PHY 564
Advanced Accelerator Physics
Lecture 22

Free Electron Lasers I: Introduction and FELs in Small Gain Regime

Vladimir N. Litvinenko
Yichao Jing
Gang Wang

CENTER for ACCELERATOR SCIENCE AND EDUCATION
Department of Physics & Astronomy, Stony Brook University
Collider-Accelerator Department, Brookhaven National Laboratory

Outline

- Introduction
  - What is free electron laser (FEL)
  - Applications and some FEL facilities
  - Basic setup
  - Different types of FEL
- How FEL works
  - Electrons' trajectory and resonant condition
  - Analysis of FEL process at small gain regime (Oscillator)

Introduction I: What is free electron lasers

- A free-electron laser (FEL), is a type of laser whose lasing medium consists of very-high-speed electrons moving freely through a magnetic structure, hence the term free electron.
- The free-electron laser was invented by John Madey in 1971 at Stanford University.
- Advantages:
  - ✓ Wide frequency range
  - ✓ Tunable frequency
  - ✓ May work without a mirror (SASE)
- Disadvantages: large, expensive

Introduction II: Applications and FEL facilities

- Medical, Biology (small wavelength and short pulse are required for imaging proteins), Military (~Mwatts)…
- FEL Facilities (~33):

FREE ELECTRON LASERS

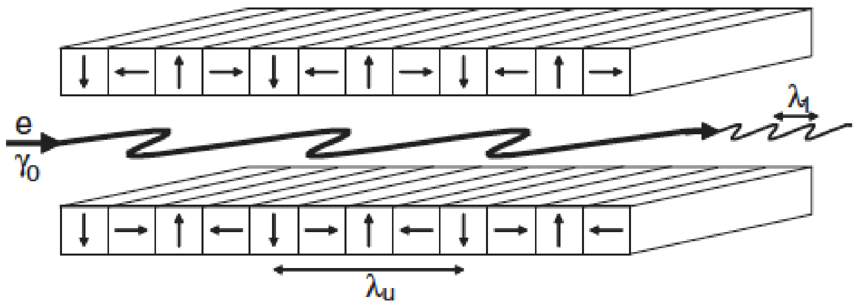
LOCATION	NAME	WAVELENGTHS	TYPE	STATUS
RIKEN (Japan)	SACLA FEL	0.63 - 3 Å	Linac	operating user facility
SLAC-SSRL (USA)	LCLS FEL	1.2 - 15 Å	Linac	operating user facility
DESY (Germany)	FLASH FEL	4.1 - 45 nm	SC Linac	operating user facility
ELETTRA Trieste, Italy	FERMI	4 - 100 nm	Linac	operating user facility
SDL(NSLS) Brookhaven (USA)	HGHG FEL	193 nm	Linac	operating experiment
Duke Univ. NC (USA)	OK-4	193 - 400 nm	storage ring	operating user facility
iFEL (Japan)	3 2 1 4 5	230 nm - 1.2 μm 1 - 6 μm 5 - 22 μm 20 - 60 μm 50 - 100 μm	linac	operating user facility
Univ. of Hawaii (USA)	MK-V	1.7 - 9.1 μm	linac	operating experiment
Vanderbilt TN (USA)	MK-III	2.1 - 9.8 μm	linac	no longer operating
Radboud University (Netherlands)	FLARE FELIX1 FELIX2	327 - 420 μm 3.1 - 35 μm 25 - 250 μm	linac	operating user facility
Stanford CA (USA)	SCA-FEL FIREFLY	3-10 μm 15-65 μm	SC-linac	no longer operating
LURE - Orsay (France)	CLIO	3 - 150 μm	linac	operating user facility
Jefferson Lab VA (USA)		3.2 - 4.8 μm 363 - 438 nm	SC-linac	operating user facility
Science Univ. of Tokyo (Japan)	FEL-SUT	5 - 16 μm	linac	operating user facility
FZ Rossendorf (Germany)		4-22 μm 18-250 μm		operating user facility
UCSB CA (USA)	FIR-FEL MM-FEL 30 μ-FEL	63 - 340 μm 340 μm - 2.5 mm 30 - 63 μm	electrostatic	operating user facility
ENEA - Frascati (Italy)		3.6 - 2.1mm	microtron	operating user facility
ETL - Tsukuba (Japan)	NIJI-IV	228 nm	storage ring	operating experiment
IMS - Okazaki (Japan)	UVSOR	239 nm	storage ring	operating experiment
Dortmund, Univ. (Germany)	Felicita 1	470 nm	storage ring	operating expriment
LANL NM (USA)	AFEL RAFEL	4 - 8 μm 16 μm	linac	operating experiment
Darmstadt Univ. (Germany)	IR-FEL	6.6 - 7.8 μm	SC-linac	operating experiment
IHEP (China)	Beijing FEL	5 - 25 μm	linac	operating experiment
CEA - Bruyeres (France)	ELSA	18-24 μm	linac	operating experiment
ISIR - Osaka (Japan)		21-126 μm	linac	operating experiment
JAERI (Japan)		22 μm 6 mm	SC-linac induction linac	operating experiment
Univ. of Tokyo (Japan)	UT-FEL	43 μm	linac	operating experiment
ILE - Osaka (Japan)		47 μm	linac	operating experiment
LASTI (Japan)	LEENA	65 - 75 μm	linac	operating experiment
KAERI (Korea)		80 - 170 μm 10 mm	microtron electrostatic	operating experiment
Budker Inst. Novosibirsk, Russia		110 - 240 μm	linac	operating experiment
Univ. of Twente (Netherlands)	TEU-FEL	200-500 μm	linac	operating experiment
FOM (Netherlands)	Fusion FEM			no longer operating
Tel Aviv Univ. (Israel)		3 mm	electrostatic	operating experiment

[1]So far only operating FEL oscillators with wavelength < 10 mm are included.
[2]"user facility" means a dedicated scientific research facility open to outside researchers.
[3]Order is first by type of facility and second roughly by wavelength.

Introduction III: Basic Setup

Planar undulator

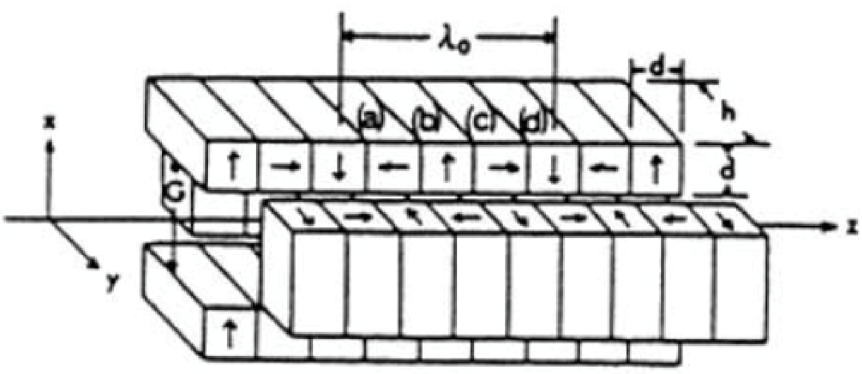
$$B_y(x,y,z) = B_0 \sin(k_u z)$$
for $x, y << $ gap size

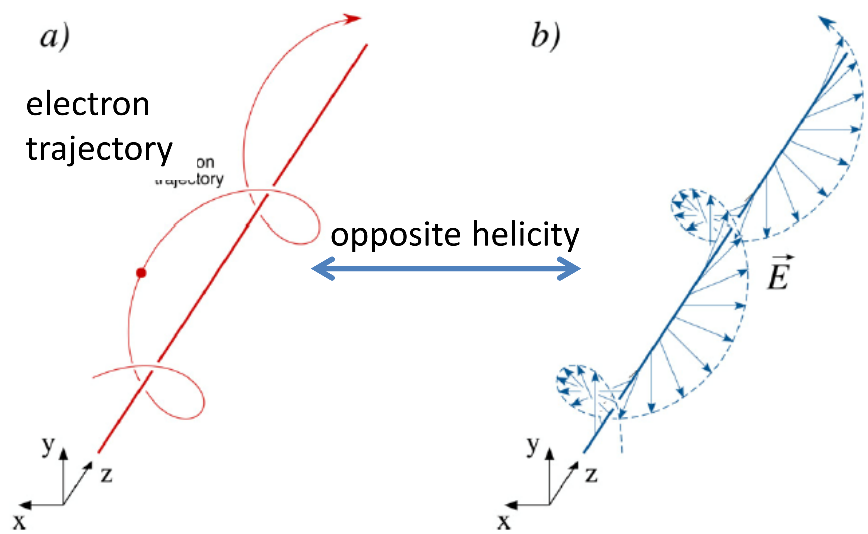


Helical wiggler for CeC PoP

Helical undulator

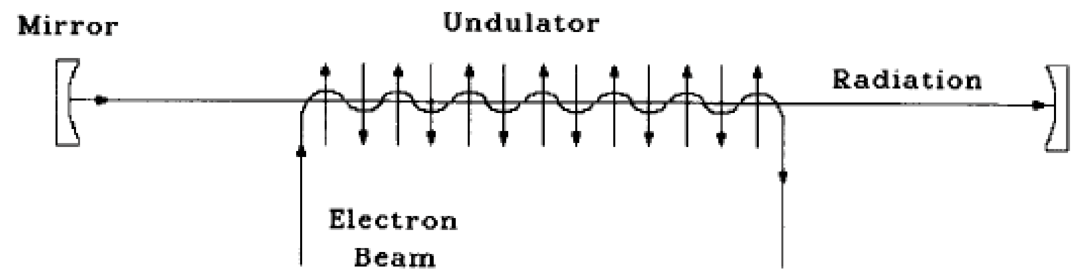
$$B_x(x,y,z) = B_0 \cos(k_u z)$$
$$B_y(x,y,z) = B_0 \sin(k_u z)$$
for $x, y << $ gap size



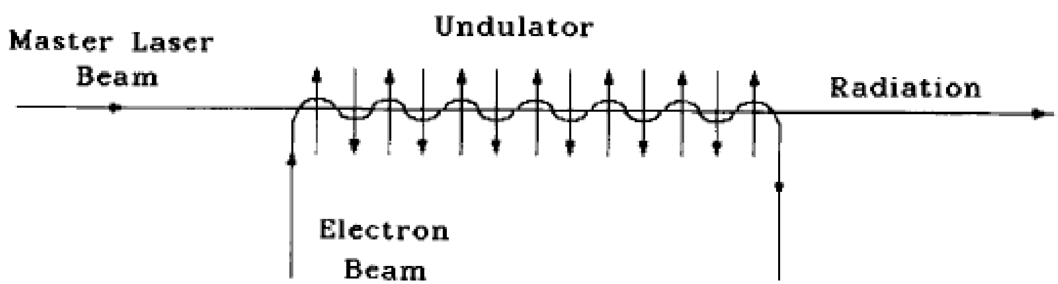


Introduction IV: different types of FEL

FEL Oscillator
(Low gain regime)

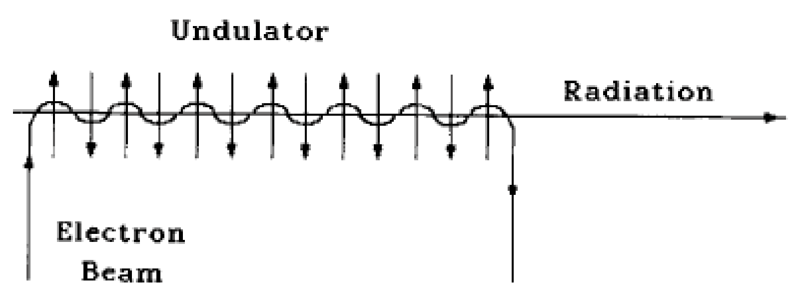


FEL Amplifier
(High gain regime)



SASE FEL
(High gain regime)

Self-Amplified Spontaneous Emittion (SASE)



Unperturbed Electron motion in helical wiggler (in the absence of radiation field)

$$\vec{B}_w(x,y,z) = B_w\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{F}(x,y,z) = -e\vec{v}\times\vec{B} = -ev_z\hat{z}\times\vec{B} = -ev_z B_w\left[\cos(k_u z)\hat{y} + \sin(k_u z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt} = m\gamma\frac{dv_x}{dt} = -ev_z B_w\sin(k_u z) \qquad \frac{d(m\gamma v_y)}{dt} = m\gamma\frac{dv_y}{dt} = -ev_z B_w\cos(k_u z)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad \tilde{v} \equiv v_x + iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt} = -iev_z B_w\left(\cos(k_u z) - i\sin(k_u z)\right) = -iev_z B_w e^{-ik_u z}$$

$$m\gamma\frac{d\tilde{v}}{dt} = m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dz} = -iev_z B_w e^{-ik_u z} \Rightarrow m\gamma\frac{d\tilde{v}}{dz} = -ieB_w e^{-ik_u z}$$

Undulator parameter, also called a_w

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

Electron rotation angle in undulator:

$$\theta_s = K/\gamma$$

$$\frac{\tilde{v}(z)}{c} = \frac{-ieB_w}{mc\gamma}\int e^{-ik_u z_1}dz_1 = \frac{eB_w}{mc\gamma k_u}e^{-ik_u z} = \frac{K}{\gamma}e^{-ik_u z}$$

*Assume the initial velocity of the electron make the integral constant vanishing.

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right] \qquad v_z = const. \qquad \vec{x}(z) = \int_0^z \vec{v}(t_1)dt_1 + \vec{x}(z=0)$$

Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

Consider a circularly polarized electromagnetic wave (plane wave is an assumption for 1D analysis, which is usually valid for near axis analysis) propogating along z direction

$$\vec{E}_\perp(z,t) = E\left[\cos(kz-\omega t)\hat{x} + \sin(kz-\omega t)\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos\left(k(z-ct)\right)\hat{x} + \sin\left(k(z-ct)\right)\hat{y}\right] \qquad \omega = kc$$

Energy change of an electron is given by

$$\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp \cdot \vec{E}_\perp$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \frac{c}{v_z}\cos(\psi) \approx -eE\theta_s \cos(\psi)$$

Pondermotive phase:

$$\psi = k_u z + k(z-ct)$$

To the leading order, electrons move with constant velocity and hence $z = v_z(t-t_0)$

Resonant Radiation Wavelength

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

We define the resonant radiation wavelength such that

$$k_w + k_0 - k_0\frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w\left(\frac{c}{v_z} - 1\right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2/c^2 = 1 - \left(v_z^2 + v_\perp^2\right)/c^2 + v_\perp^2/c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2}\left(1 + K^2\right)$$

FEL resonant frequency:

$$\boxed{\lambda_0 \approx \frac{\lambda_w\left(1 + K^2\right)}{2\gamma^2}}$$

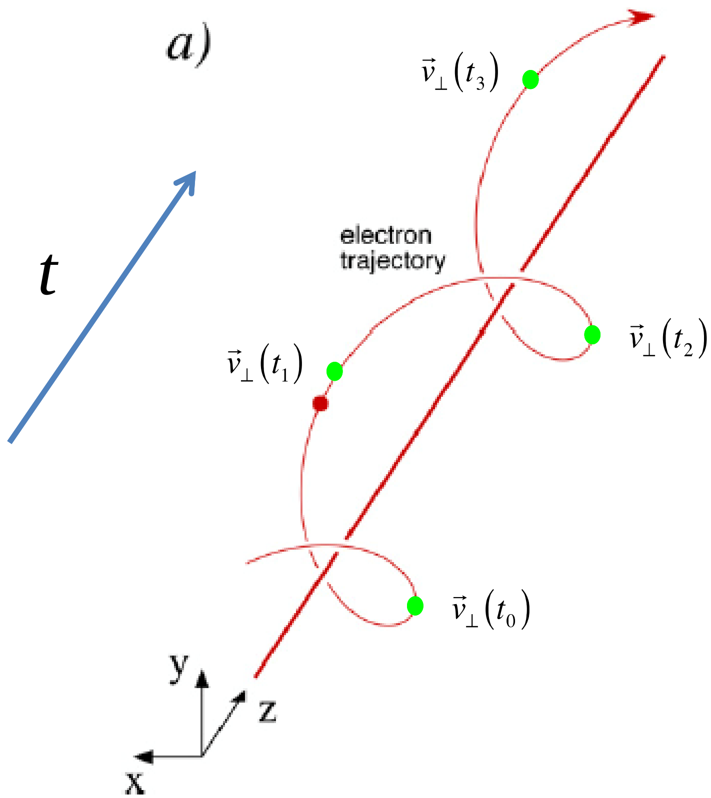
$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

At resonant frequency, the rotation of the electron and the radiation field is synchronized in the x-y plane and hence the energy exchange between them is most efficient.

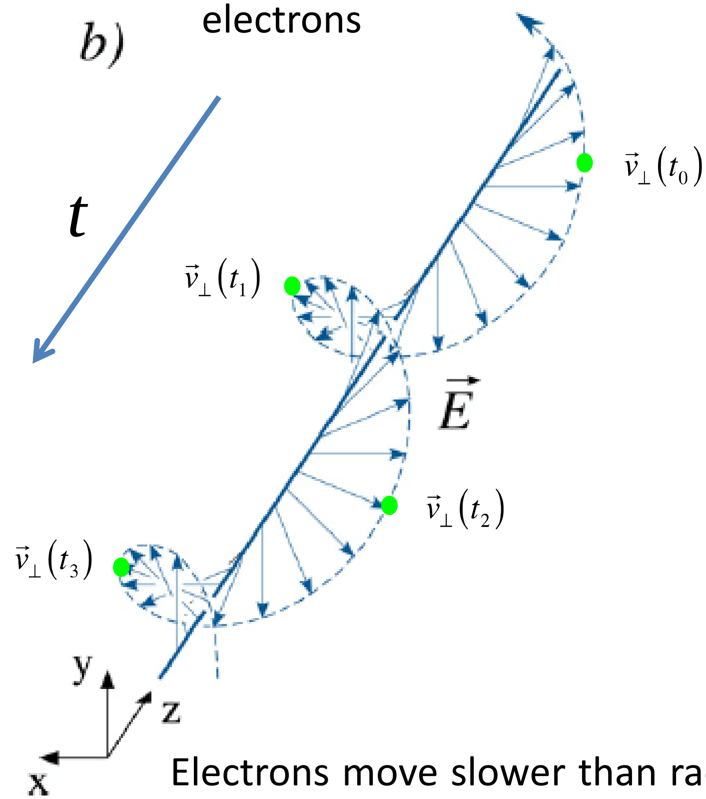
Helicity of radiation at synchronization

The synchronization requires opposite helicity of radiation with respect to the electrons' trajectories.

Electrons' trajectories



Radiation field observed by electrons



$$t_0 < t_1 < t_2 < t_3$$

Electrons move slower than radiation and hence see the radiation wave slipping ahead. As a result, the rotation direction of the radiation field seen by an electron is the same as its own rotation direction.

Longitudinal equation of motion

In the presence of the radiation field, the longitudinal equation of motion of an electron read

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos(\psi) \qquad \psi = k_w z + k(z - ct)$$

$\mathcal{E}_0$ is the average energy of the beam.

$$\frac{d}{dz}\psi = k_w + k - \frac{\omega}{v_z(\mathcal{E})}$$

$$\approx k_w + k - \omega\left[\frac{1}{v_z(\mathcal{E}_0)} + (\mathcal{E} - \mathcal{E}_0)\frac{d}{d\mathcal{E}}\frac{1}{v_z}\right]$$

$$\approx k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)} + \frac{\omega}{\gamma_z^2 c}\frac{(\mathcal{E} - \mathcal{E}_0)}{\mathcal{E}_0}$$

$$\frac{d}{d\mathcal{E}}\frac{1}{v_z} = \frac{1}{mc^3}\frac{d}{d\gamma}\frac{1}{\beta_z} = \frac{1}{mc^3}\frac{d\gamma_z}{d\gamma}\frac{d}{d\gamma_z}\frac{1}{\beta_z}$$

$$\gamma_z^2 = \frac{\gamma^2}{(1 + K^2)} \qquad \frac{d\gamma_z}{d\gamma} = \frac{\gamma}{\gamma_z(1 + K^2)}$$

$$\frac{d}{d\gamma_z}\frac{1}{\beta_z} = -\frac{1}{2\beta_z^3}\frac{d}{d\gamma_z}\left(1 - \frac{1}{\gamma_z^2}\right) = -\frac{1}{\beta_z^3\gamma_z^3}$$

$$\Rightarrow \begin{cases} \dfrac{dP}{dz} = -eE\theta_s \cos(\psi) \\ \dfrac{d}{dz}\psi \approx C + \dfrac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{cases}$$

Energy deviation: $P \equiv \mathcal{E} - \mathcal{E}_0$

Detuning parameter: $C \equiv k_w + k - \dfrac{\omega}{v_z(\mathcal{E}_0)}$

Low Gain Regime: Pendulum Equation

$$\left.\begin{aligned}\frac{dP}{dz}&=-eE\theta_s\cos(\psi)\\ \frac{d}{dz}\psi&=C+\frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P\end{aligned}\right\}\Rightarrow\qquad \frac{d^2}{dz^2}\psi+\frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\cos(\psi)=0$$

We assume that the change of the amplitude of the radiation field, E, is negligible and treat it as a constant over the whole interaction.

$$\frac{d^2}{d\hat{z}^2}\psi+\hat{u}\cos(\psi)=0\qquad \hat{u}=\frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\qquad \hat{z}=\frac{z}{l_w}$$

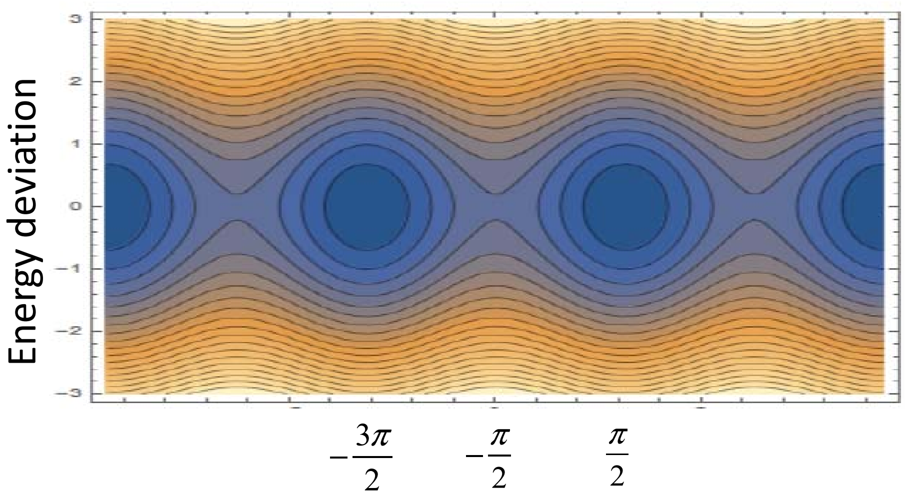
Pendulum equation:

$$\boxed{\frac{d^2}{d\hat{z}^2}\left(\psi+\frac{\pi}{2}\right)+\hat{u}\sin\left(\psi+\frac{\pi}{2}\right)=0}$$

Low Gain Regime: Similarity to Synchrotron Oscillation

FEL

ψ is the angle between the transverse velocity vector and the radiation field vector and hence there is no energy kick for $\psi = \pi/2$



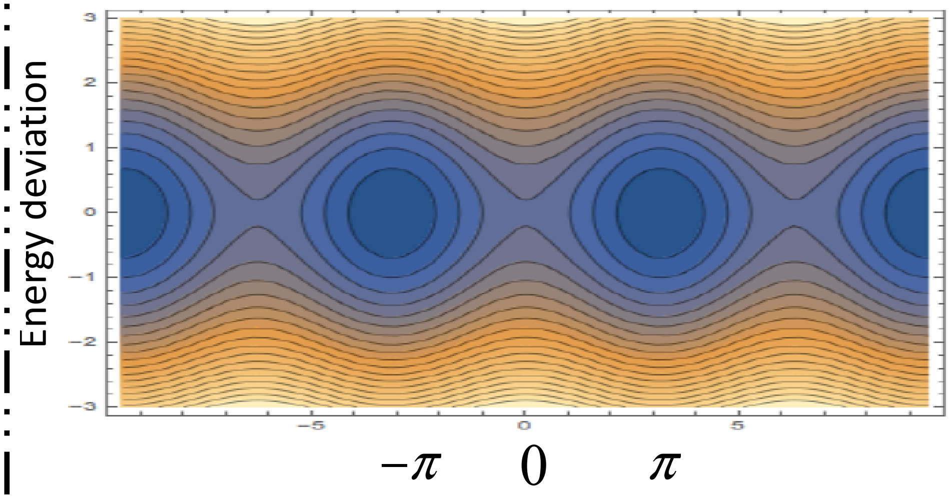
Pondermotive phase, ψ

$$\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0$$

$$\hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \psi = k_u z + k(z - ct)$$

Synchrotron Oscillation

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin(k_o h_{rf}\tau);$$

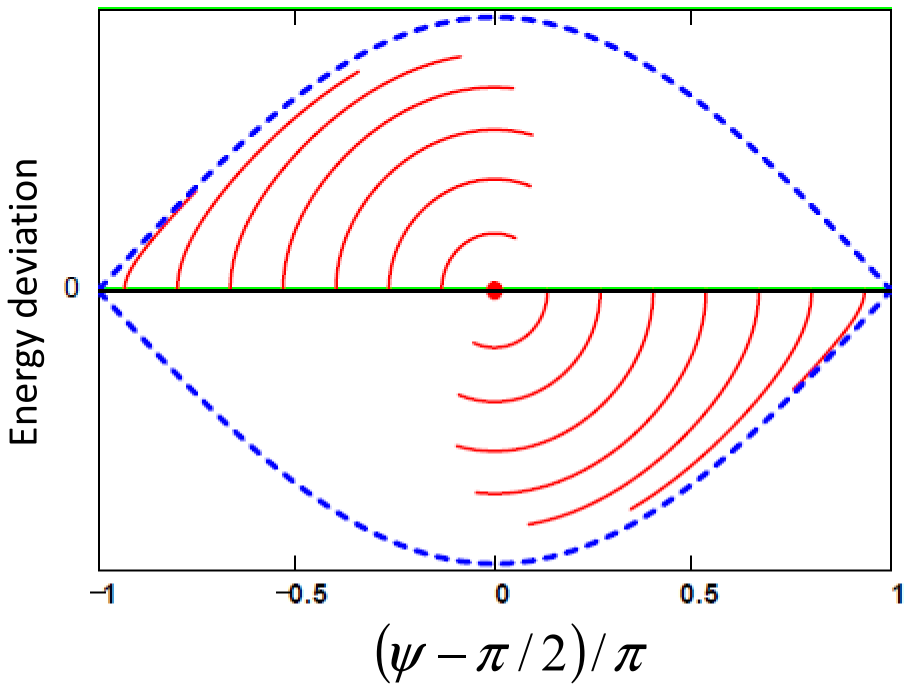


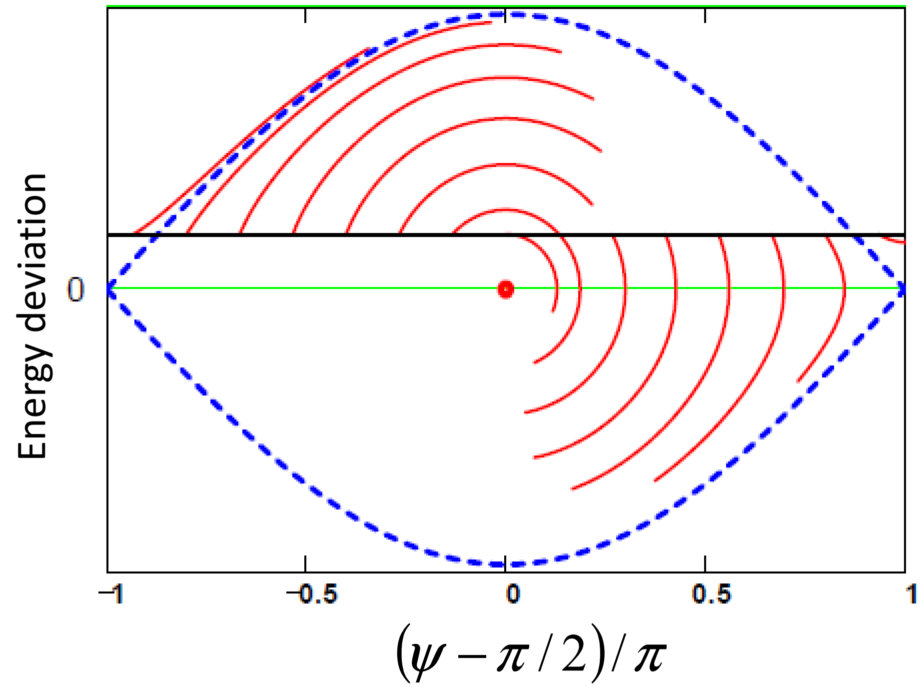
RF phase, ϕ_{rf}

$$\frac{d^2\phi_{rf}}{ds^2} = u_{rf}\sin\phi_{rf}$$

$$u_{rf} = \eta\frac{1}{C}\frac{eV_{RF}k_0 h_{rf}}{p_0 c} \qquad \phi_{rf} = k_0 h_{rf}\tau$$

Low Gain Regime: Qualitative Observation





The average energy of the electrons is right at resonant energy:

$$\lambda_0 \approx \frac{\lambda_w\left(1+K^2\right)}{2\gamma^2} \Rightarrow \gamma=\gamma_0=\sqrt{\frac{\lambda_w\left(1+K^2\right)}{2\lambda_0}}$$

The average energy of the electrons is slightly above the resonant energy:

$$\gamma=\gamma_0+\Delta\gamma$$

With positive detuning, there is net energy loss by electrons.

*Plots are taken from talk slides by Peter Schmuser.

Low Gain Regime: Derivation of FEL Gain

Change in radiation power density (energy gain per seconds per unit area):

$$\Delta\Pi_r = c\varepsilon_0\left(E_{ext}+\Delta E\right)^2 - c\varepsilon_0 E_{ext}^2 \approx 2c\varepsilon_0 E_{ext}\Delta E$$

Average change rate in electrons' energy per unit beam area:

$$\Delta\Pi_e = \frac{j_0\langle P\rangle}{e}$$

*The average, <…>, is over all electrons in the beam.

Energy deviation at entrance

Pondermotive phase at entrance

$$\langle P(z)\rangle = \int_{-\infty}^{\infty} dP_0 \int_0^{2\pi} d\psi_0\, f(P_0,\psi_0) P(P_0,\psi_0,z)$$

Assuming radiation has the same cross section area as the electron beam, we obtain the change in electric field amplitude:

$$\Delta\Pi_r + \Delta\Pi_e = 0 \Rightarrow \boxed{\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e}}$$

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s\cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\varepsilon_0}P \end{aligned}\right\} \Rightarrow \langle P\rangle = -eE\theta_s\left\langle \int_0^1 \cos[\psi(\hat{z})]d\hat{z}\right\rangle$$

Low Gain Regime: Derivation of FEL Gain

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos\psi = 0$$

$$\psi(\hat{z}) = \psi(0) + \psi'(0)\hat{z} - \hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\psi(\hat{z}_2)\,d\hat{z}_2 \qquad (1)$$

Assuming that all electrons have the same energy and uniformly distributed in the Pondermotive phase at the entrance of FEL: $P_0 = 0$ and $f(\psi_0) = \dfrac{1}{2\pi}$.

The zeroth order solution for phase evolution is given by ignoring the effects from FEL interaction:

$$\left.\begin{aligned} &\frac{dP}{dz} = -eE\theta_s\cos(\psi) \\ &\frac{d}{dz}\psi = C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{aligned}\right\} \Rightarrow \frac{d}{d\hat{z}}\psi = \hat{C} \Rightarrow \begin{cases} \psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} \\ \psi'(0) = \hat{C} \end{cases} \qquad \hat{C} \equiv Cl_w$$

Inserting the zeroth order solution back into eq. (1) yields the 1st order solution:

$$\psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0, \hat{z}) \qquad \Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

Low Energy Regime: Derivation of FEL Gain

$$\Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

$$= -\frac{\hat{u}}{\hat{C}^2}\left\{\int_0^{\hat{C}\hat{z}} \sin(\psi_0 + x_1)dx_1 - \hat{C}\hat{z}\sin\psi_0\right\} = \frac{\hat{u}}{\hat{C}^2}\left[\cos\left(\psi_0 + \hat{C}\hat{z}\right) - \cos\psi_0 + \hat{C}\hat{z}\sin\psi_0\right]$$

$$\langle P\rangle = -eEl_w\theta_s\left\langle \int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0,\hat{z})\right]d\hat{z}\right\rangle$$

← Average energy loss of electrons

$$= eE\theta_s l_w\left\langle \int_0^1 \sin\left[\psi_0 + \hat{C}\hat{z}\right]\sin(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle - eE\theta_s l_w\left\langle \int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\cos(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle$$

$$\approx eE\theta_s l_w\left\langle \int_0^1 \Delta\psi(\psi_0,\hat{z})\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle - \frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\int_0^{2\pi}\cos\left[\psi_0 + \hat{C}\hat{z}\right]d\psi_0$$

$$\left\langle \int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle = 0$$

$$= \frac{eE\theta_s l_w}{2\pi}\frac{\hat{u}}{\hat{C}^2}\int_0^1 d\hat{z}\left\{\hat{C}\hat{z}\cos\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\sin^2\psi_0 d\psi_0 - \sin\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\cos^2\psi_0 d\psi_0\right\}$$

$$= -eE\theta_s l_w\frac{\hat{u}}{\hat{C}^3}\left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

Low Energy Regime: Derivation of FEL Gain

Growth in the amplitude of radiation field:

$$\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e} = \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3 E_{ext}}{I_A}\frac{2}{\hat{C}^3}\left(1-\frac{\hat{C}}{2}\sin\hat{C}-\cos\hat{C}\right)$$

$$\hat{u} = \frac{l_w^2 e E_{ext}\theta_s\omega}{\gamma_z^2 c\gamma m c^2}$$

$$I_A = \frac{4\pi\varepsilon_0 m c^3}{e}$$

The gain is defined as the relative growth in radiation power:

$$g_s = \frac{\left(E_{ext}+\Delta E\right)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau\cdot f\left(\hat{C}\right)$$

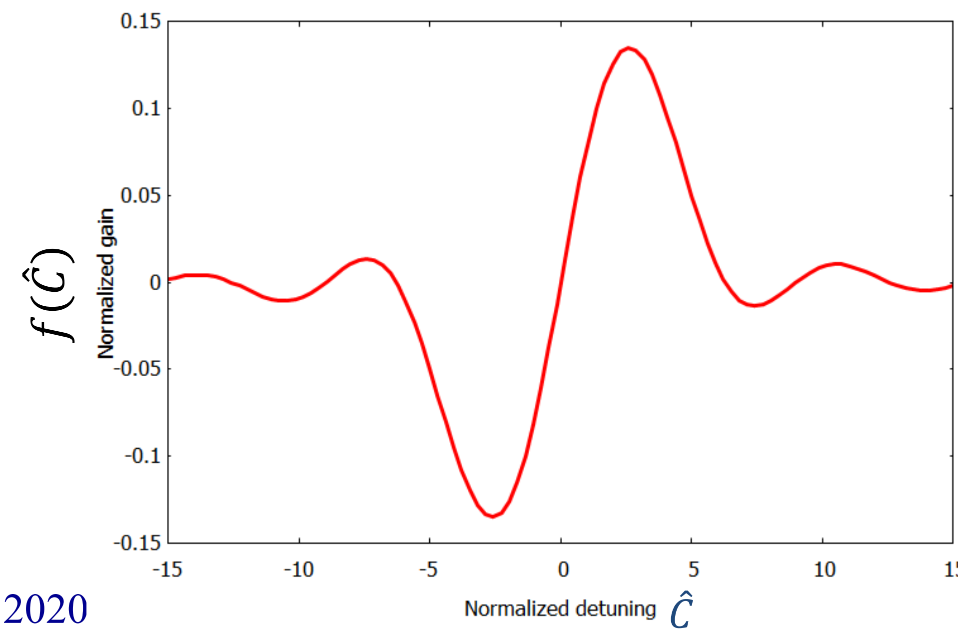
As observed earlier, there is no gain if the electrons has resonant energy.

$$\tau \equiv \frac{2\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A}$$

Cubic in FEL length

$$f\left(\hat{C}\right) = \frac{2}{\hat{C}^3}\left(1-\cos\hat{C}-\frac{\hat{C}}{2}\sin\hat{C}\right)$$

$$= -2\frac{d}{d\hat{C}}\frac{\sin^2\left(\hat{C}/2\right)}{\hat{C}^2}$$



References:

[1] ‘The Physics of Free Electron Lasers’ by E.L. Saldin, E.A. Schneidmiller and M.V. Yurkov;

[2] ‘Laser Handbook’, VOL 6 by W.B. Colson, C. Pellegrini and A. Renieri;

What we learned today

- What is a free electron laser? What are its advantages and disadvantages?
- We derived the trajectories of electrons inside a helical undulator of a free electron laser.
- We derived the resonant condition for a free electron laser to work, which determines the resonant wavelength of the free electron laser;
- We derived the gain of a free electron laser working in the low gain regime.