PHY 564
Advanced Accelerator Physics
Lecture 1 part 2

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Lecture 1: Particles In Electromagnetic Fields.

Fundamentals of Hamiltonian Mechanics

http://en.wikipedia.org/wiki/Hamilton_principle

1.0. Least-Action Principle and Hamiltonian Mechanics

Let us refresh our knowledge of some aspects of **the Least-Action Principle** (LAP is humorously termed the *coach potato principle*) **and Hamiltonian Mechanics**. **The Principle of Least Action** is the most general formulation of laws governing the motion (evolution) of systems of particles and fields in physics. In mechanics, it is known as **the Hamilton's Principle,** and states the following**:**

1) A mechanical system with n degrees of freedom is fully characterized by a monotonic generalized coordinate, ***t***, the full set of n coordinates $q = \{q_1, q_2, q_3 \ldots q_n\}$ and their derivatives $\dot{q} = \{\dot{q}_1, \dot{q}_2, \dot{q}_3 .. \dot{q}_n\}$ that are denoted by dots above a letter. We study the dynamics of the system with respect to ***t***. All the coordinates, $q = \{q_1, q_2, q_3 \ldots q_n\}$; $\dot{q} = \{\dot{q}_1, \dot{q}_2, \dot{q}_3 .. \dot{q}_n\}$ should be treated as a functions of ***t*** that itself should be treated as an independent variable.

2) Each mechanical system can be fully characterized by the **Action Integral**:

$$S(A,B) = \int_A^B L(q,\dot{q},t)\,dt \qquad (1)$$

that is taken between two events A and B described by full set of coordinates* (q,t). The function under integral $L(q,\dot{q},t)$ is called the system's **Lagrangian function.** Any system is fully described by its action integral.

After that, applying **Lagrangian mechanics involves just** n second -order ordinary differential equations**:** $\ddot{q} = f(q,\dot{q})$.

We can find these equations, setting variation of δS_{AB} to zero:

$$\delta S_{AB} = \delta\left(\int_A^B L(q,\dot{q},t)dt\right) = \int_A^B \left\{\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right\}dt = \int_A^B \left\{\frac{\partial L}{\partial q}\delta q dt + \frac{\partial L}{\partial \dot{q}}\delta dq\right\} =$$

$$\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_A^B + \int_A^B \left\{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right\}\delta q dt = 0 \qquad ; \qquad (2)$$

and taking into account $\delta q(A) = \delta q(B) = 0$. Thus, we have integral of the function in the brackets, multiplied by an arbitrary function $\delta q(t)$ equals zero.

Therefore, we must conclude that the function in the brackets also equals zero and thus obtain **Lagrange's equations:**

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \qquad (3)$$

Explicitly, this represents a set of n second-order equations

$$\frac{d}{dt}\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} = \frac{\partial L(q,\dot{q},t)}{\partial q_i} \qquad \Leftrightarrow \sum_{j=1}^{n}\left(\ddot{q}_j \frac{\partial^2 L(q,\dot{q},t)}{\partial \dot{q}_i \partial \dot{q}_j} + \dot{q}_j \frac{\partial^2 L(q,\dot{q},t)}{\partial \dot{q}_i \partial q_j} + \frac{\partial^2 L(q,\dot{q},t)}{\partial \dot{q}_i \partial t} \right) = \frac{\partial L(q,\dot{q},t)}{\partial q_i}.$$

The partial derivative of the Lagrangian over $\dot{q}$ is called generalized (**canonical**) momentum:

$$P^i = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} \;\; or \;\; P = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\;; \qquad (4)$$

and the partial derivative of the Lagrangian over q is called the generalized force: $f^i = \dfrac{\partial L(q,\dot{q},t)}{\partial q_i}$: (4)

can be rewritten in more familiar form: $\dfrac{dP^i}{dt} = f^i$. Then, <u>by a definition</u>, the energy (Hamiltonian) of the system is:

$$H = \sum_{i=1}^{n} P^i \dot{q}_i - L \equiv \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L; \; L = \sum_{i=1}^{n} P^i \dot{q}_i - H. \qquad (5)$$

Even though the Lagrangian approach fully describes a mechanical system it has some significant limitations. It treats the coordinates and their derivatives differently, and allows only coordinate transformations $q' = q'(q,t)$. There is more powerful method, the <u>**Hamiltonian or Canonical Method**</u>. The Hamiltonian is considered as a function of coordinates and momenta, which are treated equally. Specifically, pairs of coordinates with their conjugate momenta (4) (q_i,P_i) or (q^i,P_i) are called canonical pairs. The Hamiltonian method creates many links between classical and quantum theory wherein it becomes an operator. Before using the Hamiltonian, let us prove that it is really function of (q,P,t): i.e., that the full differential of the Hamiltonian is

$$dH(q,P,t) = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial P^i} dP^i \right) + \frac{\partial H}{\partial t} dt. \qquad (6)$$

Using equation (5) explicitly, we can easily prove it:

$$dH = d\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - dL \equiv \sum_{i=1}^{n}\left\{\frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \dot{q}_i d\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt\right\} =$$

$$\sum_{i=1}^{n}\left\{\dot{q}_i d\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt\right\} = \sum_{i=1}^{n}\left\{\dot{q}_i dP^i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt\right\} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial P^i}dP^i\right) + \frac{\partial H}{\partial t}dt.$$

wherein we substitute $d(\partial L/\partial \dot{q}_i) = dP^i$ with the expression for generalized momentum. In addition to this proof, we find some ratios between the Hamiltonian and the Lagrangian:

$$\left.\frac{\partial H}{\partial q_i}\right|_{P=const} = -\left.\frac{\partial L}{\partial q_i}\right|_{\dot{q}=const};\ \left.\frac{\partial H}{\partial t}\right|_{P,q=const} = -\left.\frac{\partial L}{\partial t}\right|_{q,\dot{q}=const};\ \dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial P^i};$$

wherein we should very carefully and explicitly specify what type of partial derivative we use. For example, the Hamiltonian is function of (q, P, t): thus, partial derivative on q must be taken with constant momentum and time. For the Lagrangian, we should keep $\dot{q}, t = const$ to partially differentiate on q .

The last ratio gives us the first Hamilton's equation, while the second one comes from Lagrange's equation (5-11):

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial P^i};$$
$$\frac{dP^i}{dt} = \frac{d}{dt}\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} = \frac{dP^i}{dt} = \left.\frac{\partial L}{\partial q_i}\right|_{\dot{q}=const} = -\left.\frac{\partial H}{\partial q_i}\right|_{P=const}; \quad (7)$$

both of which are given in compact form below in (11).