 1. 10 points Motion of non-radiating charged particle in constant uniform magnetic field is a well known spiral:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = \frac{e}{c}H\left[\hat{e}_x v_y - \hat{e}_y v_x\right]; \vec{H} = \hat{e}_z H$$

$$\mathrm{E} = c\sqrt{m^2c^2 + \vec{p}^2} = conts;\ \gamma = const; v = const;$$

$$p_z = const;\ z = v_{oz}t + z_o;$$

$$p_x^2 + p_y^2 = const;\ p_x + ip_y = p_\perp e^{i\varphi(t)} = m\gamma v_\perp e^{i\varphi(t)}$$

simple substitution gives:

$$m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = -i\frac{e}{c}Hv_\perp e^{i\varphi(t)}$$

$$r_\perp = x + iy = i\omega m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt}$$

$$\varphi(t) = \omega t + \varphi_o;\ \omega = -\frac{eH}{m\gamma c}$$

and trajectory: $z = v_{oz}t + z_o; x + iy = v_\perp / \omega \cdot e^{i\omega t}$. Do not forget to apply Re or Im to all necessary formulae. Use analytical extension of the Lorentz transformation to complex values by going into a reference frame with x-velocity going approaching infinity $\beta \Rightarrow \infty; \chi \to 0; \chi\beta \to 1$. Show that transverse electric field becomes a magnetic field (with an imaginary value) and visa versa. Follow this path and transfer 4-coordinates to that frame. Use analytical extension of *exp, sin, cos* to complex values and transform the solution above in that for motion in constant magnetic field. Compare it with known solution is your favorite EM book .

Solution: Let's, formally, apply Lorentz transformation as it is not limited to velocities lower than the speed of the light. In this sense, we are analytically extending Lorentz transformation into all range if the velocities: both real and imaginary.

Lorentz transformation with speed exceeding the speed of the light are analytic extensions of real transformations for $\beta > 1$ and two choices of sign (±):

$$L = \pm\begin{bmatrix} i\chi & i\chi\beta & & \\ i\chi\beta & i\chi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}; \chi = 1/\sqrt{\beta^2 - 1}$$

Applying this transformation to a pure electric field E_y, we got

$$E'_y = \pm i\chi E'_y; H'_z = \pm i\chi\beta E'_y;$$

and by applying

$$\beta \Rightarrow \infty; \chi \to 0; \chi\beta \to 1$$

we got desirable transformation:

$$E'_y = 0; H'_z = \pm iE_y$$

with imagine magnetic field instead of real electric field.

Our solutions for pure magnetic case should extended (I use specific initial conditions):

$$z' = v_{oz}t'; \omega = -\frac{eH'_z}{p_o c};$$
$$x' = v_\perp / \omega \cdot \sin(\omega t);$$
$$y' = v_\perp / \omega \cdot \cos(\omega t)$$

.

using coordinate and fields transformation have two branches

$$t' = \pm ix; x' = \pm it, z' = z; y' = y \quad and \quad H'_z = \pm iE_y$$
$$E' = \pm ip_x; p'_x = \pm iE; p'_y = p_y; p'_z = p_z;$$

into (I use $+i$ branch): $t' = +ix; x' = -it, z' = z; y' = y \quad and \quad H'_z = iE_y$:

$$z = iv_{oz}x; \omega = -i\frac{eE_y}{p_o c};$$
$$-it = v_\perp / \omega \cdot \cos(i\omega x + \varphi);$$
$$y = v_\perp / \omega \cdot \sin(i\omega x + \varphi)$$

.

And choosing constants easily transformable to well know result for an uniform electric field (we have to rename some of the constants of motion):

$$\frac{z}{x} = \frac{v_{oz}}{v_{ox}}; \ \Omega = \frac{eE_y}{p_o c};$$
$$t = \frac{E_o}{ecE_y} \cdot \sinh\left(\frac{eE_y}{p_o c}x\right);$$
$$y = \frac{E_o}{eE_y} \cdot \cosh\left(\frac{eE_y}{p_o c}x\right).$$

Naturally we used that $\sin(i\varphi) = -i\sinh\varphi; \cos(i\varphi) = \cosh\varphi$.

The goal of this problem was to demonstrate close connection of Lorentz transformations, special relativity and E&M fields. Not only that fields are transform into each other but also that solutions for particle's trajectory are analytical extensions of each other.

Problem 2. 4 points

Find maximum energy of a charged particle (with unit charge *e*!) which can be circulating in Earth's larges possible storage ring: the one going around Earth equator with radius of 6,384 km.

First, find it for storage ring using average bending magnetic field of a super-conducting magnet with strength of 10 T (100 kGs).

Second, find it for a very strong DC electric dipole fields of 10 MV/m.

Compare these energies with current largest (27 km in circumference) circular collider, LHC, circulating 6.5 TeV (1 TeV = 10^{12} eV).

Hint: assume that particles move with speed of the light. Check the final result for protons having rest mass of 938.27 MeV/c^2

Solution:

Magnetic field:

$$\frac{d\vec{p}}{dt} = -\left[\vec{\Omega} \times \vec{p}\right]; \Omega = \frac{|\vec{v}|}{R}; \vec{v} \perp \vec{H}; \ \left|\frac{d\vec{p}}{dt}\right| = \left|\frac{d\vec{p}}{ds}\right| |\vec{v}| = \frac{pv}{R} = \frac{e}{c} v \left|\vec{B}\right|$$

$$pc = eBR$$

Using ratio from the class notes: 1 Tm is 0.3 GeV (0.3*10^9 eV). It means that 1 T km is 0.3 TeV. Then we get

$$E \cong pc = eBR = 19{,}150 \ \ TeV \sim 1.9 \cdot 10^{16} \ \ eV$$

which is ~2,700 higher than LHC energy. One should note that there will be another problem, which we will study when we look into synchrotron radiation. Still, a long way to go! Relativistic factor for proton is >2 10^7.

Electric field:

$$\frac{d\vec{p}}{dt} = -\left[\vec{\Omega} \times \vec{p}\right]; \Omega = \frac{|\vec{v}|}{R}; \vec{v} \perp \vec{H}; \ \left|\frac{d\vec{p}}{dt}\right| = \left|\frac{d\vec{p}}{ds}\right| |\vec{v}| = e\left|\vec{E}\right|$$

$$pc \cong eER = 63.84 \ TeV \sim 6 \cdot 10^{13} eV$$

Just short of 10-fold higher than LHC, but 300-fold lower that possible with magnets.