Center for Accelerator Science and Education

Hadron Beam Cooling in Particle Accelerators

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We had considered parameterization of stable particles motion in a storage ring using eigen vectors of round trip matrix. A quick walk through our findings:

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s)\cdot X,\ \mathbf{H}(s+C) = \mathbf{H}(s); \tag{M3.1}$$

$$\mathbf{T}(s) = \mathbf{M}\left(s|s+C\right) \tag{M3.2}$$

$$\det[\mathbf{T} - \lambda_i \cdot \mathbf{I}] = 0 \tag{M3.3}$$

$$\mathbf{T}\cdot Y_k = \lambda_k \cdot Y_k; \qquad \lambda_k = e^{i\mu_k}; \qquad k = 1,2....n \tag{M3.4}$$

$$X = \sum_{i=1}^{2n} a_i Y_i \equiv \mathbf{U}\cdot A, \quad \mathbf{U} = [Y_1......Y_{2n}], \qquad A^T = [a_1......a_{2n}]. \tag{M3.5}$$

$$\mathbf{T}\cdot\mathbf{U} = \mathbf{U}\cdot\Lambda,\ \Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & ... & .... \\ 0 & .... & \lambda_{2n} \end{bmatrix} \tag{M3.6}$$

$$\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U} = \Lambda,\ or\ \ \mathbf{T} = \mathbf{U}\cdot\Lambda\cdot\mathbf{U}^{-1} \tag{M3.7}$$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_{j\neq k} = 0; \quad Y_k^T\cdot\mathbf{S}\cdot Y_j = 0;\ . \tag{M3.8}$$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_k = 2i, \tag{M3.9}$$

$$\mathbf{U}^T\cdot\mathbf{S}\cdot\mathbf{U} \equiv \tilde{\mathbf{U}}^T\cdot\mathbf{S}\cdot\tilde{\mathbf{U}} = -2i\mathbf{S},\ \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S}\cdot\mathbf{U}^T\cdot\mathbf{S}. \tag{M3.10}$$

$$\tilde{Y}_k(s_1) = \mathbf{M}\left(s\middle|s_1\right)\tilde{Y}_k(s) \iff \frac{d}{ds}\tilde{Y}_k = \mathbf{D}\left(s\right)\cdot\tilde{Y}_k \tag{M3.11}$$

$$\tilde{Y}_k(s) = Y_k(s)e^{i\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \qquad \psi_k\left(s+C\right) = \psi_k\left(s\right) + \mu_k \tag{M3.12}$$

$$\tilde{\mathbf{U}}(s_1) = \mathbf{M}\left(s\middle|s_1\right)\tilde{\mathbf{U}}(s) \iff \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}\left(s\right)\cdot\tilde{\mathbf{U}} \tag{M3.13}$$

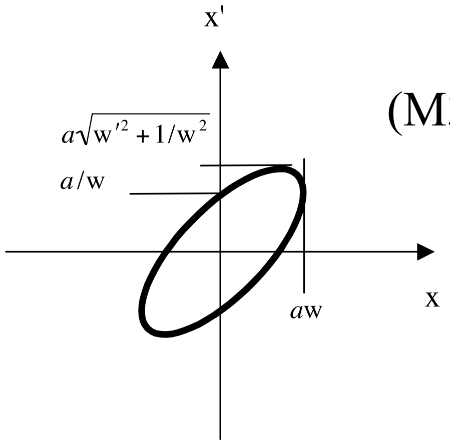
$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s)\cdot\Psi(s),\ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \ldots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \tag{M3.14}$$

$$X_o = \sum_{i=1}^{2n} a_i Y_i \Rightarrow X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a_k^{*}\tilde{Y}_k^{*}\right) \equiv \operatorname{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv$$
$$\frac{1}{2}\tilde{\mathbf{U}}\cdot A = \frac{1}{2}\mathbf{U}\cdot\Psi\cdot A = \frac{1}{2}\mathbf{U}\cdot\tilde{A} \tag{M3.15}$$

$$a_i = \frac{1}{2i}\tilde{Y}_i^{*T}SX;\ \tilde{a}_i \equiv a_i e^{i\psi}{}_i = \frac{1}{2i}Y_i^{*T}SX;$$
$$A = 2\tilde{\mathbf{U}}^{-1}\cdot X = -i\Psi^{-1}\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X;\ \tilde{A} = \Psi A = -i\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X. \tag{M3.16}$$

1D

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Y e^{i\psi} \quad \text{(M3.17)}$$



The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(a e^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot \left(\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - sin(\psi(s) + \varphi)/\mathrm{w(s)} \right) \end{aligned} \quad \text{(M3.8)}$$

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta. \quad \text{(M3.19)}$$

$$\alpha \equiv -\beta'/2 \equiv -\mathrm{w}\,\mathrm{w}', \; \gamma \equiv \frac{1+\alpha^2}{\beta}. \quad \text{(M3.20)}$$

$$\begin{aligned} x &= a \cdot \sqrt{\beta(\mathrm{s})} \cdot \cos(\psi(s) + \varphi) \\ x' &= -\frac{a}{\sqrt{\beta(\mathrm{s})}} \cdot \left(\alpha(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \right) \end{aligned} \quad \text{(M3.21)}$$

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu; \quad \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \mathbf{J}^2 = -\mathbf{I} \quad \text{(M3.22)}$$

2D

$$X=\begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix}=\mathrm{Re}\,\tilde{a}_1 Y_1+\mathrm{Re}\,\tilde{a}_2 Y_2=\mathrm{Re}\,a_1\tilde{Y}_1+\mathrm{Re}\,\tilde{a}_2\tilde{Y}_2 \tag{M3.23}$$

$$Y_k=R_k+iQ_k;\quad \tilde{Y}_k=\begin{bmatrix} w_{kx}e^{i\psi_{kx}} \\ (u_{kx}+iv_{kx})e^{i\psi_{kx}} \\ w_{ky}e^{i\psi_{ky}} \\ (u_{ky}+iv_{ky})e^{i\psi_{ky}} \end{bmatrix};\quad \psi_{kx}(s+C)=\psi_{kx}(s)+\mu_k;\psi_{ky}(s+C)=\psi_{ky}(s)+\mu_k;$$

$$w_{kx}v_{kx}+w_{ky}v_{ky}=1; \tag{M3.24}$$

Conditions: there are

$$Y_k^{*T}SY_k=2i;\quad Y_1^{*T}SY_2=0;\ Y_1^TSY_2=0;\quad \theta_k=\psi_{kx}-\psi_{ky}$$

$$a)\ \ w_{1x}v_{1x}=w_{2y}v_{2y}=1-q\quad \Rightarrow v_{1x}=\frac{1-q}{w_{1x}};\ \ v_{2y}=\frac{1-q}{w_{2y}}$$

$$b)\ \ w_{1y}v_{1y}=w_{2x}v_{2x}=q\quad \Rightarrow v_{2x}=\frac{q}{w_{2x}};\ \ w_{1y}=\frac{q}{w_{1y}}$$

$$c)\ \ c=w_{1x}w_{1y}\sin\theta_1=-w_{2x}w_{2y}\sin\theta_2$$

$$d)\ \ d=w_{1x}\left(u_{1y}\sin\theta_1-v_{1y}\cos\theta_1\right)=-w_{2x}\left(u_{2y}\sin\theta_2-v_{2y}\cos\theta_2\right)$$

$$e)\ \ e=w_{1y}\left(u_{1x}\sin\theta_1+v_{1x}\cos\theta_1\right)=-w_{2y}\left(u_{2x}\sin\theta_2+v_{2x}\cos\theta_2\right) \tag{M3.25}$$

$$Y_1=\begin{bmatrix} w_{1x}e^{i\varphi_{1x}} \\ \left(u_{1x}+i\dfrac{q}{w_{1x}}\right)e^{i\varphi_{1x}} \\ w_{1y}e^{i\varphi_{1y}} \\ \left(u_{1y}+i\dfrac{1-q}{w_{1y}}\right)e^{i\varphi_{1y}} \end{bmatrix};\quad Y_2=\begin{bmatrix} w_{2x}e^{i\varphi_{2x}} \\ \left(u_{2x}+i\dfrac{1-q}{w_{2x}}\right)e^{i\varphi_{2x}} \\ w_{2y}e^{i\varphi_{2y}} \\ \left(u_{2y}+i\dfrac{q}{w_{2y}}\right)e^{i\varphi_{2y}} \end{bmatrix} \tag{M3.26}$$

3D

$$|\lambda_k| = 1;\ \lambda_k = e^{i\mu_k};\ \mu_k = 2\pi Q_k;\ k = 1,2,3 \qquad \text{(M3.27)}$$

$$X = \begin{bmatrix} x \\ P_x \\ y \\ P_y \\ \tau \\ P_\tau \end{bmatrix} = \operatorname{Re}\tilde{a}_1 Y_1 + \operatorname{Re}\tilde{a}_2 Y_2 + \operatorname{Re}\tilde{a}_3 Y_3 = \operatorname{Re} a_1 \tilde{Y}_1 + \operatorname{Re} a_2 \tilde{Y}_2 + \operatorname{Re} a_3 \tilde{Y}_3 \qquad \text{(M3.28)}$$

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i \dfrac{q_{kx}}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i \dfrac{q_{ky}}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i \dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}} \right) e^{i\chi_{k\tau}} \end{bmatrix}; Y_k(s+C) = Y_k(s); T(s) Y_k(s) = e^{i\mu_k} Y_k(s); k = 1,2,3 \qquad \text{(M3.29)}$$

$$Y_k^{\ T} S Y_j = 0;\quad Y^{*\ T}_{\ j} S Y_k = 2i\delta_{kj}; \qquad \text{(M3.30)}$$

15 relations on the component of the eigen vectors, with the simples being:

$$q_{kx} + q_{ky} + q_{k\tau} = 1;\ k = 1,2,3 \qquad \text{(M3.31}$$

3D

In most of the cases, when synchrotron oscillations are slow, normalized 3D eigen vectors are simplified to

$$Y_k = \begin{bmatrix} Y_{\beta k} \\ \eta^T \mathbf{S}_{4x4} Y_{\beta k} \\ 0 \end{bmatrix}; k = 1,2; \;\; Y_3 = \frac{1}{\sqrt{|\eta_\tau / \mu_3|_3}} \begin{bmatrix} \eta \\ -i|\eta_\tau / \mu_3| \\ 1 \end{bmatrix}; \mu_3 \equiv \mu_s = 2\pi Q_s \qquad \text{(M3.32)}$$

We will use this very frequently. Please note that longitudinal and horizontal degrees of freedom are always coupled in any of storage ring – bending of trajectory does it. But vertical oscillation frequently are decoupled, the we have even easier eigen vectors:

$$Y_x = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}_x' + \frac{i}{\mathrm{w}_x} \\ 0 \\ 0 \\ \eta_x \left(\mathrm{w}_x' + \frac{i}{\mathrm{w}_x} \right) - \eta_x' \mathrm{w}_x \\ 0 \end{bmatrix}; \; Y_y = \begin{bmatrix} 0 \\ 0 \\ \mathrm{w}_y \\ \mathrm{w}_y' + \frac{i}{\mathrm{w}_y} \\ 0 \\ 0 \end{bmatrix}; Y_s = \frac{1}{\sqrt{|\eta_\tau / \mu_3|}} \begin{bmatrix} \eta_x \\ \eta_x' \\ 0 \\ 0 \\ -i|\eta_\tau / \mu_3| \\ 1 \end{bmatrix} \qquad \text{(M3.33)}$$

Complete parameterization developed in previous lecture can be used to solve most (if not all) of standard problems in accelerator. Incomplete list is given below:

1. Dispersion
2. Orbit distortions
3. AC dipole (periodic excitation)
4. Tune change with quadrupole (magnets) changes
5. Chromaticity
6. Beta-beat
7. Weak coupling
8. Synchro-betatron coupling
9. Beyond Hamiltonian system - weak (slow) damping
10. ….and diffusion
11. ……

We do not plan to go through all these examples while focusing on general methodology and use selected examples to demonstrate power of the symplectic linear parameterization. We will use complex form of parameterization since it gives more transparent frequency content of the oscillations, but one can do similar exercise using real notations – after all results in real life are in real notations.

Sample I. Let's start from simplest problems such as dispersion and closed orbit. We found a general form of parameterization of linear motion in Hamiltonian system, which is solution of homogeneous linear equations, where ***B*** is constant vector:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ X = \tilde{\mathbf{U}}(s)\cdot B \qquad \text{(M3-34)}$$

A standards problems is a solution of inhomogeneous equations:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + F(s); \qquad \text{(M3-35)}$$

It can be done analytically by varying the constant ***B:***

$$X = \tilde{\mathbf{U}}(s)B(s) \Rightarrow \tilde{\mathbf{U}}\cdot B' = F(s) \Rightarrow B' = \tilde{\mathbf{U}}^{-1}(s)F(s) \Rightarrow B(s) = B_o + \int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

A general solution is a specific solution of inhomogeneous equation plus arbitrary solution of the homogeneous – result you expect in linear ordinary differential equations (in this case with s-depended coefficients):

$$X(s) = \tilde{\mathbf{U}}(s)A_o + \tilde{\mathbf{U}}(s)\int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;\ \ \tilde{\mathbf{U}}^{-1} = \frac{i}{2}\mathbf{S}\cdot\tilde{\mathbf{U}}^T\cdot\mathbf{S} \qquad \text{(M3-36)}$$

For a periodic force (orbit distortions, dispersion function) $F(s+C)=F(s)$ one can fine periodic solution $X(s+C)=X(s)$:

$$X(s)=\tilde{\mathbf{U}}(s)B(s);\ B(s)=A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;$$

$$X(s)=X(s+C)\Rightarrow\tilde{\mathbf{U}}(s)A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\tilde{\mathbf{U}}(s+C)A_o+\tilde{\mathbf{U}}(s+C)\int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

$$\tilde{\mathbf{U}}(s+C)=\mathbf{T}(s)\tilde{\mathbf{U}}(s)=\tilde{\mathbf{U}}(s)\Lambda;\ \int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi+\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;$$

$$\tilde{\mathbf{U}}(s)B(s)=\tilde{\mathbf{U}}(s)\Lambda B(s)+\tilde{\mathbf{U}}(s)\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi \qquad \text{(M3-37)}$$

$$\tilde{\mathbf{U}}^{-1}(s)\times\left\{\tilde{\mathbf{U}}(s)B(s)-\tilde{\mathbf{U}}(s)\Lambda B(s)=\tilde{\mathbf{U}}(s)\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\right\}\Rightarrow$$

$$(\mathbf{I}-\Lambda)B(s)=\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\equiv\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\Rightarrow B(s)=(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

$$X(s)=\tilde{\mathbf{U}}(s)B(s)=\tilde{\mathbf{U}}(s)(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

It is easy to see that $X(s+C)=X(s)$ exists if none of the eigen values is not equal 1 – otherwise matrix $(\mathbf{I}-\Lambda)$ would have zero determinant and can not be inverted!

Specific examples: Orbit distortions caused by the field errors, transverse dispersion.

When the conditions for the equilibrium particle and the reference trajectory are slightly violated:

$$X^T = \{x, P_1, y, P_3, \tau, \delta\}; F^T = \left\{0, \frac{e}{c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right), 0, \frac{e}{c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right), 0, 0\right\}$$

$$K(s) \equiv \frac{1}{\rho(s)} - \frac{e}{p_o c}\left(B_y\big|_{ref} + \frac{E_o}{p_o c}E_x\big|_{ref}\right) - f_x; \quad f_x = \frac{e}{p_o c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right); \quad . \quad \text{(M3-38)}$$

$$\frac{e}{p_o c}B_x\Big\|\left(_{ref} - \frac{E_o}{p_o c}E_y\big|_{ref}\right) = -f_y = \frac{e}{p_o c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right)$$

Plugging (M3-38) into(M3-37) will give one the periodic closed orbit for such a case. For transverse dispersion the finding reduces to

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + g_x x\delta + g_y y\delta$$

with

$$F = S\frac{\partial H}{\partial X} = \left\{0, -g_x, 0, -g_y, 0, -\frac{m^2c^2}{p_o^3}\right\}^T. \quad \text{(M3-39)}$$

1D ACCELERATOR

$$\tilde{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ \mathrm{w}' + \frac{i}{\mathrm{w}} & \mathrm{w}' - \frac{i}{\mathrm{w}} \end{bmatrix} \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix}; \tilde{U}^{-1} = \frac{i}{2} \begin{bmatrix} e^{-i\psi} & 0 \\ 0 & e^{i\psi} \end{bmatrix} \begin{bmatrix} \mathrm{w}' - \frac{i}{\mathrm{w}} & -\mathrm{w} \\ -\mathrm{w}' \text{-} \frac{i}{\mathrm{w}} & \mathrm{w} \end{bmatrix};$$

$$X(s) = \frac{i}{2} \int_{s-C}^{s} \begin{bmatrix} \mathrm{w(s)} & \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s}) + \frac{i}{\mathrm{w(s)}} & \mathrm{w}'(\mathrm{s}) - \frac{i}{\mathrm{w(s)}} \end{bmatrix} \cdot \begin{bmatrix} \left(1 - e^{i\mu}\right)^{-1} e^{i\psi(\mathrm{s}) - i\psi(\xi)} & 0 \\ 0 & \left(1 - e^{-i\mu}\right)^{-1} e^{-i\left(\psi(\mathrm{s}) - i\psi(\xi)\right)} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{w}'(\xi) - \frac{i}{\mathrm{w}(\xi)} & \text{-}\mathrm{w}(\xi) \\ \mathrm{w}'(\xi) - \frac{i}{\mathrm{w}(\xi)} & \mathrm{w}(\xi) \end{bmatrix} F(\xi)\, d\xi$$

$X^T = \{x, x'\}; F^T = \dfrac{e}{p_o c} \delta B_y \{0,1\}$ - orbit; $F^T = K(s)\{0,1\}$ for dispersion, i.e. $F^T = f(s)\{0,1\}$

$$X(s) = \int_{s-C}^{s} \left[\frac{\mathrm{Re}\left(\mathrm{w(s)}\mathrm{w}(\xi) e^{i(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)} \left(\frac{e^{-i\mu/2} - e^{i\mu/2}}{-i} \right)^{-1} \right)}{\ldots\ldots} \right] f(\xi)\, d\xi$$

i.e. $$x(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2} \oint_C f(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)\, d\xi \qquad \text{(M3-40)}$$

First example: orbit distortion

$$f_x(s) = -\frac{e\delta B_y(s)}{p_o c}; \quad f_y(s) = \frac{e\delta B_x(s)}{p_o c}$$

$$\delta x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_y}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi$$

$$\delta y(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_x}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \qquad \text{(M3-41)}$$

but this is not the end of the story for horizontal motion! What about change of the orbiting time?

Second example: Dispersion

$$f_x(s) = K_o(s)\pi_l = K_o(s)\pi_\tau/\beta_o;$$

$$x(s) = \eta_x(s)\cdot\pi_l = \eta_x(s)\cdot\pi_\tau/\beta_o;$$

$$\eta_x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C K_o(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \quad \text{(M3-42)}$$

Sample II: Beta-beat – 1D case

Let's consider a case when we are designing a circular accelerator comprised of various parts and want parameterization parameters (in this case envelope function w) to have a specific s-dependence. For example, we want it to fit a long periodic arc of an accelerator or generate a very specific w(s) – consistent with equations of motion - to satisfy a specific function needed from accelerator: minimize emittance, achromatic lattice….

It is simple fact that any solution can be expanded upon the eigen vectors of periodic system (FODO cell repeated again and again is an example). Let 's consider that at azimuth $s=s_o$ initial value of "injected" eigen vector **V** being different from the periodic solution **Y**. We expand it as

$$V(s_o) = aY_k(s_o) + bY_k^*(s_o) = \begin{bmatrix} v_o \\ v_o' + \dfrac{i}{v_o} \end{bmatrix}; Y_k = \begin{bmatrix} w_o \\ w_o' + \dfrac{i}{w_o} \end{bmatrix}$$

$$a = \frac{1}{2i}\ Y_k^{*T}(s_o) S V(s_o)\ ;\ b = \frac{1}{-2i}\ Y_k^{T}(s_o) S V(s_o)$$

$$a = \frac{1}{2i}\left\{ v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} + \frac{w_o}{v_o}\right)\right\}; b = -\frac{1}{2i}\left\{ v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} - \frac{w_o}{v_o}\right)\right\};$$

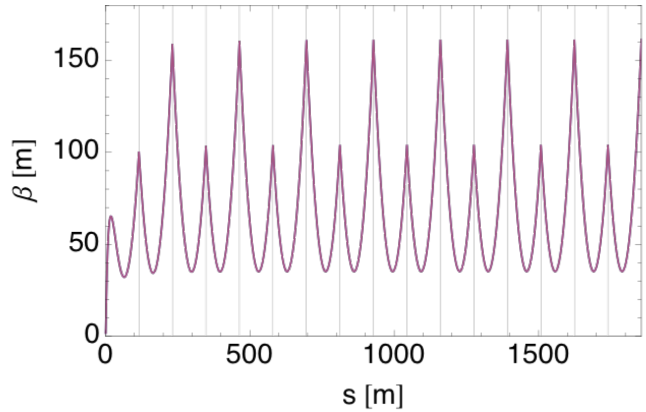
$$\frac{d}{ds}\tilde{Y}(s) = \mathbf{D}(s)\cdot \tilde{Y}(s); \quad \tilde{Y}(s) = Y(s)e^{i\psi(s)}; Y(s+C) = Y(s)$$

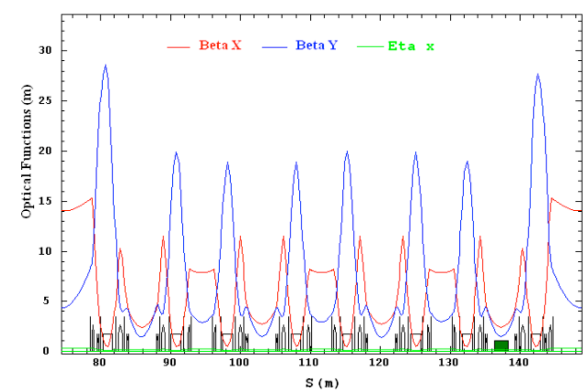
It is self-evident that

$$\tilde{V}' = D\tilde{V}; \quad \tilde{V}(s) = a\tilde{Y}_k(s) + b\tilde{Y}_k^*(s) = \begin{bmatrix} v \\ v' + \dfrac{i}{v} \end{bmatrix} e^{i\varphi} = Y_k = a\begin{bmatrix} w \\ w' + \dfrac{i}{w} \end{bmatrix} e^{i\psi} + b\begin{bmatrix} w \\ w' - \dfrac{i}{w} \end{bmatrix} e^{-i\psi}$$

$$|v|^2 = \frac{|w|^2}{4}\left|ae^{i\psi} + be^{-i\psi}\right|^2 = \frac{|w|^2}{4}\left(|a|^2 + |b|^2 - 2\,\mathrm{Re}\left(ab^* e^{2i\psi}\right)\right)$$

i.e. beta-function will beat with double of the betatron phase.





Sample III: Perturbation theory (ala quantum mechanics)

Small variation of the linear Hamiltonian terms (including coupling)

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot X = \left(\mathbf{SH}(s) + \varepsilon \mathbf{SH}_1(s)\right)\cdot X$$

$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s)\tilde{Y}_k(s); k = 1,..,n. \qquad \text{(M3-42)}$$

Assuming that changes are very small we can express the changes in the eigen vectors using basis of (15):

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\ *} + \varepsilon \sum_{j\neq k}\left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}^*_j\right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_{1k}^{\ *} = \tilde{Y}_k^{\ *} e^{-i\delta\phi_k} + \varepsilon c^*_{\ k}\tilde{Y}_k + \varepsilon \sum_{j\neq k}\left(a^*_{\ kj}\tilde{Y}^*_j + b^*_{\ kj}\tilde{Y}_j\right) + O(\varepsilon^2); \qquad \text{(M3-43)}$$

$$\frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

We need substitute the expansion of the new eigen vectors into the differential equation and to keep first order term of ε

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj} \tilde{Y}_j + b_{kj} \tilde{Y}_j^{\,*} \right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_k' e^{i\delta\phi_k} + \delta\phi_k' \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k' \tilde{Y}_k^{\,*} + \varepsilon c_k \tilde{Y}_k'^{*} + \varepsilon \sum_{j\neq k} \left(a_{kj}' \tilde{Y}_j + b_{kj}' \tilde{Y}_j^{\,*} \right) + \varepsilon \sum_{j\neq k} \left(a_{kj} \tilde{Y}_j' + b_{kj} \tilde{Y}_j'^{*} \right) =$$

$$\mathbf{D}\left(\tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj} \tilde{Y}_j + b_{kj} \tilde{Y}_j^{\,*} \right) \right) + \varepsilon \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k} + O(\varepsilon^2)$$

$$\tilde{Y}_j' = \mathbf{D}\tilde{Y}_k; \tilde{Y}_j'^{*} = \mathbf{D}\tilde{Y}_j^{\,*}.$$

and all terms in red cancel each other leaving us with

$$\delta\phi_k' \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k' \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj}' \tilde{Y}_j + b_{kj}' \tilde{Y}_j^{\,*} \right) = \varepsilon \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k}$$

which we can split into individual equations for each component using symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^{T*} S \tilde{Y}_j = -\tilde{Y}_k^T S \tilde{Y}_j^{\,*} = 2i\delta_{ik};\ \tilde{Y}_k^T S \tilde{Y}_j = \tilde{Y}_k^{T*} S \tilde{Y}_j^{\,*} = 0$$

Multiplying by $\tilde{Y}_m^{\,*}S$ or $\tilde{Y}_m S$ from the left yields:

$$-2\delta\phi_k' = \varepsilon \tilde{Y}^*_k \mathbf{SD}_1(s)\tilde{Y}_k \rightarrow \delta\phi' = \frac{\varepsilon}{2} Y_k^{\,*T}\mathbf{H}_1(s)Y_k; \quad \mathbf{SD}_1 = -\mathbf{H}_1;$$

$$-2ic' = \tilde{Y}^T_k \mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow c' = \frac{1}{2i} Y_k^{\,T}\mathbf{H}_1(s)Y_k e^{i(2\psi_k+\delta\phi_k)} \cong \frac{1}{2i} Y^T{}_k \mathbf{H}_1 Y_k e^{2i\psi_k}$$

$$2ia_{kj}' = \tilde{Y}^{T*}_j \mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow a_{kj}' = \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)} \cong \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k-\psi_j\right)}; j \neq k$$

$$-2ib_{kj}' = \tilde{Y}^T_j \,\mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow b_{kj}' = \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)} \cong \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k+\psi_j\right)}; j \neq k.$$

with solutions in form of integrals:

$$\delta\phi(s) = \phi_o + \frac{\varepsilon}{2}\int_0^s Y_k^{\,*T}\mathbf{H}_1 Y_k \, d\xi; \; c(s) = c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)};$$

$$a_{kj} = a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)}; b_{kj} = b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)};$$

$$Y_{1k}(s) = \tilde{Y}_{1k} e^{-i\left(\psi_k+\delta\phi_k\right)} = Y_k + \varepsilon c_k Y_k^{\,*} e^{-i\left(2\psi_k+\delta\phi_k\right)}\left(c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i\left(2\psi_k+\delta\phi_k\right)} \right) +$$

$$\varepsilon \sum_{j\neq k} \left(\begin{array}{l} Y_j e^{-i\left(\psi_k-\psi_j+\delta\phi_k\right)} \left(a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)} \right) + \\ Y^*{}_j e^{-i\left(\psi_k+\psi_j+\delta\phi_k\right)} \left(b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)} \right) \end{array} \right) + O(\varepsilon^2)$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C) = \tilde{Y}_{1k}(s)e^{i\mu_{1k}}; \mu_{1k} = \mu_k + \frac{\varepsilon}{2}\int_0^C Y_k^{*T}\mathbf{H}_1 Y_k\, d\xi;$$

into periodic functions, we need to choose the initial conditions

$$d(s) = e^{-i\theta(s)}\left(d_o - \frac{1}{2i}\int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right); f(\xi+C) = f(\xi).$$

to make a coefficient looking

$$e^{-i\theta(s+C)}\left(d_o + \int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}\right) = e^{-i\theta(s)}\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)} = \left(e^{i\Delta\theta(C)} - 1\right)\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right) = \frac{1}{e^{i\Delta\theta(C)} - 1}\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}.$$

Final expression is:

$$\tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_{1k}(s) = Y_k + \varepsilon \frac{Y_k^{\,*} e^{-i(2\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(2\mu_k+\delta\mu_k)}\right)} \int_s^{s+C} d\xi Y_k^{\,T} \mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)} +$$

$$\varepsilon \sum_{j\neq k} \left(\begin{array}{l} -\dfrac{Y_j e^{i(\psi_j-\psi_k-\delta\phi_k)}}{2i\left(1-e^{i(\mu_k-\mu_j+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{\,*T} \mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} + \\ \dfrac{Y_j^{\,*} e^{-i(\psi_j+\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(\mu_j+\mu_k+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{\,T} \mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \end{array} \right) + O(\varepsilon^2) \qquad \text{(M3-44)}$$

We should note, that while it was easy to keep $\delta\mu_k, \delta\phi_k$ in the final expression, it belongs to the next order correction and generally speaking should be dropped.

One should be aware of the resonant case $e^{i(\mu_k-\mu_i)} = 1$, including parametric resonance $e^{2i\mu_k} = 1$, when one should solve self-consistently the set of (14-14). It is well known case well described in weak coupling resonance case or in the case of parametric resonance.

Sample IV: small variation of the gradient. It can come from errors in quadrupoles or from a deviation of the energy from the reference value or orbit distortions $\delta x, y$. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$H_1 = \delta K_1 \frac{z^2}{2}; z = \{x, y\};\ \pi_l = p/p_o - 1$$
$$\delta K_{1x,y} = \mp\delta\left(\frac{e}{pc}\frac{\partial B_y}{\partial x}\right) = \mp\left(\frac{e}{pc}\delta\frac{\partial B_y}{\partial x}\right) - K_1\pi_l \mp \frac{e}{pc}\frac{\partial^2 B_y}{\partial x^2}\left(D_x\pi_l + \delta x, y\right) + o\left(\pi_l^2\right) \quad \text{(M3-45)}$$

Plugging our parameterization into the residual Hamiltonian we get:

$$z = w(s)\sqrt{2I}\cos\left(\psi(s) + \varphi\right)$$
$$H_1 = \delta K_1(s)\cdot w^2(s)\cdot I\cdot\cos^2\left(\psi(s) + \varphi\right) \quad \text{(M3-46)}$$

The easiest way is to average the Hamiltonian (on the phase of fast betatron oscillation – our change is small! And does not effect them strongly) to have a well-know fact that the beta-function is also a Green function (modulo 4π) of the tune response on the variation of the focusing strength.

$$\left\langle H_1\right\rangle = \frac{\left\langle \delta K_1(s)\cdot w^2(s)\right\rangle}{2}\cdot I \equiv \frac{\left\langle \delta K_1(s)\cdot\beta(s)\right\rangle}{2}\cdot I; \left\langle\varphi'\right\rangle = \frac{\partial\left\langle H_1\right\rangle}{\partial I} = \frac{\left\langle\delta K_1(s)\cdot\beta(s)\right\rangle}{2};$$
$$\Delta\varphi = \frac{1}{2}\oint\delta K_1(s)\cdot\beta(s)\,ds;\ \Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\oint\delta K_1(s)\cdot\beta(s)\,ds; \quad \text{(M3-47)}$$

Direct way will be to put it into the equations (43) and to find just the same, that <I'>=0 and the above result. Finally, putting a weak thin lens as a perturbation gives a classical relation:

$$\delta K_1(s) = \frac{1}{f}\delta\left(s - s_o\right) \Rightarrow \Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\frac{\beta_o(s)}{f} \quad \text{(M3-48)}$$

Going beyond Hamiltonian system – taking dissipation into account

Let's consider that an additional linear term is no longer a Hamiltonian

$$\frac{dX}{ds} = (\mathbf{D}(s) + \varepsilon \mathbf{d}(s)) \cdot X;\ \mathbf{D} = \mathbf{SH}; Trace[\mathbf{D}] = 0; Trace[\mathbf{d}] \neq 0 \qquad \text{(M3-49)}$$

e.g. the overall motion is no longer symplectic

$$X(s) = \mathbf{R}(s) X_o \to \frac{d\mathbf{R}}{ds} = (\mathbf{D} + \varepsilon \mathbf{d})\mathbf{R} \to \frac{d \det[\mathbf{R}(s)]}{ds} = Trace[\mathbf{d}(s)]$$
$$\det[\mathbf{R}(s)] = \varepsilon \int_o^s Trace[\mathbf{d}(\xi)] d\xi; \qquad \text{(M3-50)}$$

Such contributions can come form natural dissipative (or anti-dissipative) processes such as radiation reaction (synchrotron radiation damping), ionization cooling or from special accelerator systems, such as electron or stochastic cooling. Here we are not specifying what is the source of the non-Hamiltonian force and only assume that it is linear.

Similarly to regular parameterization, we can assume that motion can be expanded as a set of eigen modes

$$X(s) = \tilde{V}(s)\chi(s) \cdot B = \sum_{k=1}^{2n} \tilde{V}_k(s) e^{\chi_k(s)} b_k; \det \tilde{V}(s) = 1;\ \det\left(\tilde{V}(0)\chi(0)\right) = 1;$$

$$\tilde{V}(s)\chi(s) = \mathbf{R}(s)\tilde{V}(0)\chi(0);\ \ \det \chi(s) = \prod_{k=1}^{2n} e^{\chi_k(s)} = \exp\left(\sum_{k=1}^{2n} \chi_k(s)\right);$$

$$\frac{d}{ds}\det\left(\tilde{V}(s)\chi(s)\right) = \frac{d}{ds}\left(\det \tilde{V}(s) \det \chi(s)\right) = \frac{d}{ds}\det\left(\mathbf{R}(s)\tilde{V}(0)\chi(0)\right) = \frac{d}{ds}\left(\det \mathbf{R}(s)\right) = Tr\mathbf{D};$$

$$\frac{d}{ds}\det \chi(s) = \sum_{k=1}^{2n} \chi_k'(s) = Tr\left((\mathbf{D}(s) + \varepsilon \mathbf{d}(s))\right) = \varepsilon Tr[\mathbf{d}(s)].$$

than (M3-50) became

$$\frac{d}{ds}\sum_{k=1}^{2n} \chi_k(s) = \varepsilon Trace[\mathbf{d}(s)]$$
$$\sum_{k=1}^{2n} \chi_k(s) = \varepsilon \int_o^s Trace[\mathbf{d}(\xi)] d\xi; \qquad \text{(M3-51)}$$

which is commonly know as a the sum of decrements theorem: sum of the decrements (or increments!) of all eigen modes is equal to the integral of the trace of the dissipative matrix. This is to a degree the most trivial and well known relation for ordinary differential equation.

What is more interesting is to find decrements (increments) of the amplitudes of individual modes. Rewriting already established expansion (M3-34)

$$X(s) = \frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s) = \mathrm{Re}\sum_{k=1}^{n} Y_k(s) e^{i(\psi_k(s)+\varphi_k)} a_k(s); \frac{d}{ds}\tilde{\mathbf{U}}(s) = \mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s)$$
$$\mathrm{Re}\sum_{k=1}^{n} Y_k(s) e^{i(\psi_k+\varphi_k)} \frac{da_k}{ds} = \varepsilon\mathbf{d}\cdot\mathrm{Re}\sum_{m=1}^{n} Y_m e^{i(\psi_m+\varphi_m)} a_m; \qquad \text{(M3-52)}$$

Using symplectic orthogonality of the eigen vectors we get equations of the evolution for individual amplitudes:

$$\frac{da_k}{ds} = \frac{\varepsilon}{2i}\cdot e^{-i(\psi_k+\varphi_k)}\left(\sum_{m=1}^{n} Y_k^{*T}(\mathbf{Sd})Y_m e^{i(\psi_m+\varphi_m)} a_m + Y_k^{*T}(\mathbf{Sd})Y_m^{*} e^{-i(\psi_m+\varphi_m)} a_m^{*}\right); \qquad \text{(M3-53)}$$

Hence, the perturbation can slightly change the eigen modes (as we discussed above in *ala quantum* perturbation) and phase of oscillations – the right side is not necessarily a real number.

But the main effect of-interest is in change of the amplitude of the oscillations, which comes from a simple averaging of (M3-53). Since

$$\Delta\psi_k = \psi_k(s+C) - \psi_k(s) = \mu_k$$
$$\Delta(\psi_k \pm \psi_m) = \mu_k \pm \mu_m \;;$$

the only non-oscillating term in (14-29) is $Y_k^{*T}(\mathbf{Sd})Y_k$ and averaging yields

$$\left\langle \frac{da_k}{ds} \right\rangle = \frac{\varepsilon}{2i} Y^{*T}{}_k(s)(\mathbf{Sd}(s))Y_k(s)\langle a_k\rangle;$$
$$\langle a_k\rangle(s) = \langle a_k\rangle_o \exp\left[-\frac{\varepsilon}{2i}\int_0^s Y^{*T}{}_k(\xi)\cdot\mathbf{S}\cdot\mathbf{d}(\xi)\cdot Y_k(\xi)d\xi\right]; \qquad \text{(M3-54)}$$

At no surprise, we arrived to an equation nearly identical to (M3-48) with only exception that we did not assumed that motion is Hamiltonian. Indeed, if

$$if \quad \varepsilon \mathbf{d}(s) = \mathbf{S}\delta\mathbf{H}_1$$

$$\langle a_k \rangle(s) = \langle a_k \rangle_o \exp\left[\frac{1}{2i}\int_0^s Y^{*T}{}_k \ (\xi)\delta\mathbf{H}_1 Y_m(\xi)d\xi\right];$$

$$\Delta\varphi = \frac{1}{2}\int_0^s Y^{*T}{}_k \ (\xi)\delta\mathbf{H}_1 Y_k(\xi)d\xi$$

It should not be surprising – we are solving more or less the same problem using more or less the same method of varying constants.

The most useful form of (M3-54) is calculation of dumping (or anti-damping) coefficients

$$|a_k| \cong |a_{k0}| e^{-\frac{\xi_k s}{C}}$$
$$\xi_k = -\frac{\varepsilon}{2}\int_0^C \mathrm{Im}\left(Y^{*T}{}_k \ (s)\left(\mathbf{Sd}(s)\right)Y_k(s)\right)ds; \qquad \text{(M3-55)}$$

Naturally, the sum of the decrements is determined by the trace of the matrix. What is non-trivial is that we can re-distribute some (if not all) decrements between various modes of oscillations using coupling between them.

As indicated above, we combine the real and imaginary parts:

$$a_k e^{i\varphi} \cong a_{k0} \cdot e^{\frac{s}{C}(i\Delta\mu - \xi_k)}$$
$$i\Delta\mu - \xi_k = \frac{\varepsilon}{2}\int_0^C \left(Y^{*T}{}_k \ (s)\left(\mathbf{Sd}(s)\right)Y_k(s)\right)ds; \qquad \text{(M3-56)}$$

We will use this expression now and again.

Again 1D case

It gives us know fact that damping of the amplitude of the oscillation is ½ of the dissipative term in $x'' - \xi_o x' + K_1(s)x = 0$:

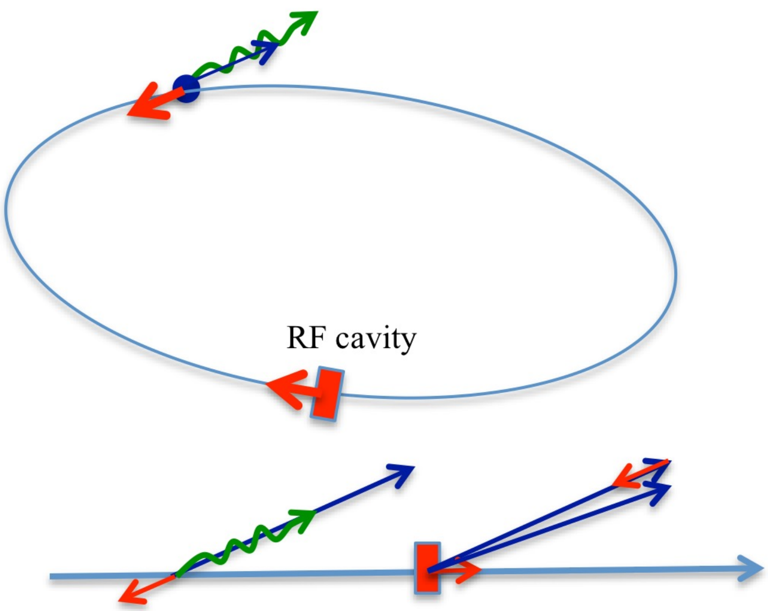
$$\varepsilon\mathbf{d} = \begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}$$

$$\xi_x = -\frac{1}{2}\mathrm{Im}\begin{bmatrix} \mathrm{w} & \mathrm{w}' - \frac{i}{\mathrm{w}} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}\begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}\mathrm{Im}\begin{bmatrix} -\mathrm{w}' + \frac{i}{\mathrm{w}} & w \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}.$$

By the way, the real part of the expression gives

$$\varphi'_x = \frac{1}{2}\mathrm{w}'_x \mathrm{w}_x \frac{\xi_o}{2}$$

while being interesting academically, it does not play too much role in the accelerators.
We will return to damping when considering synchrotron radiation effects in accelerators.



How radiation cools beam in a storage ring: vertical motion: Particle radiate in the direction of the motion and RF cavity restores only longitudinal part of the momentum.

Sample VI: Going beyond Hamiltonian system – random kicks

Particle in accelerators frequently experience a sudden events, which change their momenta essentially in instance. Naturally, there are no sudden changes of position – it would require not infinite force, but also a finite time to change position.

Examples of such processes include: radiation of a photon (so called quantum fluctuation of radiation), scattering on residual gas or on other particles inside the beam. The later is called intra-beam scattering and is one of limiting factors in attaining small beam emittances.

Again, let's just add an additional term in our equation of motion (M3-34):

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + DP(s); DP(s) = \sum_a \delta P_a \cdot \delta(s - s_a) \qquad \text{(M3-57)}$$

which has similar appearance as (M3-35) but has very different nature – it represents a random process, not a regular continuous force. Nevertheless, we can directly find the change of the oscillation amplitude and phase at each random kick:

$$\sum_{k=1}^{n} e^{i\psi_k(s_a)} Y_k(s_a)\delta\left(a_k e^{i\varphi}\right)_{s_a} = \delta P_a \to \delta\left(a_k e^{i\varphi}\right)_{s_a} = e^{-i\psi_k(s_a)}\frac{1}{2i} Y_k^{T*}(s_a)\mathbf{S}\delta P_a;$$

$$a_k(s) e^{i\varphi} = a_{ok} + \sum_{s_a<s} e^{-i\psi_k(s_a)}\frac{1}{2i} Y_k^{T*}(s_a)\mathbf{S}\delta P_a; \qquad \text{(M3-58)}$$

Naturally, the exact result depends of a realization of the random process. But statistically we can write the average change if the actions:

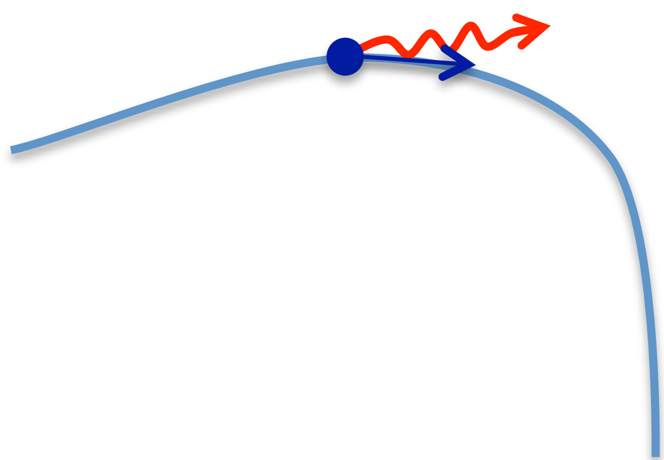
$$J_k = \frac{a_k^2}{2} \to \delta J_k = \frac{(a_k + \delta a_k)^2 - a_k^2}{2} = 2a_k \delta a_k + (\delta a_k)^2 \quad \text{(M3-59)}$$

Now we need to look on the average picture again:

$$\tilde{a}_k = a_k e^{i\varphi}; \delta \tilde{a}_k = e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$

$$\delta |\tilde{a}_k|^2 \to (\tilde{a}_k + \delta \tilde{a}_k)(\tilde{a}_k^* + \delta \tilde{a}_k^*) - \tilde{a}_k \tilde{a}_k^* = |\delta \tilde{a}_k|^2 + 2 \operatorname{Re} \tilde{a}_k^* \delta \tilde{a}_k \quad \text{(M3-60)}$$

$$\tilde{a}_k^* \delta \tilde{a}_k = a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$



Since the kicks occur at random locations and the phase of the oscillation is randomized.

Hence,

$$\left\langle \tilde{a}_k^* \delta \tilde{a}_k \right\rangle = \left\langle a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right\rangle = 0 \qquad \text{(M3-61)}$$

and

$$\left\langle \delta J_k \right\rangle = \left\langle \frac{\delta a_k^2}{2} \right\rangle = \frac{1}{2} \left\langle \left| \delta \tilde{a}_k \right|^2 \right\rangle = \frac{1}{8} \left| Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right|^2 \qquad \text{(M3-62)}$$

Now we need to introduce probability of the random kick δP at azimuth s to write an statistical average growth of the oscillation amplitude:

$$\left\langle \frac{dJ_k}{ds} \right\rangle = \frac{1}{8} \left\langle \left| Y_k^{T*}(s) \mathbf{S} \delta P \right|^2 \cdot \phi(s, \delta P) \right\rangle = D_k(s) \qquad \text{(M3-63)}$$

This growth is called diffusion (or random walk). It has interesting characteristic that amplitude of oscillations growth proportionally to the square root of time – e.g. the action grows linearly.

Again, we will discuss values for specific processes later. What is interesting now is to combine damping and diffusion. To do this we need to tone that without diffusion

$$\frac{dJ_k}{ds} = \frac{1}{2} \frac{da_k^2}{ds} = a_k \frac{da_k}{ds} = -2\xi_k J_k \qquad \text{(M3-64)}$$

and adding diffusion we get to

$$\frac{d\left\langle J_k \right\rangle}{ds} = -2\xi_k(s) \left\langle J_k \right\rangle + D_k(s);$$

$$\left\langle J_k(s) \right\rangle = J_{ok} e^{-2\int_o^s \xi_k(z)dz} + \int_0^s e^{-2\int_z^s \xi_k(u)du} D_k(z) dz; \qquad \text{(M3-65)}$$

In storage rings it is frequently that the processes are very slow and you can average the damping and the diffusion over the circumference

$$\langle D_k \rangle = \langle D_k(s) \rangle_C ; \langle \xi_k \rangle = \langle \xi_k(s) \rangle_C$$

$$\langle J_k(s) \rangle = e^{-2\langle \xi_k \rangle s} \left(J_{ok} + \langle D_k \rangle \int_0^s e^{2\langle \xi_k \rangle z} dz \right) = J_{ok} e^{-2\langle \xi_k \rangle s} + \frac{\langle D_k \rangle}{2\langle \xi_k \rangle} \left(1 - e^{-2\langle \xi_k \rangle s} \right); \qquad \text{(M3-66)}$$

and stationary action at large s (many turns) being

$$\langle J_k(s) \rangle \to \frac{\langle D_k \rangle}{2\langle \xi_k \rangle} \qquad \text{(M3-67)}$$

This formula is very useful for both calculating and estimating the beam emittances in presence of diffusion and damping.

Note, that an anti-damping $\langle \xi_k \rangle < 0$ will cause exponential growth of the oscillating amplitude and is almost as bad as instability of oscialltions. Hence, this is important for accelerators where damping plays significant role in the beam dynamics, e.g. damping (anti-damping) time is much smaller or compatible with the beam life-time in the accelerator.

Remarkably, I know about one storage ring (VEPP-4 in Novosibirsk), which was initially built for proton-antiproton collisions but then will turned into electron-position collider. Since protons do not radiate any significant part of radiation, synchrotron radiation decrements were not important and neglected during design. When the switch to electrons and positrons, which have damping times of millisecond, did occurred, it turned out that synchrotron radiation will damp one degree of freedom and anti-damp the other… It was required to add an additional radiation device into the lattice (a strong wiggler) to solve this important problem.

Perturbation method. We discussed a number of ways how both the parameterization of the motion in linear Hamiltonian system can be used to solve variety of standard problems arising in accelerator physics. Some of them were exact solutions (like orbit distortions or dispersion function), but some of them were clearly perturbative and relied on averaging over fast oscillations. The later, while intuitively understandable, requires some more discussions – and this is what we start doing today. Let's consider an additional (not necessarily a simple, constant or linear, but definitely a weak) term in our equations of motion

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + \varepsilon F(X,s); \tag{M3-68}$$

Using our already well established parameterization, we can always write:

$$X = \frac{1}{2}\tilde{\mathbf{U}}(s)A(s) = \operatorname{Re}\sum_{k=1}^{n} a_k(s)Y_k(s)e^{i(\psi_k(s)+\varphi_k(s))}; A(s) = \begin{bmatrix} \dots \\ a_k e^{i\varphi_k} \\ a_k e^{-i\varphi_k} \\ \\ \end{bmatrix}; \tag{M3-69}$$

$$\tilde{\mathbf{U}}(s)\frac{d}{ds}A = \varepsilon F\left(\tilde{\mathbf{U}}(s)A(s),s\right) \Leftrightarrow \frac{d}{ds}a_k e^{i\varphi_k} = \varepsilon\frac{e^{-i\psi_k(s)}}{i}Y_k^{*T}(s)\mathbf{S}F\left(\tilde{\mathbf{U}}(s)A(s),s\right)$$

If one likes real form of the equations, it can be written as

$$\begin{aligned} &\frac{d}{ds}a_k e^{i\varphi_k} = \left(a_k' + i\varphi_k' a_k\right)e^{i\varphi_k} = \varepsilon\frac{e^{-i\psi_k}}{i}Y_k^{*T}\mathbf{S}F; \\ &\frac{d}{ds}a_k e^{-i\varphi_k} = \left(a_k' - i\varphi_k' a_k\right)e^{-i\varphi_k} = -\varepsilon\frac{e^{i\psi_k}}{i}Y_k^{T}\mathbf{S}F; \\ &\frac{da_k}{ds} = \varepsilon\operatorname{Im}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right]; a_k\cdot\frac{d\varphi_k}{ds} = \varepsilon\operatorname{Re}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right]; \end{aligned} \tag{M3-70}$$

In analytical mechanics, these equations for constant of motion in linear system are called "reduced" or "slow" equations when ε is so small that it significantly affect the motion only when right side of equation has constant terms, e.g. either the phase or amplitude of oscillations can grow in time, not just simply oscillate.

As an example, let's consider a 1D motion with write side having a power of x:

$$F=\begin{bmatrix} 0 \\ f(s)x^m \end{bmatrix}; x = a\mathrm{w}\cos(\psi+\varphi)$$

$$\frac{da}{ds}=\varepsilon f a^m \mathrm{w}^m \sin(\psi+\varphi)\cdot\cos^m(\psi+\varphi); \frac{d\varphi}{ds}=\varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}(\psi+\varphi); \tag{M3-71}$$

The equations (M3-71) are non-linear and do not have explicit analytical solution in general case (we know that it can be parameterized for n=1). Let's now consider a periodical system:

$$\psi(s)+\mu \to \psi(s)=\chi(s)+\frac{\mu s}{C};$$

$$\chi(s+C)=\chi(s)\mathrm{w}(s+C)=\mathrm{w}(s); f(s+C)=f(s); \psi(s+C);$$

$$\frac{da}{ds}=\varepsilon f a^m \mathrm{w}^m \sin\left(\frac{\mu s}{C}+\chi+\varphi\right)\cdot\cos^m\left(\chi(s)+\frac{\mu s}{C}+\varphi\right); \frac{d\varphi}{ds}=\varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}\left(\frac{\mu s}{C}+\chi+\varphi\right);$$

Considering that slow variable are nearly constant, we have on the right side terms oscillating with phase advancing as $(k\mu \pm 2\pi j)\frac{s}{C} = 2\pi\frac{s}{C}(kQ \pm j); -m \le k \le m; j - \text{int}\,eger$. Only when $kQ \pm j = 0$ (or close to zero – see next) we have a stationary growth. Otherwise, the oscillating terms will average.

One can intuitively expand the variation of constants a power of the infinitesimal ε

$$a = a_o + \sum_{k=1} a_k \varepsilon^k; \varphi = \varphi_o + \sum_{k=1} \varphi_k \varepsilon^k$$
$$\frac{da}{ds} = \varepsilon a^m \mathrm{w}^m \sin(\psi + \varphi)\cdot\cos^m(\psi + \varphi); \frac{d\varphi}{ds} = \varepsilon \mathrm{w}^m a^{m-1}\cos^{m+1}(\psi + \varphi); \tag{M3-72}$$

and explore it further. But this will bring us to a method developed by Bogolyubov and Metropolsky (N. Bogolubov N. (1961). *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Paris: Gordon & Breach. ISBN 978-0-677-20050-7.) in analytical mechanics. You can find a straightforward, but rather long derivation in the book – here we will only discuss the results.

Let's start from an equation of motion with a small (infinitesimally) perturbation for a linear system with deduced equation of

$$\frac{dA}{ds} = \varepsilon F(X,s); \tag{M3-72}$$

than the first order perturbation can be written as

$$A = \xi(s) + \varepsilon \tilde{F}(\xi,s); \frac{d}{ds}\xi(s) = \langle F(\xi,s) \rangle;$$
$$\langle F(A,s) \rangle = \frac{1}{S}\int_{s}^{s+S} \langle F(A = const, s) \rangle ds; \;\; \tilde{F} = \int \left(F - \langle F \rangle \right) ds; \tag{M3-73}$$

What is quite remarkable, that they also derived a second order perturbation:

$$A = \xi(s) + \varepsilon \tilde{F}(\xi,s) + \varepsilon^2 \overbrace{\left\{ \left(\tilde{F}\frac{\partial}{\partial \xi} \right) F \right\}}^{\sim} - \varepsilon^2 \frac{\partial \tilde{\tilde{F}}}{\partial \xi} \langle F(\xi,s) \rangle;$$
$$\frac{d}{ds}\xi(s) = \varepsilon \left\langle F(\xi,s) + \varepsilon \tilde{F} \right\rangle \approx \varepsilon \left\langle \left(1 + \varepsilon \left(\tilde{F}\frac{\partial}{\partial \xi} \right) \right) F(\xi,s) \right\rangle. \tag{M3-74}$$

These equations were used and still used to derive number of analytical expressions. You can check that quadrupole errors will result in expression that we already derived, but more interesting are cases of non-linear terms in equations.

An example: Octupole term in horizontal motion

Let's consider a 4th order term (non-linear) in our 1D Hamiltonian:

$$H = \frac{p_x^2}{2} + K_1(s)\frac{x^2}{2} + o(s)\frac{x^4}{4} \tag{M3-75}$$

with transformation to action angle variables

$$x = \sqrt{2I}\cdot \mathrm{w}(s)\cdot\cos(\psi(s)+\varphi);\ p_x = x'; \\ \{x, p_x\} \to \{\varphi, I\}. \tag{M3-76}$$

which removes the linear part of the Hamiltonian leaving only nonlinear term, which we need to express using action-angle variables:

$$H_1(\varphi, I, s) = o(s)\mathrm{w}^4(s) I^2 \cos^4(\psi(s)+\varphi); \\ 2\cos^2\theta = 1+\cos 2\theta;\ 2(1+\cos 2\theta)^2 = 3+4\cos 2\theta + \cos 4\theta; \\ H_1 = o\mathrm{w}^4 \frac{I^2}{2}\left(\frac{3}{4} + \cos(2\psi+2\varphi) + \frac{\cos(4\psi+4\varphi)}{4}\right); \tag{M3-77}$$

with equations of motion being

$$\varphi' = \frac{\partial H_1}{\partial I} = Io(s)\mathrm{w}^4(s)\left(\frac{3}{4} + \cos(2\psi+2\varphi) + \frac{\cos(4\psi+4\varphi)}{4}\right); \\ I' = -\frac{\partial H_1}{\partial \varphi'} = I^2 o(s)\mathrm{w}^4(s)\left(\sin(2\psi+2\varphi) + \frac{\sin(4\psi+4\varphi)}{2}\right); \tag{M3-78}$$

Staying away from the parametric (second order) and 4th order resonances $4Q \neq \pm k$ we can first average ((M3-78) noting that oscillating term yelled zero in this approximation

$$\langle I' \rangle = 0 \to \bar{I} = const;\ \ \bar{\varphi}' = \frac{3}{4}\bar{I}\left\langle o(s)\mathrm{w}^4(s)\right\rangle \equiv \frac{3}{4}\bar{I}\left\langle o(s)\beta^2(s)\right\rangle;$$

$$\bar{\varphi} = \varphi_o + \frac{\partial\mu}{\partial I}\bar{I}\frac{s}{C} \equiv \varphi_o + \frac{\partial\mu}{\partial I}\frac{a^2}{2};\ \ \frac{\partial\mu}{\partial I} = \frac{3}{4}\int_C o(s)\beta^3(s)ds; \qquad \text{(M3-79)}$$

e.g. while amplitude of oscillations remain constant, phase advance per turn (e.g. oscillation frequency) start depends on the square of amplitude of oscillations

$$Q = Q_o + \frac{1}{2\pi}\frac{\partial\mu}{\partial I}\frac{a^2}{2}. \qquad \text{(M3-80)}$$

Hence, octupole term in the Hamiltonian makes oscillations inharmonic. We always add oscillating terms

$$I = \bar{I} + \tilde{I};\ \ \varphi = \bar{\varphi} + \tilde{\varphi};$$

$$\tilde{I}(s) = \bar{I}^2 \int^s o(\xi)\mathrm{w}^4(\xi)\left(\sin\left(2\psi(\xi) + 2\bar{\varphi}\right) + \frac{\sin\left(4\psi(\xi) + 4\bar{\varphi}\right)}{2}\right)d\xi; \qquad \text{(M3-81)}$$

$$\tilde{\varphi} = \bar{I}\int^s o(\xi)\mathrm{w}^4(\xi)\left(\cos\left(2\psi(\xi) + 2\bar{\varphi}\right) + \frac{\cos\left(4\psi(\xi) + 4\bar{\varphi}\right)}{4}\right)d\xi;$$

which can be evaluated and expressed in terms oscillating with double and quadruple betatron frequency. Since we are considering quadrupole term being a perturbation, away from the resonance these oscillations are small. Naturally, one can go one extra step and write second order perturbation terms, which will be proportional to second order of quadrupole strength and higher order of action and harmonics of betatron frequencies. While it is possible, expression become rather long and are not very “educational.

It is not all… But already, not too shabby for a single parameterization

$$X(s)=\frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s)=\mathrm{Re}\sum_{k=1}^{n}Y_k(s)e^{i(\psi_k(s)+\varphi_k)}a_k(s);$$

$$\frac{d}{ds}\tilde{\mathbf{U}}(s)=\mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s);\tilde{\mathbf{U}}=\left[...Y_k e^{i\psi_k},Y^*{}_k e^{-i\psi_k},...\right];k=1,..,n$$

$$\tilde{\mathbf{U}}^T\mathbf{S}\tilde{\mathbf{U}}=2i\mathbf{S}.$$

What we found that we can calculate tune shift and tune spread caused by non-linearities, damping and diffusion coefficients. We introduced important notion of Canonically conjugate actions and phases for each degree of oscillations

$$\left(I_k=\frac{a_k{}^2}{2},\varphi_k\right)$$

We also found that in Hamiltonian system actions remain constant (it is actually true for nonlinear systems as well – look Lectures 25 and 26 in our Advanced Accelerator course), while phases are spread uniformly by tune spreads caused by energy spread (chromaticity), dependence on particle's action (octupole terms in magnets, it driven by sextupole terms, but in a second order of perturbation) and non-linear effects in beam-beam effects. This is the main reason why we can assume – with a large degree of confidence that phases of particles are evenly distributed from 0 to 2π.

Let's go to next step –finding solutions for particle distributions.

Fokker-Plank equation. Distribution function of particles.

This probabilistic approach for Fokker-Plank equation follows §83 from *Thermodynamics, Statistical Physics and Kinetics* by Yu. Rumer and M. Ryvkin, Nauka, RAN, 2001 [TSPK]. English translation: Mir, Moscow, 1980

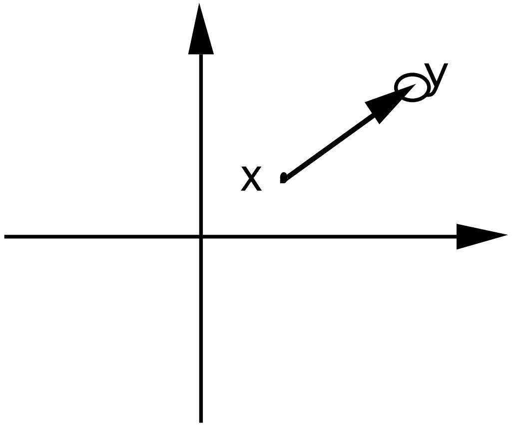
1. Particles described by a distribution function in the Phase Space $x=\{\vec{r},\vec{P}\}$:

$$f\left(\vec{r},\vec{P},t\right)\equiv f(x,t); x=\{\vec{r},\vec{P}\}: \quad \int f\left(\vec{r},\vec{P},t\right)d\vec{r}\,d\vec{P}=1:$$

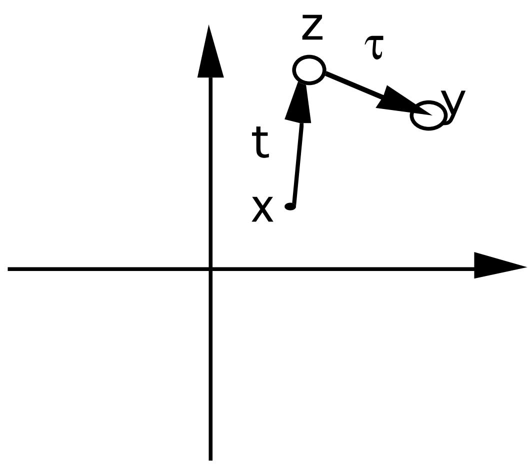
$$\Rightarrow \rho(\vec{r},t)=\int f\left(\vec{r},\vec{P},t\right)d\vec{P}; \;\; n\left(\vec{P},t\right)=\int f\left(\vec{r},\vec{P},t\right)d\vec{r};$$

2. Markov's chain: no dependence on pre-history of the event=> ***hypothesis****: correlations exist only between two consequent events: the probability to "move" from point x in the phase space to y during time τ depends only on $\{x,y,t,\tau\}$:*

$$dw=W\left(y,x\middle|\tau,t\right)dy; \quad dy=d\vec{r}_y d\vec{P}_y.$$



What probability to move from *point x in the phase space to y* during time $t+\tau$ trough an intermediate point z ?



There t is time for move $x \Rightarrow z$; τ is time for move $z \Rightarrow y$. Two events are independent and total probability is product of two probabilities: $x \Rightarrow z$ and then $z \Rightarrow y$:

$$W\left(y,z\middle|\tau,t_o+t\right)dy\,W\left(z,x\middle|t,t_o\right)dz$$

To find probability $W\left(y,x\middle|t+\tau,t_o\right)$ it is sufficient to integrae over all z :

$$W\left(y,x\middle|t+\tau,t_o\right)=\int dz W\left(y,z\middle|\tau,t_o+t\right)W\left(z,x\middle|t,t_o\right). \qquad \text{(M3-82)}$$

This is Smolukhovsky equation. **Fokker-Plank Equation** can be derived from ((M3-82) in following form ($t_o=0$):

$$W\left(y,x\middle|t+\tau,0\right)=\int dz W\left(y,z\middle|\tau,t\right)W\left(z,x\middle|t,0\right). \qquad \text{(M3-83)}$$

Lets consider an analytical (integrable) function $g(x)$, which is limited in all phase space and goes to zero with all it derivatives at the infinity (i.e. we use a finite system):

$$g(x) \;\Rightarrow\; g(x)\to 0;\;\; \frac{\partial^n g(x)}{\prod_{k=1}^{n}\partial x_{i_k}} \to 0;\;\; |x|\to\infty.$$

We should keep in mind that $g(x)$ can be a distribution function and these properties are natural for finite system with finite energy: $g(x)=0\ when \left|\vec{r}\right|>r_{\max}\ or\ \left|\vec{P}\right|>P_{\max}\ (E_{\max})$. Multiplying (17-2) by $g(y)$ and integrating it give us:

$$\int g(y)W\left(y,x\middle|t+\tau,0\right)dy=\iint g(y)W\left(y,z\middle|\tau,t\right)dyW\left(z,x\middle|t,0\right)dydz. \tag{M3-84}$$

$g(y)$ can be expanded into Taylor series:

$$g(y)=g(z)+\frac{\partial g}{\partial z_i}\left(y_i-z_i\right)+\frac{1}{2}\frac{\partial^2 g}{\partial z_i\partial z_k}\left(y_i-z_i\right)\left(y_k-z_k\right)+\ldots..$$

(summation is assumed on repeated indexes)

$$\int g(y)W\left(y,x\middle|t+\tau,0\right)dy=$$
$$\iint\left(g(z)+\frac{\partial g}{\partial z_i}\left(y_i-z_i\right)+\frac{1}{2}\frac{\partial^2 g}{\partial z_i\partial z_k}\left(y_i-z_i\right)\left(y_k-z_k\right)+\ldots..\right)W\left(y,z\middle|\tau,t\right)dyW\left(z,x\middle|t,0\right)dydz. \tag{M3-85}$$

Taking into account that:

$$\int g(z)W(y,x|\tau,t)dy = g(z); \;\; \int W(y,x|\tau,t)dy \equiv 1;$$

$$\int g(y)W(y,x|t+\tau,0)dy - \int g(z)W(z,x|t,0)dz = \int g(y)\{W(y,x|t+\tau,0) - W(y,x|t,0)\}dy$$

we can rewrite (17-4) in following from:

$$\int g(y)\frac{W(y,x|t+\tau,0) - W(y,x|t,0)}{\tau}dy -$$

$$-\int a_i^{(\tau)}(y,t)\frac{\partial g}{\partial y_i}W(y,x|t,0)dy$$

$$-\int b_{ik}^{(\tau)}(y,t)\frac{\partial^2 g}{\partial y_i \partial y_k}W(y,x|t,0)dy - \ldots\ldots = 0.$$

where we introduce following notations:

$$a_i^{(\tau)}(y,t) = \frac{1}{\tau}\int (z_i - y_i)W(z,y|\tau,t)dz;$$

$$b_{ik}^{(\tau)}(y,t) = \frac{1}{\tau}\int (z_i - y_i)(z_k - y_k)W(z,y|\tau,t)dz;$$

....

2n-D vector $a=\left\{a_i^{(\tau)}\right\}$ is an "average speed" particles' point on the in Poincaré plot in the phase space. $b_{ik}^{(\tau)}$ is 2n-D tensor representing correlations between variations of i's and k's components of $x=\{\vec{r},\vec{P}\}$ with the tensor's trace giving RMS drift of the point $b_{ii}^{(\tau)}(y,t)=\frac{1}{\tau}\int\left(z_i-y_i\right)^2 W\left(z,y\middle|\tau,t\right)dz$.

Integrating by parts (here we use the boundary condition for finite system!):

$$\int a_i^{(\tau)}(y,t)\frac{\partial g}{\partial y_i}W\left(y,x\middle|t,0\right)dy=$$

$$\int\frac{\partial}{\partial y_i}\{a_i^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)g(y)\}dy-\int g(y)\frac{\partial}{\partial y_i}\{a_i^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)\}dy;$$

$$\int b_{ik}^{(\tau)}(y,t)\frac{\partial^2 g}{\partial y_i\partial y_k}W\left(y,x\middle|t,0\right)dy=\int\frac{\partial}{\partial y_i}\{b_{ik}^{(\tau)}(y,t)\frac{\partial g}{\partial y_k}W\left(y,x\middle|t,0\right)\}dy-$$

$$-\int\frac{\partial}{\partial y_i}\{g(y)\frac{\partial}{\partial y_k}[b_{ik}^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)]\}dy+\int g(y)\frac{\partial^2}{\partial y_i\partial y_k}\{b_{ik}^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)\}dy.$$

$$\int\frac{\partial}{\partial y_i}h(y)\prod_{k=1,\ldots 6}dy_k=\int\prod_{k\neq i}dy_k\int\frac{\partial}{\partial y_i}h(y)dy_k=\int\prod_{k\neq i}dy_k\{h(y_{k\neq i},y_i=+\infty)-h(y_{k\neq i},y_i=-\infty)\}=0.$$

and finally:

$$\int g(y)\left\{\frac{\partial W(y,x|t,0)}{\partial t}+\frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t)W(y,x|t,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_k}\left[b_{ik}^{(\tau)}(y,t)W(y,x|t,0)\right]\right\}dy=0.$$

$g(y)$ is arbitrary function which requires the expression in the brackets to be zero:

$$\frac{\partial W(y,x|t,0)}{\partial t}+\frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t)W(y,x|t,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_k}\left[b_{ik}^{(\tau)}(y,t)W(y,x|t,0)\right]=0 \qquad \text{(M3-86)}$$

This is called mono-molecular kinetic equation of Fokker and Plank. What about the distribution function? The Fokker Plank equation for the distribution function can be derived from this using connection between distribution function $f(x,t)\equiv f(\vec{r},\vec{p},t)$ and probability $W(y,x|\tau,t)$: deviation of the particles density in phase space volume dx during time t is equal to the difference between number of particles left this point and arrived into this point:

$$\left[f(x,t)-f(x,0)\right]dx=dx\int\left[W(x,z|t,0)f(z,0)-W(z,x|t,0)f(x,0)\right]dz \qquad \text{(M3-87)}$$

Remembering that $\int W(z,x|t,0)dz=1$, we get

$$f(x,t)=\int W(x,z|t,0)f(z,0)dz \qquad \text{(M3-88)}$$

which shows that multiplication on $W(y,x|t,0)$ and integrating over the phase space equivalent to a propagation in the phase space by (x-z) and in time by t.

Thus, multiplying (M3-86) by $f(x,0)$ and integrating over x we obtaining Fokker-Plank equation for the distribution function:

$$\frac{\partial f(y,t)}{\partial t} + \frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t) f(y,t)\right] - \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[b_{ik}^{(\tau)}(y,t) f(y,t)\right] = 0 \qquad \text{(M3-89)}$$

This equation also can be written as continuity equations in the phase space:

$$\frac{\partial f(y,t)}{\partial t} + \frac{\partial j_k}{\partial y_k} = 0; \quad j_k = \left[a_k^{(\tau)}(y,t) f(y,t)\right] - \frac{1}{2}\frac{\partial}{\partial y_i}\left[b_{ik}^{(\tau)}(y,t) f(y,t)\right]. \qquad \text{(M3-90)}$$

This is the final form of the of Fokker-Plank equation, where we just should recognize the terms such as motion of the particle and diffusion coefficients D:

$$\frac{\partial f(y,t)}{\partial t} + \frac{\partial}{\partial y_i}\left[\frac{dy_i(y,t)}{dt} f(y,t)\right] - \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[D_{ik}(y,t) f(y,t)\right] = 0$$

Finally, nobody told us to use time as independent variable, *s* is as good!

$$\frac{\partial f(y,s)}{\partial s} + \frac{\partial}{\partial y_i}\left[\frac{dy_i(y,s)}{ds} f(y,s)\right] - \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[D_{ik}(y,s) f(y,s)\right] = 0 \qquad \text{(M3-91)}$$

Effects damping and difusion on particle's distribution

I. Oscillator

Before we embark on detail studies of radiation effects on the beams in accelerators, let's look on a very simple model of harmonic oscillator:

$$H = \frac{P^2}{2m} + k\frac{x^2}{2} \quad or \quad h = \frac{p^2}{2} + \omega^2\frac{x^2}{2} \;;\; \omega = \sqrt{\frac{k}{m}} \qquad \text{(M3-92)}$$

described by differential equations

$$x' = \frac{\partial h}{\partial p} = p;\; p' \equiv x'' = -\frac{\partial h}{\partial x} = -\omega^2 x;\; x = A\cdot\cos(\omega t + \varphi);\; p = -A\omega\cdot\sin(\omega t + \varphi)$$

$$X' = \begin{bmatrix} x' \\ p' \end{bmatrix} = S\frac{\partial h}{\partial X} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\cdot X;\; X = \operatorname{Re} aYe^{i(\omega t+\varphi)};\; Y = \begin{bmatrix} 1/\sqrt{\omega} \\ i\sqrt{\omega} \end{bmatrix};\; \left\{ I = \frac{a^2}{2}, \varphi \right\} \qquad \text{(M3-93)}$$

Let's add a weak friction $\varepsilon << \omega$:

$$p = x';\; p' = -\omega^2 x - 2\alpha p;\; X' = D\cdot X = \begin{vmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{vmatrix}\cdot X;$$

$$\det\begin{bmatrix} \lambda & -1 \\ \omega^2 & \lambda + 2\alpha \end{bmatrix} = \lambda(\lambda + 2\alpha) + \omega^2 = 0;\; \lambda = -\alpha \pm i\omega_1;\quad \omega_1 = \sqrt{\omega^2 - \alpha^2}; \qquad \text{(M3-94)}$$

$$x = A\cdot e^{-\alpha t}\cdot\cos(\omega_1 t);\; p = -A\cdot e^{-\alpha t}\left(\alpha\cdot\cos(\omega_1 t) + \sin(\omega_1 t)\right);$$

$$X = \operatorname{Re} aYe^{\lambda t + i\varphi} = a\cdot e^{-\alpha t}\operatorname{Re} Ye^{i\omega_1 t};\; Y = \begin{bmatrix} 1/\sqrt{\lambda} \\ i\sqrt{\lambda} \end{bmatrix};$$

which make <u>a very small change</u> to the frequency of the oscillations, but make free oscillations slowly decaying.

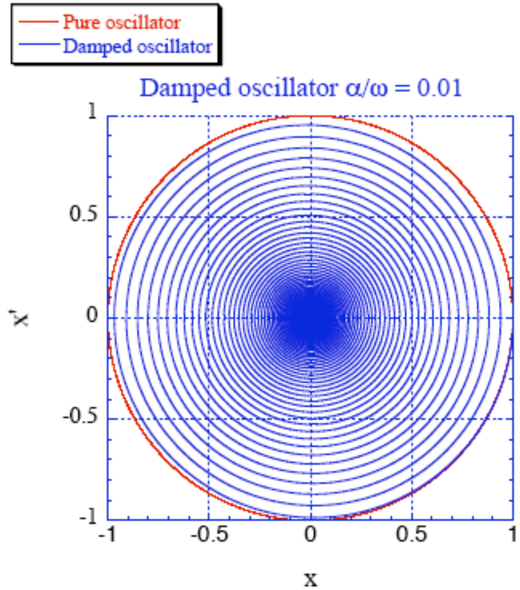
Note that damping decrement α is only a half of that of simple decay:

$$p' = -2\alpha p \rightarrow p = p_o e^{-2\alpha t}.$$

This is the result of oscillations, where, time-averaged, only half energy is in the kinetic energy $p^2/2$, which decays. The potential energy decays only through its coupling to the kinetic energy via oscillations. The action of the oscillator, I, which represent the area of the phase space, decays with the simple decay rate towards zero:

$$I' = \left(\frac{a^2}{2}\right)' = -2\alpha I; I = I_o e^{-2\alpha t}, \tag{M3-95}$$

while the oscillator phase does not stationary point or any decay.



Poincaré plot of trajectories of normal and damped oscillator in dimensionless coordinates x/a; x'/ωa.

Note the second fact, that trace of matrix $D = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{bmatrix}$ gives the damping rate of the oscillator phase space volume. Let's add a random noise to the equations:

$$x' = p + \delta x(t); \;\; p' = -\omega^2 x - 2\alpha p + \delta x'(t); \;\; X' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{bmatrix} \cdot X + \begin{bmatrix} \delta x(t) \\ \delta x'(t) \end{bmatrix}; \qquad \text{(M3-97)}$$

$$\langle \delta x(t) \rangle = 0; \quad \langle \delta x'(t) \rangle = 0.$$

where $\delta x(t), \delta x'(t)$ are "sudden" and randomly distributed in time and amplitude jumps.

One can easily calculate change in the amplitude and phase of the oscillator caused by a random kick:

$$\delta\left(a e^{i\varphi}\right) = -i e^{-i\omega_1 t} Y^{*T} \cdot S \cdot \begin{bmatrix} \delta x \\ \delta x' \end{bmatrix};$$

$$\delta a + i a \delta\varphi \cong -i e^{-i(\omega_1 t + \varphi)} Y^{*T} \cdot S \cdot \left(\delta x' / \sqrt{\omega} - i \delta x \sqrt{\omega} \right) \qquad \text{(M3-98)}$$

$$\langle \delta a \rangle = 0; \langle \delta\varphi \rangle = 0; \;\; \delta I = a\delta a + \delta a^2 / 2; \langle \delta I \rangle = \left\langle \frac{\delta a^2}{2} \right\rangle = \frac{\langle \delta x'^2 \rangle / \omega + \omega \langle \delta x^2 \rangle}{2}$$

Thus, the only one thing is well determined – the average change of the action, *I*.

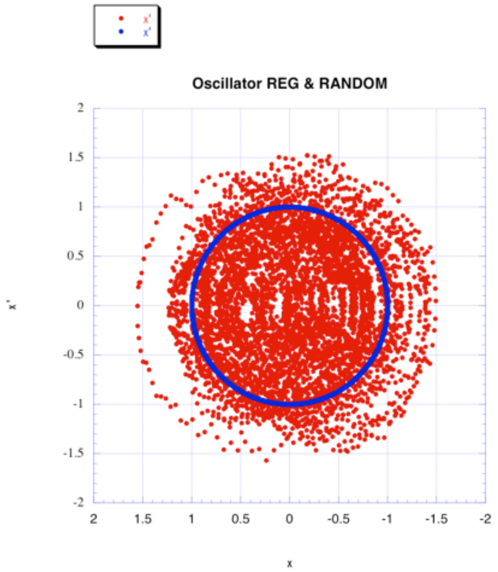
Adding damping term we have:

$$\langle I\rangle' = -2\alpha\langle I\rangle + D/2; \quad D = \langle \delta x'^2\rangle/\omega + \omega\langle \delta x^2\rangle; \tag{M3-99}$$

with stationary solution for average action (emittance) and RMS amplitude of the ensemble of oscillators:

$$\langle I\rangle = \frac{D}{4\alpha}, i.e.\ \varepsilon = \langle a^2\rangle = \frac{D}{2\alpha} = \frac{\langle \delta x'^2\rangle/\omega + \omega\langle \delta x^2\rangle}{2\alpha}, \tag{M3-100}$$

where ε is called emittance – phase space area occupied divided by π - of the ensemble of oscillators ($\varepsilon^2 = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$).



Poincaré plot of trajectories of normal and damped oscillator with random kicks (in dimensionless coordinates x/a; x'/ωa).

Poincaré plot shows few hundreds of such an oscillators starting from the same initial conditions (1,0) and going around for few damping times. Overall, a large ensemble of oscillators (or equivalently distribution of (x,p) for one oscillator in very long time – via Ergodic theorem, see **http://en.wikipedia.org/wiki/Ergodic_theory**) is described by distribution function.

Because *(I,φ)* is Canonical pair, it is natural to use them as independent variables for the distribution function, *f(I,φ,t)*. Few facts are apparent: the phases of oscillators walk randomly and because phase is cyclic function it is distributed evenly in the interval {-π, π}. Thus, there is no dependence on *φ*: $\frac{\partial f}{\partial \varphi} = 0$. Finding distribution function of the action, *f(I,t),* requires solution of Fokker-Plank equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial I}\left(f \frac{dI}{dt} \right) - \frac{1}{2}\frac{\partial^2}{\partial I^2}\left(\left\langle \frac{\delta I^2}{\tau} \right\rangle f \right) = 0 . \qquad \text{(M3-101)}$$

First, let's observe that

$$a = a_o(1-\alpha\tau) + \delta a;$$

$$I = \frac{\left| a_o(1-\alpha\tau) + \delta a \right|^2}{2} = \frac{\left(a_o{}^2(1-\alpha\tau)^2 + 2a_o(1-\alpha\tau)\operatorname{Re}\delta a + |\delta a|^2 \right)}{2};$$

$$\delta I = -2\alpha\tau I_o + a_o(1-\alpha\tau)\operatorname{Re}\delta a + \frac{|\delta a|^2}{2} + I_o(\alpha\tau)^2$$

Using coefficients definitions for Fokker-Plan equation (17-4) we can average both coefficients (using randomness of the phase of δa -> $2\left\langle \operatorname{Re}\delta a^2 \right\rangle = \left\langle |\delta a|^2 \right\rangle$):

$$\frac{dI}{dt} = \frac{\langle \delta I \rangle}{\tau} = -2\alpha I + \frac{|\delta a|^2}{2\tau} + O(\tau) \cong -2\alpha I + D$$

$$\frac{\left\langle \delta I^2 \right\rangle}{\tau} = 2I \frac{\left\langle \operatorname{Re}\delta a^2 \right\rangle}{\tau} + O\left(\tau, |\delta a|^2\right) \cong I \frac{|\delta a|^2}{\tau} = 2I \cdot D$$

We are interested in stationary solution $\frac{\partial f}{\partial t}=0$, which result in 1D ordinary differential equation:

$$\frac{d}{dI}\left(f\frac{dI}{dt}-\frac{1}{2}\frac{d}{dI}\left(\left\langle\frac{\delta I^2}{\tau}\right\rangle f\right)\right)=0\rightarrow\frac{d}{dI}\left(f(2\alpha I-D)+\frac{d}{dI}(DI\cdot f)\right)=0;$$

$$\frac{d}{dI}\left(I\left(2\alpha f+D\frac{df}{dI}\right)\right)=0\rightarrow I\left(2\alpha f+D\frac{df}{dI}\right)=const.$$

Setting I=0 and demanding the finite values for the expression in the brackets, sets *const=0*. Thus

$$2\alpha f+D\frac{df}{dI}=0\rightarrow\frac{d\ln f}{dI}=-\frac{2\alpha}{D};$$

$$\ln f=-\frac{2\alpha}{D}\cdot I+\ln c\rightarrow f=c\exp\left[-\frac{I}{I_o}\right];I_o=\frac{D}{2\alpha}. \qquad \text{(M3-101)}$$

where c is a normalization coefficient. Remembering that $I=\frac{x'^2\omega+\omega\delta x^2}{2}$, it gives us just a trivial Gaussian distribution for the oscillators

$$f(x,x')=\frac{1}{2\pi\varepsilon}e^{-\frac{a^2}{2\varepsilon}}=\frac{1}{2\pi\varepsilon}\exp\left(\frac{x^2\omega+x'^2/\omega}{2\varepsilon}\right)=\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}\frac{1}{\sqrt{2\pi}\sigma_{x'}}e^{-\frac{x'^2}{2\sigma_{x'}^2}} \qquad \text{(M3-102)}$$

$$\sigma_x=\sqrt{\varepsilon/\omega};\sigma_{x'}=\sqrt{\varepsilon\cdot\omega}$$

where we normalize it as $\iint f(x,x')dxdx'=1$.

Conclusions are easy to remember: Position independent diffusion in the presence of linear damping results in stationary Gaussian distribution of the oscillator's amplitudes, positions and velocities. Phases of individual oscillators become random. Naturally, this process takes few damping times $T_d=1/\alpha$, if initial distribution deviates from the stationary.

Now we are fully equipped to write distribution for in 6D phase space. Using well-established

$$X = \mathrm{Re}\sum_{k=1}^{3} a_k Y_k e^{i(\psi_k+\varphi_k)} \rightarrow a_k = \frac{e^{i(\psi_k+\varphi_k)}}{i} Y_k^{*T} SX;\;\; I_k = \frac{a_k^{\;2}}{2} = \frac{\left|Y_k^{\;T} SX\right|^2}{2}; \quad \text{(M3-103)}$$

we can write that particles distribution at any location s:

$$f(X) = \frac{1}{2\pi\varepsilon_1} e^{-\frac{a_1^{\;2}}{2\varepsilon_1}} \frac{1}{2\pi\varepsilon_2} e^{-\frac{a_2^{\;2}}{2\varepsilon_2}} \frac{1}{2\pi\varepsilon_3} e^{-\frac{a_3^{\;2}}{2\varepsilon_3}} = \frac{1}{\prod_{k=1}^{3} 2\pi\varepsilon_k} \exp\left[-\frac{1}{2}\sum_{k=1}^{3} \frac{\left|Y_k^{\;T}(s) SX\right|^2}{\varepsilon_k} \right] \quad \text{(M3-104)}$$

This is one of the most useful applications of the eigen vectors and their components. It worth mentioning that this distribution is positively defined quadratic for of the particles positions (x,y, τ) and corresponding Canonical momenta. In accelerator physics $\left|Y_k^{\;T}(s) SX\right|^2$ are can be also known as Courant-Snyder invariants, which they derived in 1950s for 1D case.

For 1D case and slow synchrotron oscillation it is easy for write detailed distribution

$$\left|Y_x^T(s)SX\right|^2 = \left(\mathrm{w}_x x' - \mathrm{w}'_x x\right)^2 + \frac{x^2}{\mathrm{w}_x^{\,2}} = \frac{x_\beta^{\,2} + \left(\beta_x x'_\beta + \alpha_x x_\beta\right)^2}{\beta_x}; x_\beta = x - \eta_x \pi_\tau;$$

$$f_{6D} = f_x\ f_y\ f_s; \qquad \text{(M3-105)}$$

$$f_x = \frac{1}{2\pi\varepsilon_x} e^{-\frac{x_\beta^{\,2} + \left(\beta_x x'_\beta + \alpha_x x_\beta\right)^2}{2\varepsilon_x \beta_x}};\ f_y = \frac{1}{2\pi\varepsilon_y} e^{-\frac{y^2 + \left(\beta_y y' + \alpha_y y\right)^2}{2\varepsilon_y \beta_y}};\ f_s = \frac{1}{2\pi\sigma_\delta\sigma_s} e^{-\frac{\pi_\tau^{\,2}}{2\sigma_\delta^{\,2}}} e^{-\frac{\tau^2}{2\sigma_s^{\,2}}}.$$

that you would find in most of the accelerator books. But you are now capable of doing it for any arbitrary coupling (M3-104). You also can calculate RMS beam size in any direction by integrating (M3-104) over the rest of coordinated and momenta. For example:

$$\left\langle x^2 \right\rangle = \int x^2 f(X) dX = \int x^2 dX \frac{1}{\prod_{k=1}^{3} 2\pi\varepsilon_k} \exp\left[-\frac{1}{2} \sum_{k=1}^{3} \frac{\left|Y_k^T(s)SX\right|^2}{\varepsilon_k} \right] \qquad \text{(M3-106}$$

with 1D result being

$$\left\langle x^2 \right\rangle = \beta_x \varepsilon_x + \eta_x^2 \sigma_\delta^2; \left\langle y^2 \right\rangle = \beta_y \varepsilon_y; \left\langle \tau^2 \right\rangle = \sigma_s^2;$$

$$\left\langle x'^2 \right\rangle = \varepsilon_x \frac{1+\alpha_x^2}{\beta_x} + \eta_x'^2 \sigma_\delta^2; \left\langle y'^2 \right\rangle = \varepsilon_y \frac{1+\alpha_y^2}{\beta_y}; \left\langle \pi_\tau^{\,2} \right\rangle = \sigma_\delta^2. \qquad \text{(M3-107)}$$

Vlasov equation.

While Fokker-Plank equation is an important tool in many areas of physics, a reduced version of it, Vlasov equation, is one of the most important tools in accelerator and plasma physics. In contrast with the Fokker-Plank equation, Vlasov equation does not take into account random processes, such as quantum fluctuations of spontaneous radiation or particle's scattering on each other. Nevertheless, it is one of most useful tools for studying instabilities of beams.

Instabilities in the beam are among main reasons behind increases or limitations in beam's emittances and energy spreads. But since this is not that is not the main topic of our course here, Lectures 20 and 21 in our Advanced Accelerator Physics course can be used if you want to glance thought. Otherwise, there are a large number of books devoted specifically to studies of beam instabilities.

Here we will provide you only with discussion of Vlasov equations, which could be – in principle – useful to cooling studies. As possible example, it can provide you with tools needed to investigate space charge effects in electron cooling.

In general, in addition to the describing the motion of particles in external (giving field), we need to find the fields induced by the particles moving (and accelerating) and include them into the equation of motion. This process is frequently complicated by necessity of including the boundary conditions and turns into very tough problem to crack. Still there is a number of approaches which allow to study some of the most important processes and instabilities analytically or semi-analytically. It usually involves solving self-consistently Vlasov equation and Maxwell equations:

$$div\vec{E} = 4\pi\rho;\ curl\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t};\ curl\vec{E} = +\frac{1}{c}\frac{\partial \vec{B}}{\partial t};\ div\vec{B} = 0;$$
$$\rho = \sum_p e_p \delta\left(\vec{r} - \vec{r}_p(t)\right);\ \vec{j} = \rho\vec{\mathrm{v}}. \qquad \text{(M3-108)}$$

Since you are well familiar with the later set of partial differential equations , you can guess that it is not a trivial problem and frequently requires significant simplification (assumptions) to be solved together with also not trivial partial differential equation.

Let's derive Vlasov equation for an ensemble of particles by considering a large number of them (in accelerator typical N ~ 10^{10}) evolving in the phase space. The microscopic distribution function

$$f = \sum_{part} \delta_{6D}\left(X - X_p(s)\right) \quad . \qquad \text{(M3-109)}$$

while exact, results in N individual equations, which may be solvable exactly by computers in some distant-or-not-distant future. Meanwhile, so-called particle-in-cell (PIC) code are solving these equations typically using macro-particles and other simplifications.

Smoothing (M3-109) out by using a space volume ΔV_{2n} containing large number of particles $\Delta N_p >> 1$ should allow us to introduce the distribution function:

$$f = f(X,s): \qquad dN_p = f(X,s)dX^{2n} \equiv f(X,s)dV_{2n}. \qquad \text{(M3-110)}$$

This immediately introduces the scale at which Vlasov equation is violated. While at the typical beam size σ scale there is a huge number of particles, at a typical distance between particles $\delta l \sim \sigma / \sqrt[3]{N_p}$ they scatter on each other. Hence, according detailed (and non-trivial) studies there should be a scale L, when the scattering can be neglected,

$$\delta l << L << \sigma$$

and Vlasov equation can be used. We will assume the following:

1. The local interaction of the particles in small volume dV_{2n} is negligible compared with their interaction with the rest of the ensemble;
2. The system is Hamiltonian, i.e. dissipation is absent;
3. The consequence of 1) is that we neglect scattering processes between the particles! It is important – otherwise we could not say that number of particles in the phase space volume stays constant. We discussed such Markov processes when we were deriving the Fokker-Plank equation.

Sub-ensemble of particles in small volume dV_{2n} satisfies the conditions we for derivation of Liouville's theorem. Let's draw the boundary of the infinitesimal phase-space volume around the particles. Because the phase space trajectories do not cross,

$$\left(X_1\left(s_o\right),s_o\right)=\left(X_2\left(s_o\right),s_o\right)\Leftrightarrow X_1\left(s\right)\equiv X_2\left(s\right).$$

the particles can not escape the volume. It means that phase density along the trajectory stays constant:

The number of the particles dN_p is constant;

The volume dV_{2n} is constant.

Thus,

$$f=f\left(X(s),s\right)=\frac{dN_p}{dV_{2n}}=const\ , \qquad \text{(M3-111)}$$

when $X(s)$ is the trajectory satisfying the equation of motion. The consequence of this equation is very powerful. If we follow the trajectory of the point in the phase space

$$X(s)=M:X_o(s_o)\Leftrightarrow X_o(s_o)=M^{-1}:X(s), \qquad \text{(M3-112)}$$

than particles density remains constant at that point and know initial particle's distribution $f_o\left(X,s_o\right)$ at s_o, then,

$$f\left(X,s\right)=f_o\left(M^{-1}:X,s_o\right). \qquad \text{(M3-113)}$$

It is called methods of trajectories and is used broadly from plasma physics to quantum field theory (famous Feynman's method of trajectories). While very interesting, finding (M3-112) is equivalent to solving the pair of Maxwell-Vlasov equations. Hence, let's write Vlasov equation noting the total derivative of the distribution function along the (particle) trajectory is equal zero:

$$\frac{d}{ds} f\left(X(s),s\right) = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial X}\frac{dX}{ds} = 0. \qquad \text{(M3-114)}$$

This is famous Vlasov equation, which equivalent of the Liouville theorem. Do not forget that s is the independent variable, i.e. in most of the books $s=t$! Using the Hamiltonian equations to finish the job in matrix form:

$$\frac{\partial f}{\partial s} + \frac{\partial f}{\partial X} S \frac{\partial H}{\partial X} = 0 \qquad \text{(M3-115)}$$

or in more traditional open form

$$\frac{\partial f}{\partial s} + \frac{\partial H}{\partial P_i}\frac{\partial f}{\partial Q_i} - \frac{\partial H}{\partial Q_i}\frac{\partial f}{\partial P_i} = 0. \qquad \text{(M3-116)}$$

When time is used as independent variable (e.g. most of the books), the 3-D Vlasov equation reads:

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \vec{P}}\frac{\partial f}{\partial \vec{r}} - \frac{\partial H}{\partial \vec{r}}\frac{\partial f}{\partial \vec{P}} = 0. \qquad \text{(M3-117)}$$

End of the “tools” lectures

- Goal of these three lectures was to provide you with a set of tools necessary to evaluate cooling decrements and diffusion coefficients
- Actual calculation may include multiple effects, such as partial overlaps of electron and hadron beams, which would result in cooling forces dependence on amplitudes and phases of oscitations. In this case, use of first order perturbation method with averaging over phases of oscillations would provide you with –generally speaking, a nonlinear - evolution equations for three actions.
- Similarly, averaging over phases (and sometime multiple turns) will provide you with diffusion coefficients, which in general case will depend on actions.

Additional formulae

Let's consider a case we already studies during last class: **small variation of the quadrupole gradient**. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$\delta H = \delta K_1(s)\frac{x^2}{2} = I\cdot\delta K_1(s)\beta(s)\cos^2\left(\psi(s)+\varphi\right)$$

$$\frac{d\varphi}{ds} = \frac{\partial \delta H}{\partial I} = \delta K_1\beta\cos^2(\psi+\varphi) = \delta K_1\beta\frac{1+\cos 2(\psi+\varphi)}{2};$$

$$\frac{dI}{ds} = -\frac{\partial \delta H}{\partial \varphi} = I\cdot\delta K_1(s)\beta(s)\sin 2(\psi+\varphi);$$

Using first order approximation we get:

$$\frac{d\langle\varphi\rangle}{ds} = \frac{1}{S}\int_s^{s+S}\delta K_1(s)\beta(s)\frac{1+\cos 2(\psi+\varphi)}{2} =$$

$$\frac{\langle\delta K_1(s)\beta(s)\rangle}{2} + \frac{\langle\delta K_1(s)\beta(s)\cos 2(\psi+\varphi_o)\rangle}{2}$$

$$\frac{d\langle I\rangle}{ds} = I_o\cdot\langle\delta K_1(s)\beta(s)\sin 2(\psi+\varphi_o)\rangle;$$

We already got the first term average term

$$\frac{\langle \delta K_1(s)\beta(s)\rangle}{2} = \frac{1}{2C}\int_0^C \delta K_1(s)\beta(s)$$

while the amplitude does not have obvious non-oscillating term. Oscillating terms are also of some interest - let's explore them:

$$\frac{d\tilde{\varphi}}{ds} = \mathrm{Re}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = \mathrm{Re}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o}$$

$$\frac{d\tilde{I}}{ds} = I_o \cdot \mathrm{Im}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Im}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o}$$

$$\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{k}{C}s}.$$

where we simply expanded periodic complex function into a Fourier series.

(15-11) is easy to integrate

$$\tilde{\varphi} = -\frac{C}{2\pi}\mathrm{Im}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = \mathrm{Re}\,\phi(s) e^{2i(\psi+\varphi_o)};\ \ \phi(s+C) = \phi(s)$$

$$\tilde{I} == -I_o \cdot \frac{C}{2\pi}\mathrm{Re}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Re}\,\upsilon(s) e^{2i(\psi+\varphi_o)};\ \ \upsilon(s+C) = \upsilon(s)$$

Unless the accelerator is "sitting" at a parametric resonance $2Q = \pm k$, there oscillating term simply oscillating with double betatron frequency. Otherwise, at the parametric resonance $2Q = \pm k$ both the amplitude and the phase can grow – e.g. it is an instability we have to stay away from. Parametric resonance is one you are using to increase amplitude of oscillation of a swing by periodically changing the "oscillation frequency" with your legs and body.

In general case of change in Hamiltonian of linear motion

$$H = \frac{1}{2} X^T \left(\mathbf{H}_o + \delta \mathbf{H}_1 \right) X;\; X \to \{\varphi_k, I_k\} \to \delta H_1(\varphi_k, I_k, s);$$

)

$$\Delta \mu_k = \frac{\partial}{\partial I_k} \int_o^C \left\langle \delta H_1(\varphi_k, I_k, s) \right\rangle_{\varphi_k} ds.$$

$$H_1(\varphi_k, I_k, s) = \frac{1}{2} A^T \tilde{U}^T \delta \mathbf{H}_1 \tilde{U} A =$$

$$\frac{1}{8} \left\{ \sum_{k=1}^{n} \sqrt{2 I_k} \left(Y_k e^{i(\psi_k + \varphi_k)} + Y^*{}_k e^{-i(\psi_k + \varphi_k)} \right) \right\}^T \delta \mathbf{H}_1 \left\{ \sum_{k=1}^{n} \sqrt{2 I_k} \left(Y_k e^{i(\psi_k + \varphi_k)} + Y^*{}_k e^{-i(\psi_k + \varphi_k)} \right) \right\}$$

$$\left\langle H_1(\varphi_k, I_k, s) \right\rangle_\varphi = \frac{1}{2} \sum_{k=1}^{n} I_k \operatorname{Re}\left(Y^{*T}{}_k \delta \mathbf{H}_1(s) \tilde{Y}_k \right); \frac{d\varphi_k}{ds} = \frac{\partial \langle H_1 \rangle}{\partial I_k} = \frac{1}{2} \operatorname{Re}\left(Y^{*T}{}_k \; \delta \mathbf{H}_1(s) \tilde{Y}_k \right);$$

or

$$\frac{d\varphi_k}{ds} = \frac{\partial H_1}{\partial I_k} = \frac{1}{4} \left(Y_k e^{i(\psi_k + \varphi_k)} + Y_k{}^* e^{-i(\psi_k + \varphi_k)} \right)^T \delta \mathbf{H}_1 \left(Y_k e^{i(\psi_k + \varphi_k)} + Y_k{}^* e^{-i(\psi_k + \varphi_k)} \right)$$

$$\left\langle \frac{d\varphi_k}{ds} \right\rangle = \left\langle \frac{\partial H_1}{\partial I_k} \right\rangle = \frac{1}{2} Y_k^{*T} \delta \mathbf{H}_1(s) \tilde{Y}_k.$$

with $\operatorname{Im}(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y}_k) = 0$ since

$$\left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y} \right)^* = \left(Y_k^T \; \delta \mathbf{H}_1 \tilde{Y}^* \right) = \left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y} \right)^T = \left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y} \right)$$

Finally, the tune change is just an integral:

$$\Delta Q_k = \frac{\Delta \mu_k}{2\pi} = \frac{1}{4\pi} \int_0^c Y_k^{*T}(s) \delta \mathbf{H}_1(s) \tilde{Y}_k(s) ds$$

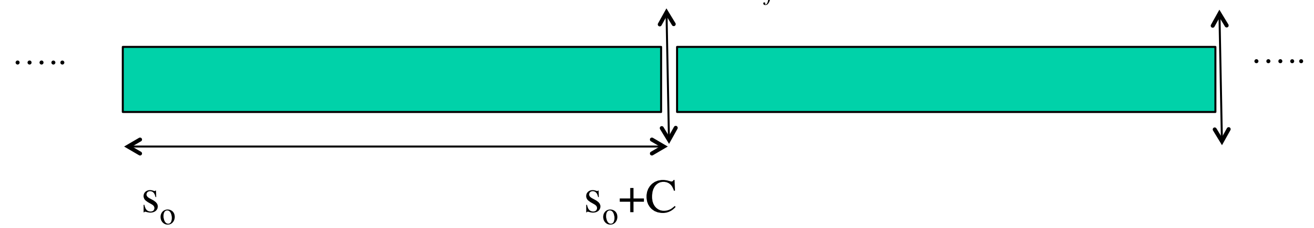
Just to drive it home: 1D case

$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \mathrm{Re}\left(\begin{bmatrix} w & w' + \frac{i}{w} \end{bmatrix}\begin{bmatrix} \delta K_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ w' + \frac{i}{w} \end{bmatrix}\right) ds =$$

$$\frac{1}{4\pi}\int_0^c w^2\,\delta K_1 ds = \frac{1}{4\pi}\int_0^c \beta(s)\,\delta K_1(s) ds$$

It worth comparing with a traditional way of doing this: introducing a weak thin lens (lumped focusing):

$$\delta K_1 \to (\delta K_1 l)\delta(s-s_o) = \frac{1}{f}\delta(s-s_o) \qquad \delta K_1 \to \delta(K_1 l)\delta(s-s_o)$$



$$T = I\cos\mu + J\sin\mu = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu + \alpha\sin\mu \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ -\delta K_1 l & 1 \end{bmatrix}\cdot T = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ \ldots\ldots & \cos\mu + \alpha\sin\mu - \beta K_1 l\sin\mu \end{bmatrix}$$

$$Trace T' = Trace T - \beta K_1 l \sin\mu;$$

$$\cos\mu' = \cos\mu - \frac{\beta\delta K_1 l}{2}\sin\mu; \beta K_1 l << 1 \to \mu' = \mu + \delta\mu$$

$$\cos(\mu + \delta\mu) \cong \cos\mu - \delta\mu\sin\mu \to \delta\mu = \frac{\beta\delta K_1 l}{2} \equiv \frac{\beta}{2f}; \delta Q = \frac{\beta}{4\pi f}.$$

Synchro-beatron coupling.

When we were discussing 3D motion and synchrotron oscillations we arrived to the following Hamiltonian

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta + \delta\mathcal{H}_\tau$$

$$\mathcal{H}_\beta = \frac{\pi_{x\beta}^{\;2} + \pi_{y\beta}^{\;2}}{2} + F\frac{x_\beta^{\;2}}{2} + N x_\beta y_\beta + G\frac{y_\beta^{\;2}}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\;2}}{2} = c_\tau\frac{\pi_\tau^{\;2}}{2}$$

$$\delta\mathcal{H}_\tau = \frac{e}{p_o c}\sum_n \frac{|E_n|\cos\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};$$

$$\tau = \tilde{\tau} + \tau_{add};\ \tau_{add} = \eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta = \eta^T \mathbf{S} X$$

$$\frac{d\pi_\tau}{ds} = -\frac{\partial(\delta H)}{\partial\tilde{\tau}} = \frac{e}{p_o c}\sum_n \frac{|E_n|\sin\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};$$

To make it solvable we superficially removed τ_{add}

$$\mathcal{H}_s = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\;2}}{2} + \frac{e}{p_o c}\frac{\mathbf{E_o}(s)\cos\left(hk_o\tau + \varphi_o\right)}{hk_o}$$

or in linear case

$$\mathcal{H}_{sL} = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\;2}}{2} - hk_o\frac{\tau^2}{2}\frac{e\mathbf{E_o}(s)}{p_o c}\cos\varphi_o$$

We identify that such “removal” is valid is RF system is located at dispersion-free place, η =0. We further simplified the situation and replaces the RF cavity with the energy kick:

$$\mathbf{E_o}(s) = V_{RF}\delta(s - s_{RF})$$

Than we found that stability $0 < u\eta_c < 2;\ u = -\dfrac{eV_{rf}hk_0}{p_o c}\cos\varphi_0;\ \ \cos\varphi_0 \pm 1$

$$\mu_s = \sin^{-1}\sqrt{u\eta_c - \frac{(u\eta_c)^2}{4}};\beta_\tau = abs\left(\frac{\eta_c}{\sin\mu_s}\right);\alpha_\tau = \frac{|u\eta_c|}{2\sin\mu_s}$$

or in the case of slow synchrotron oscillations

$$\mu_s \cong \sqrt{u\eta_c};\beta_\tau = \sqrt{\left|\frac{\eta_c}{u}\right|};\alpha_\tau = 0;$$

In this approximation (weak longitudinal focusing) we can estimate effect on the transverse betatron motion if RF system is installed where $\eta \neq 0$.

$$\delta\mathcal{H} = u\delta(s - s_{rf})\left(\frac{\tau^2 - \tilde{\tau}^2}{2}\right) = u\delta(s - s_{rf})\cdot\left(\frac{\tau^2{}_{add}}{2} - \tilde{\tau}\tau_{add}\right);$$

$$\tau_{add} = \eta^T\mathbf{S}X; X = \mathrm{Re}\left(a_1Y_1e^{i\psi_1} + a_2Y_2e^{i\psi_2}\right);\ \tilde{\tau} == \mathrm{Re}\,w_\tau a_\tau e^{i\psi_s};\psi_s = \frac{\mu_s}{C}.$$

First, let's notice that term

$$\left\langle \tilde{\tau}\tau_{add} \right\rangle = \left\langle \mathrm{Re}\left(a_1 Y_1 e^{i\psi_1} + a_2 Y_2 e^{i\psi_2} \right) \mathrm{Re}\, w_\tau a_\tau e^{i\psi_s} \right\rangle$$

contains only oscillating terms like $\psi_{1,2} \pm \psi_s$ and averages to zero. While the second term

$$\left\langle \tau^2{}_{add} \right\rangle = \left\langle \left(\frac{a_1 \eta^T \mathbf{S} Y_1 e^{i\psi_1} + a_2 \eta^T \mathbf{S} Y_2 e^{i\psi_2} + c.c.}{2} \right)^2 \right\rangle =$$

$$\frac{|a_1|^2 \left|\eta^T \mathbf{S} Y_1\right|^2 + |a_2|^2 \left|\eta^T \mathbf{S} Y_2\right|^2}{2} = I_1 \left|\eta^T \mathbf{S} Y_1\right|^2 + I_2 \left|\eta^T \mathbf{S} Y_2\right|^2$$

$$\left\langle \delta\mathcal{H} \right\rangle == u\delta\left(s - s_{rf}\right) \cdot \frac{I_1 \left|\eta^T \mathbf{S} Y_1\right|^2 + I_2 \left|\eta^T \mathbf{S} Y_2\right|^2}{2};$$

$$\varphi_k^{/} = \frac{\partial \mathcal{H}}{\partial I_2} = \frac{u}{2} \cdot \left|\eta^T \mathbf{S} Y_k\right|^2 \delta\left(s - s_{rf}\right);$$

$$\Delta\mu_k = \frac{u}{2} \cdot \left| \eta\left(s_{rf}\right)^T \mathbf{S} Y_k\left(s_{rf}\right) \right|^2.$$

Finally in combination with $\mu_s^{\;2} = u\eta_c$ we can show that betatron tunes shift is indeed

$$\Delta\mu_k = \frac{u}{2}\cdot\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2 = \frac{\mu_s^{\;2}}{2}\frac{\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2}{\eta_c}$$

proportional to $\mu_s^{\;2}$ and can be positive or negative depending on the "longitudinal mass" sign, e.g. the sign of η_c. We will see expression $\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2$ responsible for coupling between betatron and synchrony degrees of motion in many occasions. It is worth mentioning that it has dimension of length, L:

$$\eta^T \mathbf{S}Y_k \to w_{kx}\eta_{px}, w'_{kx}\eta_x, \eta_x / w_{kx};$$

$$\dim\left(\eta_x / w_{kx}\right)^2 = \dim\left(\frac{\eta_x^{\;2}}{\beta_{kx}}\right)^2 = \frac{L^2}{L} = L,$$

which just proves that previous equation is indeed has right dimensionally….