Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

What we learned:

For a distributed dipole field error:

$$X_{\rm co}(s) = \sqrt{\beta(s)} \sum_{k=-\infty}^{\infty} \frac{\nu^2 f_k}{\nu^2 - k^2} \, e^{jk\phi(s)}$$

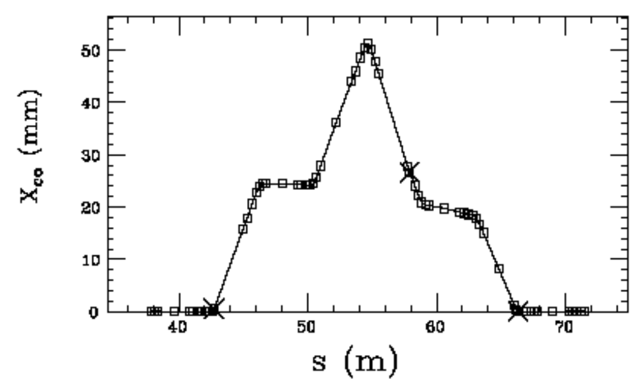
Where the field error is expanded in Fourier series $\left[\beta^{3/2}(\phi)\dfrac{\Delta B(\phi)}{B\rho}\right] = \sum_{k=-\infty}^{\infty} f_k e^{jk\phi}$

$$f_k = \frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi = \frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$\text{Sensitivity factor} \equiv \frac{\left\langle \left(X_{\rm co}(s)\right)^2 \right\rangle^{1/2}}{\theta_{rms}} \propto \sqrt{\beta(s)}$$

closed orbit bump: $X_{co}(s_f) = 0$, $X'_{co}(s_f) = 0$

$$\Delta x_{co}(s) = \left\{\sqrt{\beta_x(s_k)\beta_x(s)}\sin(\Delta\psi_x(s))\right\}\theta_k$$



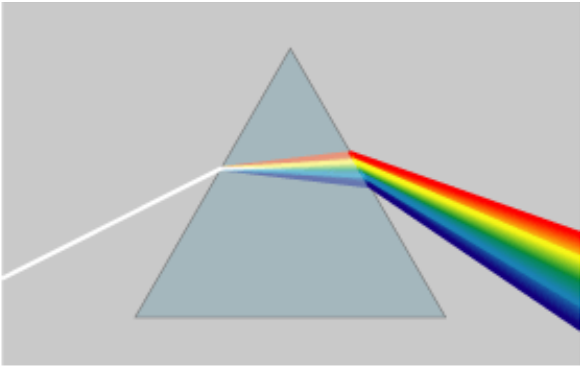
Orbit length change:

$$\Delta C = C - C_0 = \theta_0 \oint \frac{G_x(s,s_0)}{\rho} ds = D(s_0)\theta_0$$

$$\Delta C = \oint D(s_0)\frac{\Delta B_y(s_0)}{B\rho} ds_0$$

Off-momentum and dispersion

For different particle energy $$\delta = \frac{p - p_0}{p_0}$$



$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

Extend the matrix representation to 3 by 3

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole (K=0):

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathrm{L} & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

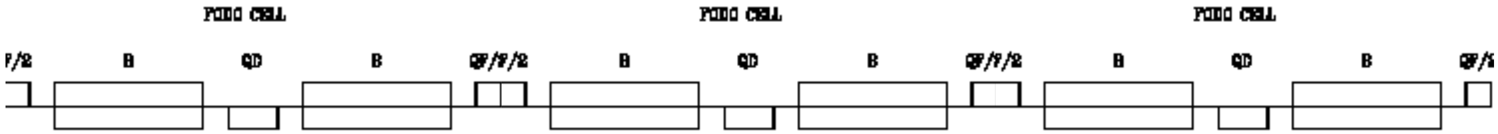
$$\theta << 1 \quad i.e. \quad L << \rho$$

For quadrupoles:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing
change K -> -K

FODO cell



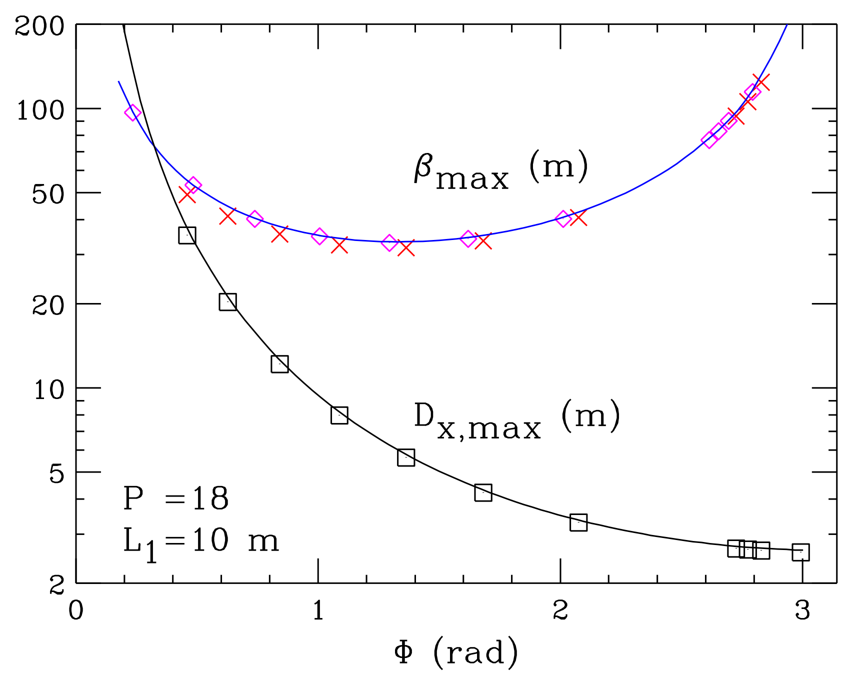
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D_F \\ D_F' \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{L^2}{2f^2} & 2L(1+\dfrac{L}{2f}) & 2L\theta(1+\dfrac{L}{4f}) \\ -\dfrac{L}{2f^2}+\dfrac{L^2}{4f^3} & 1-\dfrac{L^2}{2f^2} & 2\theta(1-\dfrac{L}{4f}-\dfrac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_F \\ D_F' \\ 1 \end{pmatrix}$$

$$D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D_F' = 0$$

$$\beta_{\max} = \frac{2L_1(1+\frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$



rf dipole

$$x'' + K(s)x = \theta_a \sin\omega_m t \sum_{n=-\infty}^{\infty} \delta(s - nC)$$

where $\theta_a = \Delta BL/B\rho$ is the kick angle and ωm is the angular frequency of the rf dipole. By Applying Floquet transformation,

$$\eta = \frac{x}{\sqrt{\beta}}, \quad \phi = \frac{1}{\nu}\int_0^s \frac{1}{\beta} ds \qquad \frac{d^2\eta}{d\phi^2} + \nu^2\eta = \frac{\nu\sqrt{\beta_0}\theta_a}{2\pi} \sum_{n=-\infty}^{\infty} \sin(n + \nu_m)\phi$$

where νm=ωm/ω0 is the modulation tune and we have relation $\delta(s - nC) = \frac{1}{|ds/d\phi|}\delta(\phi - 2\pi n)$

thus solution can be expressed as

$$\eta = A\cos\nu\phi + B\sin\nu\phi + \eta_{co}$$

$$\eta_{co} = \sum_{n=-\infty}^{\infty} \frac{\nu\sqrt{\beta_0}\theta_a}{2\pi[\nu^2 - (n + \nu_m)^2]} \sin(n + \nu_m)\phi$$

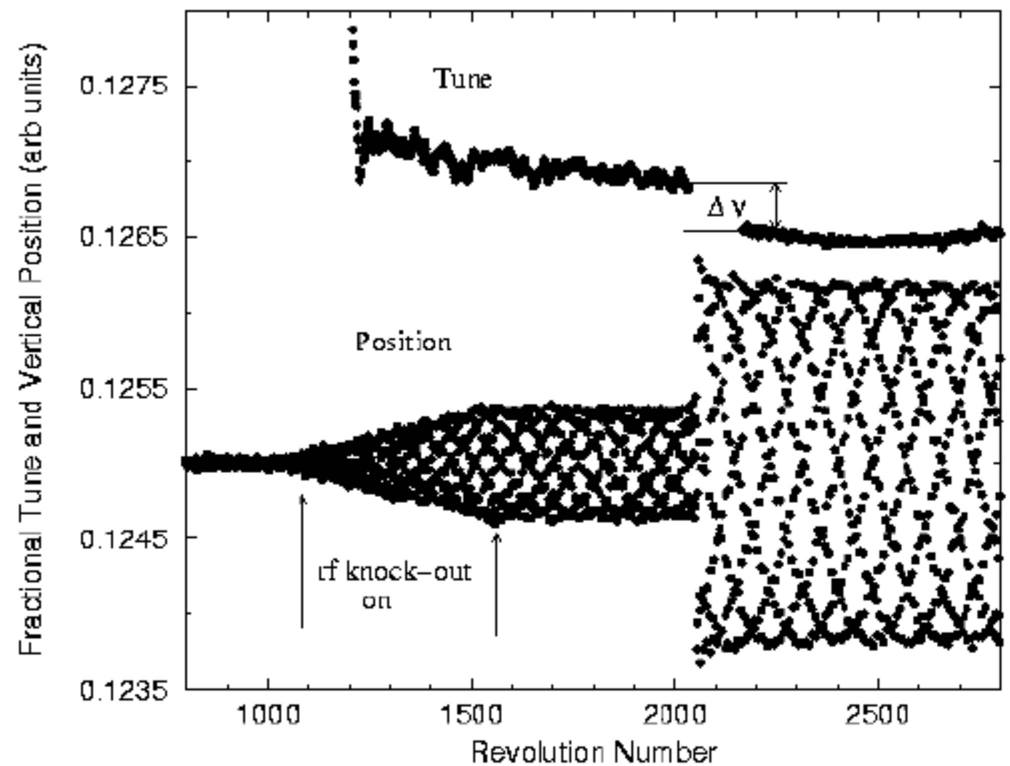
The discrete nature of the localized kicker generates error harmonics n + ν_m for all n ∈ (−∞,∞). If the betatron tune is 8.8, large betatron oscillations can be generated by an rf dipole at any of the following modulation tunes: ν_m = 0.2, 0.8, 1.2, 1.8, ... The localized repetitive kicks generate sidebands around the revolution lines.

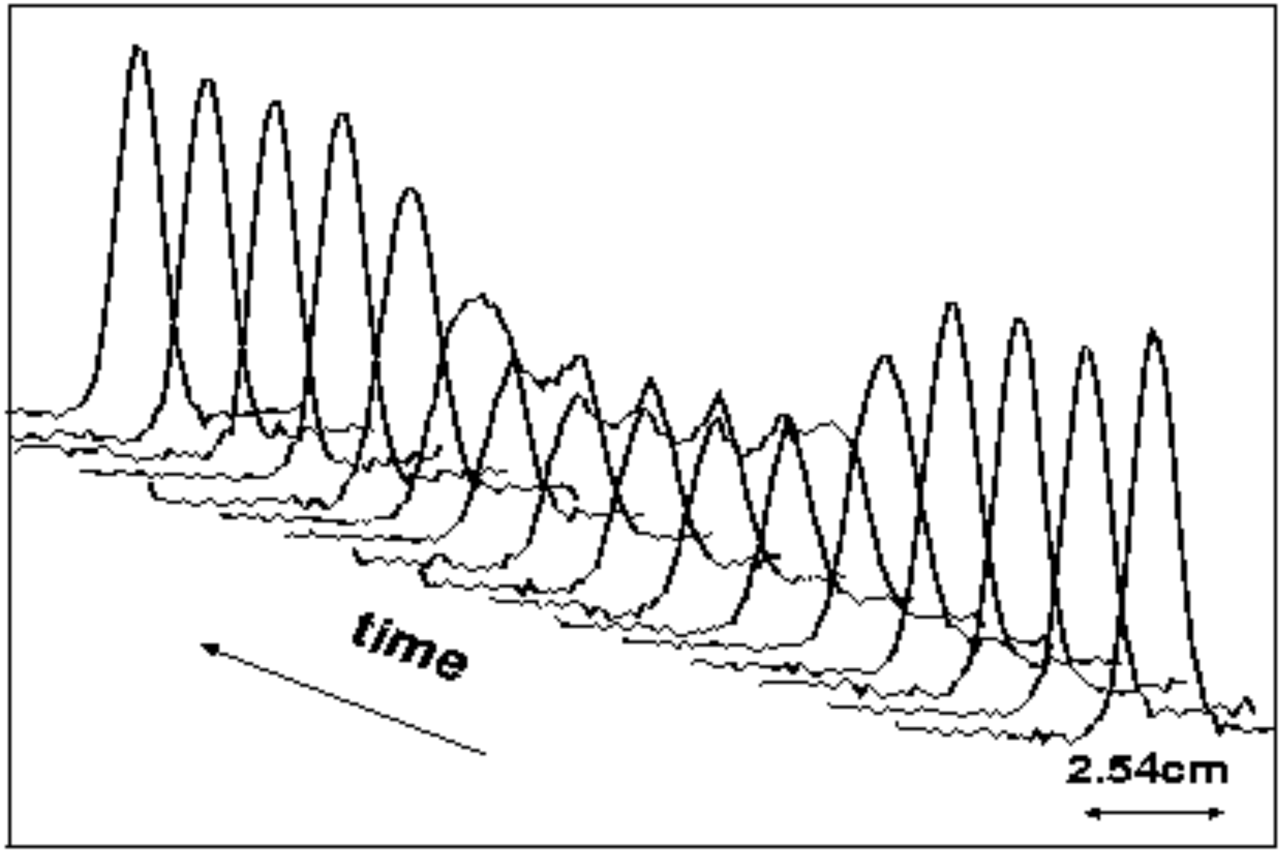
The coherent betatron motion of the beam in the presence of an rf dipole at $\nu_m \approx \nu$ (moduol 1) with initial condition x = x′ = 0 is

$$x(s) = \frac{\sqrt{\beta(s)\beta_0}\theta_a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\nu^2-(n+\nu_m)^2}[\nu\sin(n+\nu_m)\phi-(n+\nu_m)\sin\nu\phi]$$

$$\approx -\frac{\sqrt{\beta(s)\beta_0}\theta_a}{2\pi}\frac{s}{R}\cos\frac{\nu s}{R}+\cdots$$

The lower curve shows the measured vertical betatron oscillations at one BPM in the IUCF Cooler resulting from an rf dipole kicker at the betatron frequency. The rf dipole was turned on for 512 revolutions, and the beam was imparted by a one-turn kicker after another 512 revolutions. The betatron amplitude grew linearly during the rf knockout-on time. The upper curve shows the fractional part of the betatron tune obtained by counting the phase advance in the phase-space map using data of two BPMs.





The beam profile measured from an ionization profile monitor (IPM) at the AGS during the adiabatic turn-on/off of an rf dipole. The beam profile appeared to be much larger during the time that the rf dipole was on because the profile was an integration of many coherent synchrotron oscillations. After the rf dipole was adiabatically turned off, the beam profile restored back to its original shape (@RHIC BNL).

Effect of quadrupole field error:

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho} \quad \rightarrow \quad X'' + (K_0 + k)X = 0$$

We assume that the transfer matrix of the unperturbed betatron can be described by

$$J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

$$M_0(s) = I\cos\Phi_0 + J\sin\Phi_0, \quad \Phi_0 = 2\pi\nu_0$$

The perturbation of a thin quadrupole can be described by

$$m(s_1) = \begin{pmatrix} 1 & 0 \\ -k(s_1)ds_1 & 1 \end{pmatrix}$$

Sources of quadrupole field error:

(1) quadrupole and (2) closed orbit in sextupoles

Betatron tune shift

The transfer matrix of the one-turn map is $\mathbf{M}(s_1)=\mathbf{M}_0(s_1)m(s_1)$

$$M(s_1)=\begin{pmatrix} \cos\Phi_0+\alpha_1\sin\Phi_0-\beta_1\sin\Phi_0 k(s_1)ds_1 & \beta_1\sin\Phi_0 \\ -\gamma_1\sin\Phi_0-[\cos\Phi_0+\alpha_1\sin\Phi_0]k(s_1)ds_1 & \cos\Phi_0-\alpha_1\sin\Phi_0 \end{pmatrix}$$

$$\cos\Phi=\cos\Phi_0-\frac{1}{2}\beta_1 k(s_1)ds_1\sin\Phi_0$$

$$\Delta\Phi\approx\frac{1}{2}\beta_1 k(s_1)ds_1,\quad \Delta\nu\approx\frac{1}{4\pi}\beta_1 k(s_1)ds_1,\quad \Delta\nu\approx\frac{1}{4\pi}\oint\beta_1 k(s_1)ds_1$$

The betatron tune of the accelerator is changed by the quadrupole field error.

The betatron amplitude function is also changed by the quadrupole field error. The betatron amplitude function can be obtained by a one-turn map, i.e.

$$\mathrm{M}(s_2) = M(s_2 + C, s_1) m(s_1) M(s_1, s_2)$$

$$\Delta\left[\mathrm{M}(s_2)\right]_{12} = -k(s_1) ds_1 \beta_1 \beta_2 \sin \nu_0 (\phi_1 - \phi_2) \sin \nu_0 (2\pi + \phi_2 - \phi_1)$$

$$= -\frac{1}{2} k(s_1) ds_1 \beta_1 \beta_2 \left(\cos 2\nu_0 (\pi + \phi_2 - \phi_1) - \cos 2\pi\nu_0\right)$$

$$\Delta\left[\mathrm{M}(s_2)\right]_{12} = \Delta\left[\beta_2 \sin \Phi\right] \cong \Delta\beta_2 \sin \Phi_0 + \beta_2 \cos \Phi_0 \Delta\Phi$$

$$= \Delta\beta_2 \sin \Phi_0 + \frac{1}{2} \beta_1 k(s_1) ds_1 \beta_2 \cos \Phi_0$$

$$\frac{\Delta\beta_2}{\beta_2} = -\frac{1}{2 \sin \Phi_0} \beta_1 k(s_1) ds_1 \cos 2\nu_0 (\pi + \phi_2 - \phi_1)$$

For a distributed quadrupole field error, the perturbation to the betatron amplitude function becomes

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{1}{2 \sin \Phi_0} \int_s^{s+C} ds_1 \beta_1 k(s_1) \cos 2\nu_0 (\pi + \phi_2 - \phi_1)$$

$$\begin{aligned}\frac{\Delta\beta(s)}{\beta(s)} &= -\frac{1}{2\sin\Phi_0}\int_s^{s+C} k(s_1)\beta(s_1)\cos[2\nu_0(\pi+\phi-\phi_1)]\,ds_1 \\ &= -\frac{\nu_0}{2\sin\Phi_0}\int_\phi^{\phi+2\pi} k(\phi_1)\beta^2(\phi_1)\cos[2\nu_0(\pi+\phi-\phi_1)]\,d\phi_1,\end{aligned}$$

Where $\phi = \frac{1}{\nu}\int_0^s \frac{1}{\beta} ds$, it is easy to verify that

$$\frac{d^2}{d\phi^2}\left[\frac{\Delta\beta(s)}{\beta(s)}\right] + 4\nu_0^2\left[\frac{\Delta\beta(s)}{\beta(s)}\right] = -2\nu_0^2\beta^2 k(s).$$

Note that the betatron amplitude function diverges when the betatron tune is integer or half-integer!

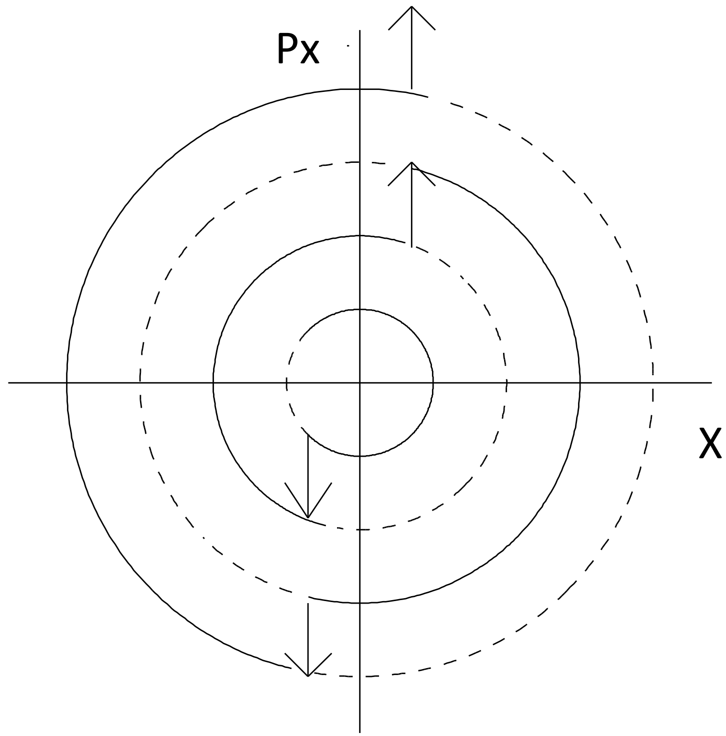
Half integer stop bands

$$\left[\nu_0\beta^2 k(s)\right] = \sum_{p=-\infty}^{\infty} J_p e^{jp\varphi},$$

Half integer stopband integral

$$J_p = \frac{1}{2\pi}\oint\left[\beta k(s)\right] e^{-jp\varphi} ds$$

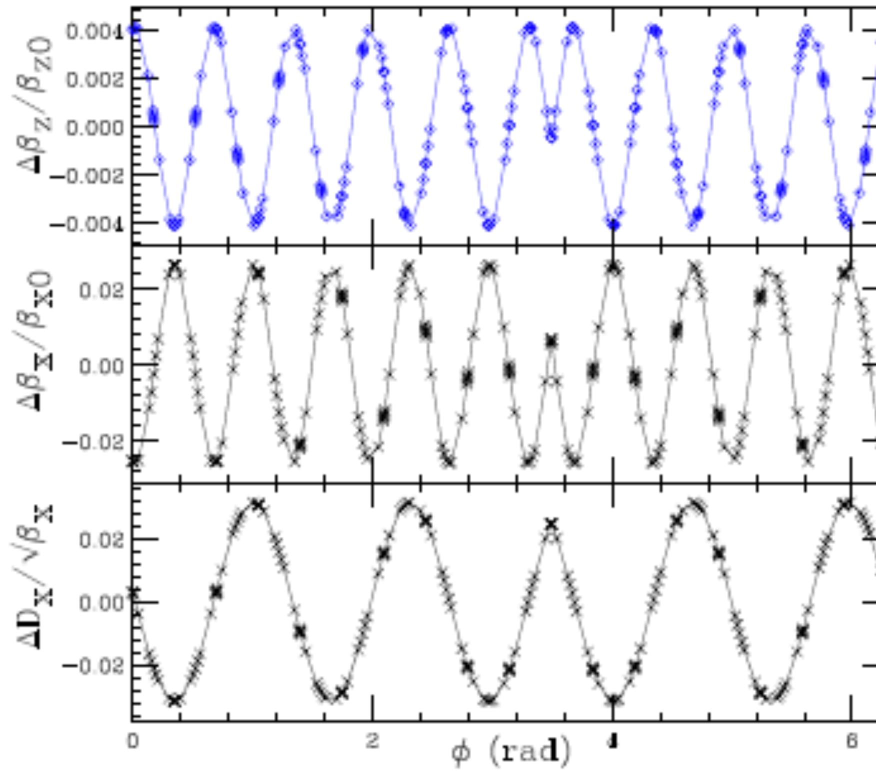
$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi}$$



Schematic plot of a particle trajectory at a half-integer betatron tune resulting from an error quadrupole kick $p_X = \beta_X \Delta X' = -\beta_X X/f$, where f is the focal length, X is the displacement from the quadrupole center, and β_X is the betatron amplitude function at the quadrupole. The quadrupole kick is proportional to the displacement X. At a half-integer betatron tune, the betatron coordinate changes sign in each consecutive revolution and the kick angles coherently add in each revolution to produce unstable particle motion.

Example of one quadrupole error in FODO cell lattice

Consider a simple accelerator lattice made of 18 FODO cells with half cell length 10-m, and dipole length 8 m bending angle 10°. The betatron tunes are set at ν_x=4.79302 and ν_z=4.78298 by quadrupoles. Now, consider an 1% decrease in focusing quadrupole strength at the end of the 10th cell.

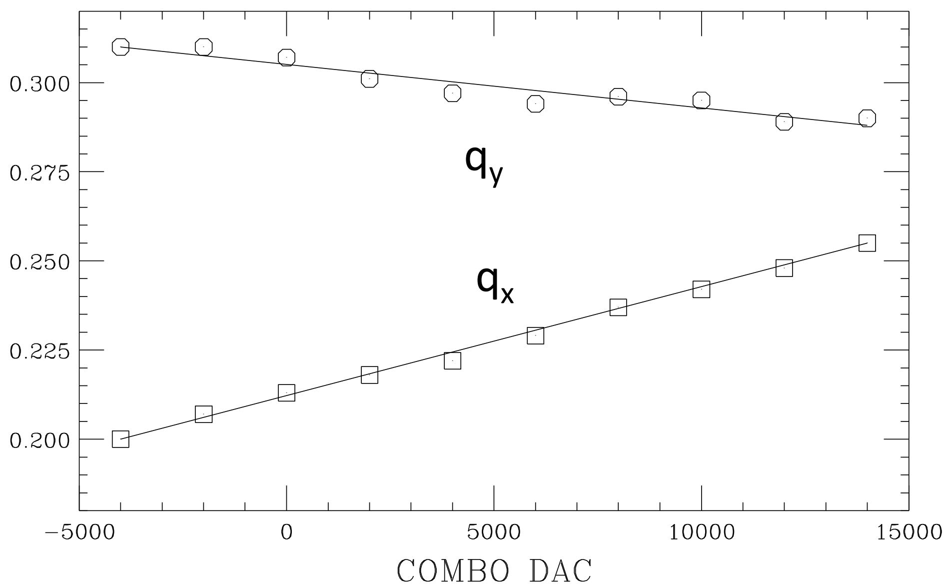


Perturbation of betatron amplitude functions vs ϕ (either ϕ_x or ϕ_y) resulting from 1% decrease in gradient strength of the 10th focusing quadrupole. The betatron amplitude function perturbation is dominated by harmonics nearest [$2\nu_x$] and [$2\nu_y$]. Since β_x/β_y~6.37 at the focusing quadrupole location, the resulting error Δβx/βx is about 6.37Δβy/βy. A single kick at the error quadrupole location can be identified in the top 2 plots. The bottom plot shows the effect of quadrupole error on dispersion function shown as $\Delta D_x/\sqrt{\beta_x}$ vs $\phi=\phi_x$. A single kick at the error quadrupole location is visible to the dispersion closed orbit.

Applications of quadrupole error

$$\Delta\nu \approx \frac{1}{4\pi}\oint \beta_1 k(s_1)\,ds_1$$

1. Betatron amplitude function measurement



$$\langle \beta_{x,y} \rangle = 4\pi \frac{\Delta\nu_{x,y}}{\Delta Kl}$$

The horizontal and vertical tunes, determined by the FFT spectrum of the betatron oscillations, vs quadrupole field strength. The slope can be used to determine the **average** betatron amplitude function in a quadrupole.

The fractional parts of betatron tunes were $q_x=4-\nu_x$ and $q_y=5-\nu_y$. The experimental result of fractional horizontal tune appeared to "increase" with the strength of the quadrupole.

Q: Is the quadrupole focusing or defocusing? At this location, what can you say about the betatron amplitude functions?

2. Tune jump

$$\Delta\nu = \frac{1}{4\pi}\oint \beta_1 \frac{\Delta B_1}{B\rho}\,ds_1$$