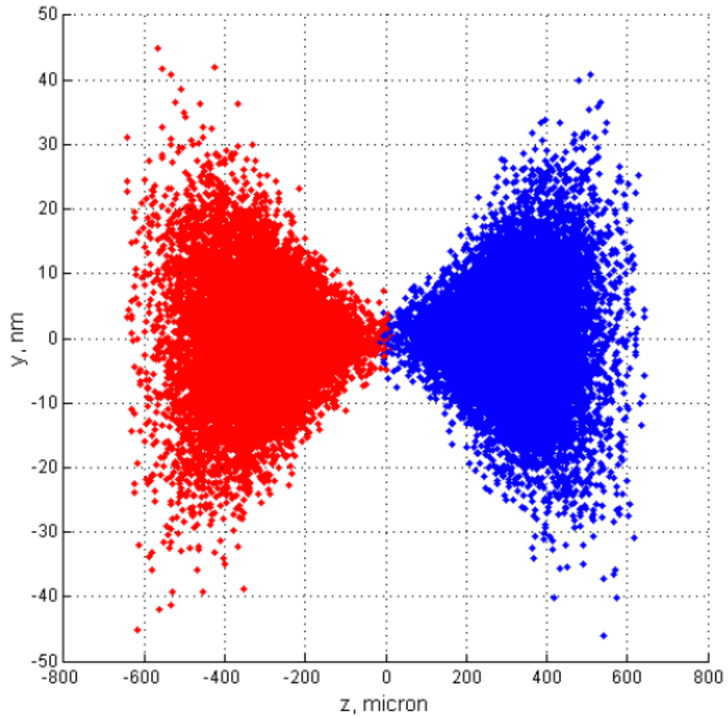
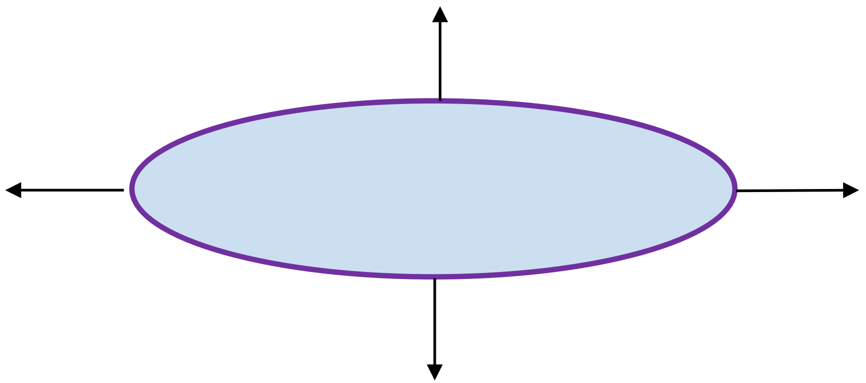
Beam-Beam Effect



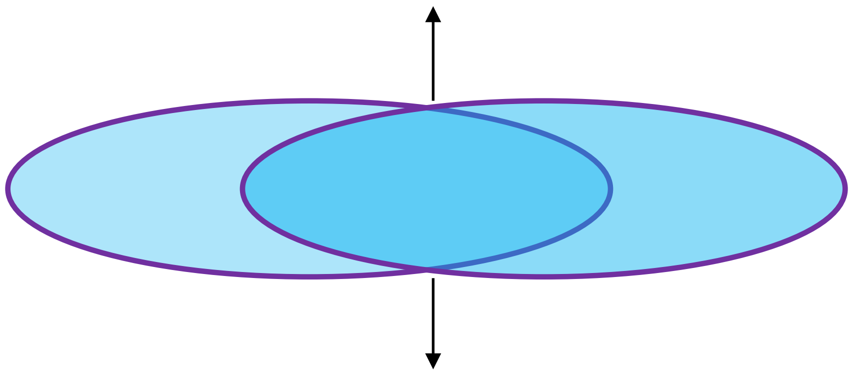
Kai Shih

(*graph cited from “Beam-Beam Effects in Particle Colliders”, Mauro Pivi, US Particle Accelerator School, 2011)

- Beam experiences mainly EM force inside a collider.
- Beam generate self EM field.
- For the force that produced by self EM field, call Space-Charge effect
- For the force that produced by other beam's EM field, call beam-beam effect
- Beam-beam effect is 2D for relativistic beams.



Space-Charge effect



beam-beam effect

- **Beam-Beam Force Derivation**
- **Beam-Beam Effect**
- **Beam-Beam Tune Shift**
- **Hour-Glass Effect**
- **Conclusion**

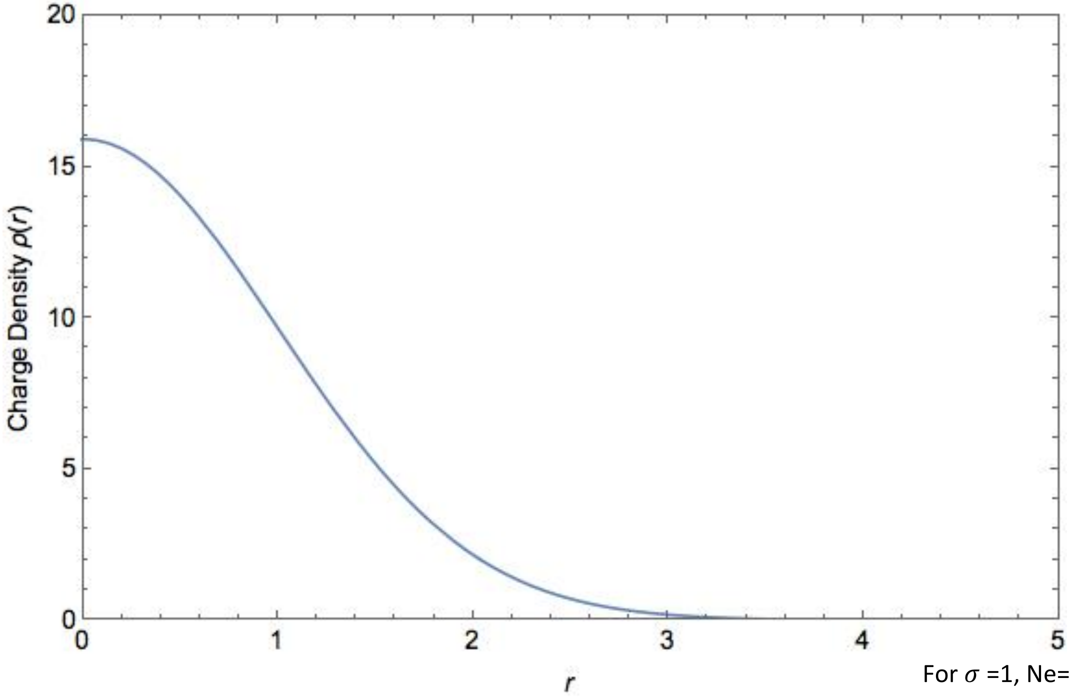
Beam-Beam Force Derivation

Let's consider a long round Gaussian Proton beam:

$$\rho(r) = \frac{Ne}{2\pi\sigma^2 L} Exp[-\frac{r^2}{2\sigma^2}]$$

The beam charge is uniform longitudinally, with total charge (Ne)

Transverse charge distribution of beam



For σ =1, Ne= 100

Assume Long Beam !

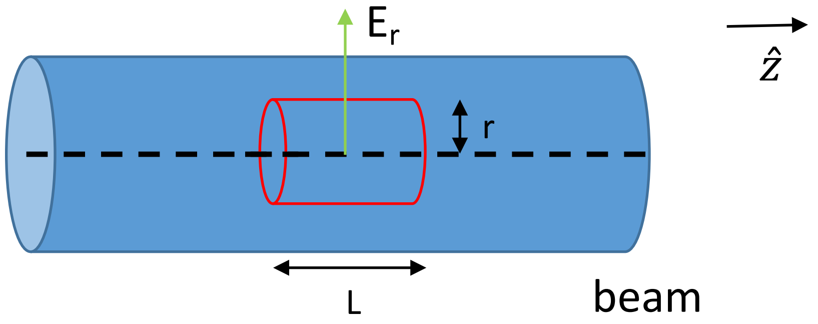
E field:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

$$E_r \times 2\pi r L = \frac{1}{\varepsilon_0} \iiint \rho(r) r d\phi dr dz$$

$$E_r \times 2\pi r L = \frac{1}{\varepsilon_0} \times 2\pi \int_0^r \frac{Ne}{2\pi\sigma^2} r' \, e^{-\frac{r'^2}{2\sigma^2}} dr'$$

$$E_r = \frac{Ne}{2\pi\varepsilon_0 r L}\left(1 - e^{-\frac{r'^2}{2\sigma^2}}\right)$$



Lab Frame

Assume Long Beam !

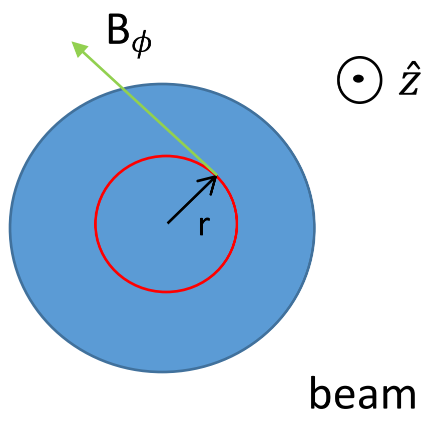
B field:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$B_\phi \times 2\pi r = \mu_0 \lambda v$$

$$B_\phi \times 2\pi r = \frac{\mu_0 v}{L} \int_0^{2\pi} \int_0^{r} \frac{Ne}{2\pi\sigma^2} r' \, e^{-\frac{r'^2}{2\sigma^2}} dr' d\phi$$

$$B_\phi = \frac{Ne\mu_0 v}{2\pi r L} \left(1 - e^{-\frac{r'^2}{2\sigma^2}}\right)$$



Self-field of the beam

$$\begin{cases} \vec{E} = \dfrac{Ne}{2\pi\varepsilon_0 rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\hat{r} \\ \vec{B} = \dfrac{Ne\mu_0 v}{2\pi rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\hat{\phi} \end{cases}$$

$\because \vec{F} = -e\,(\vec{E} + \vec{v}\times\vec{B})$

For $\beta \approx 1$, single electron

beam-beam effect:

$$\vec{F}_{bb} = -e\,(\vec{E} - v\hat{z}\times\vec{B})$$

$$= -\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\left(1 + \frac{v^2}{c^2}\right)$$

$$\boxed{\approx -\frac{Ne^2}{\pi\varepsilon_0 rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\hat{r}}$$

Space-Charge effect:

$$\vec{F}_{sc} = -e\,(\vec{E} + v\hat{z}\times\vec{B})$$

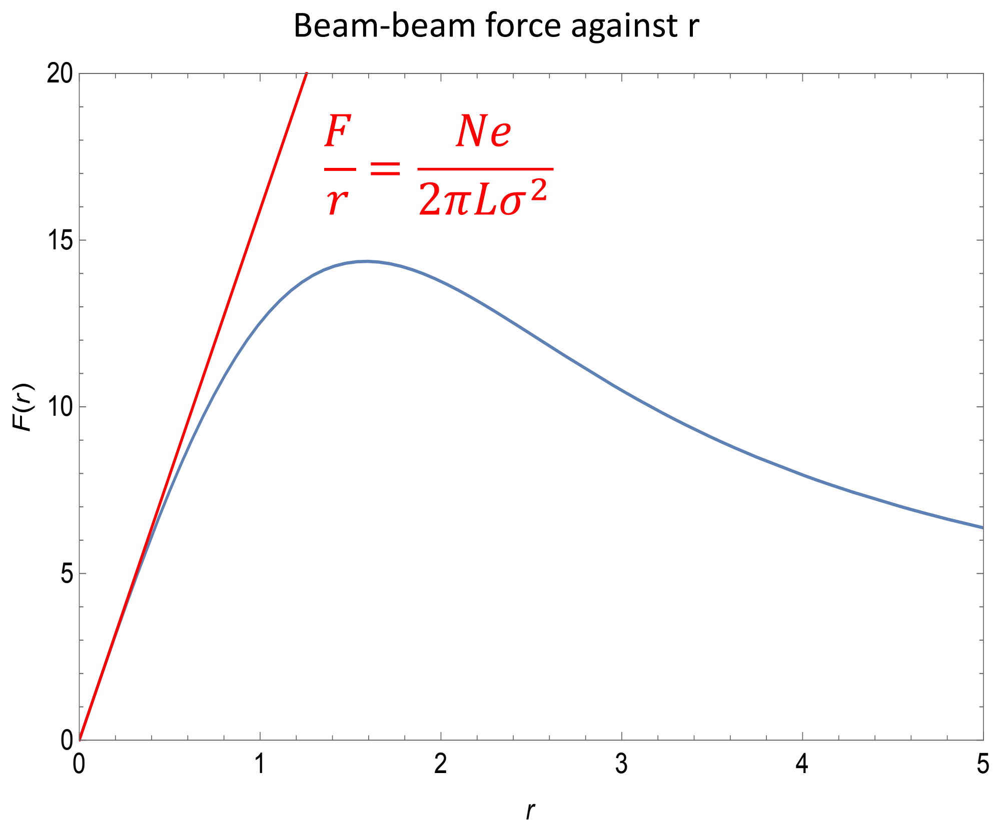
$$= -\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$= -\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\gamma^{-2}\hat{r}$$

$$\boxed{\approx 0}$$

Lab Frame

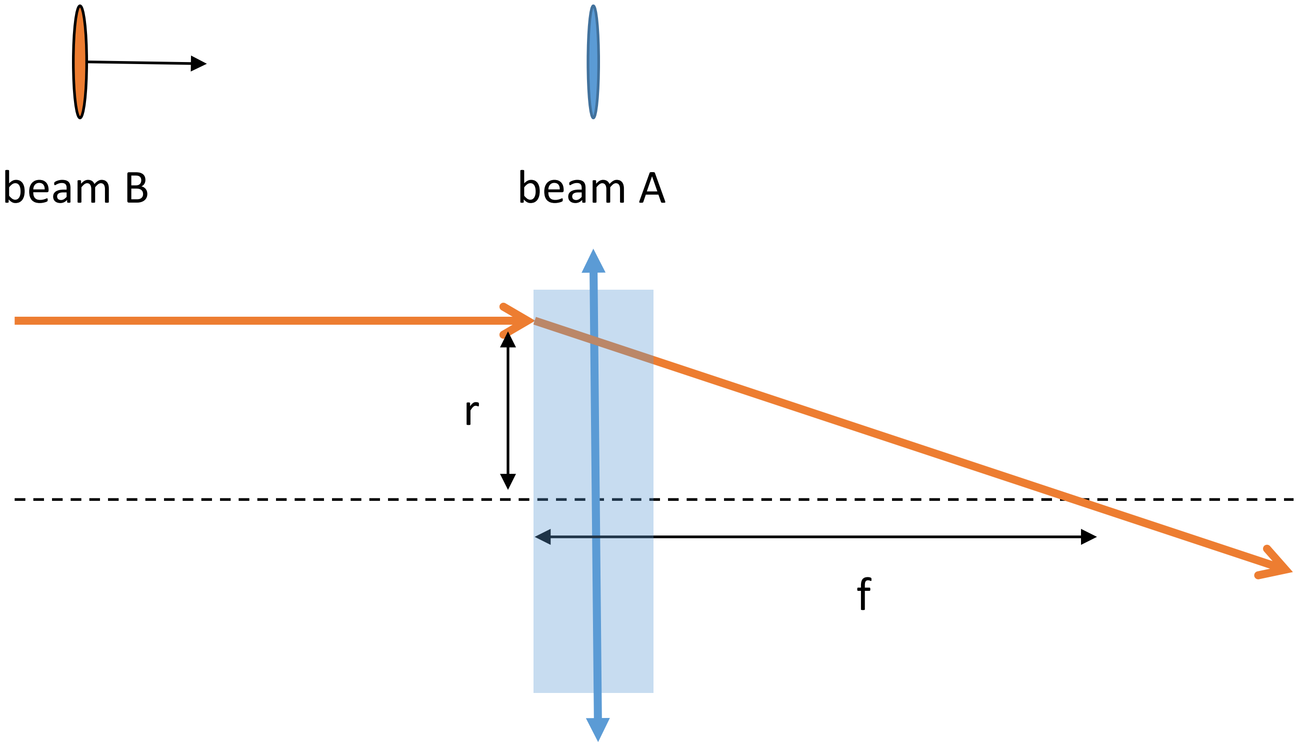
- It acts as a focusing/defocusing force, depends on the species of the beams.
- In small r, the beam-beam force is linear in the direction of +-$\hat{r}$.
- Beam-beam effect for Gaussian beam just like a small quadrupole error !



For σ =1, Ne= 100

Beam-Beam Effect

- Consider beam A, B (P^+ , e^-)is short round Gaussian beam
- It beam A act as a lens to beam B, vice versa.



$$\frac{r}{f} = \left|\frac{p_r}{p_0}\right|$$

Beam A Frame

$$\frac{r}{f} = \left|\frac{p_r}{p_0}\right|$$

$$\frac{1}{f} = \frac{1}{r} \cdot \frac{\left|\int_0^T F dt + 0\right|}{\gamma m c^2}$$

$$T \approx \frac{L}{2c}$$

$$= \frac{1}{r} \cdot \frac{|F|L}{2\gamma m c^3}$$

$$= \frac{1}{r} \cdot \frac{2Nr_e}{\gamma r}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)$$

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m c^2}$$

Assume Small r, force is linear in r.

$$\therefore \quad \frac{1}{f} \approx \frac{2Nr_e}{\gamma r^2}\left(1 - 1 + \frac{r^2}{2\sigma^2}\right)$$

$$\boxed{= \frac{N}{\sigma^2} \cdot \frac{r_e}{\gamma}}$$

!!! beam A contribution (lens)

!!! beam B contribution

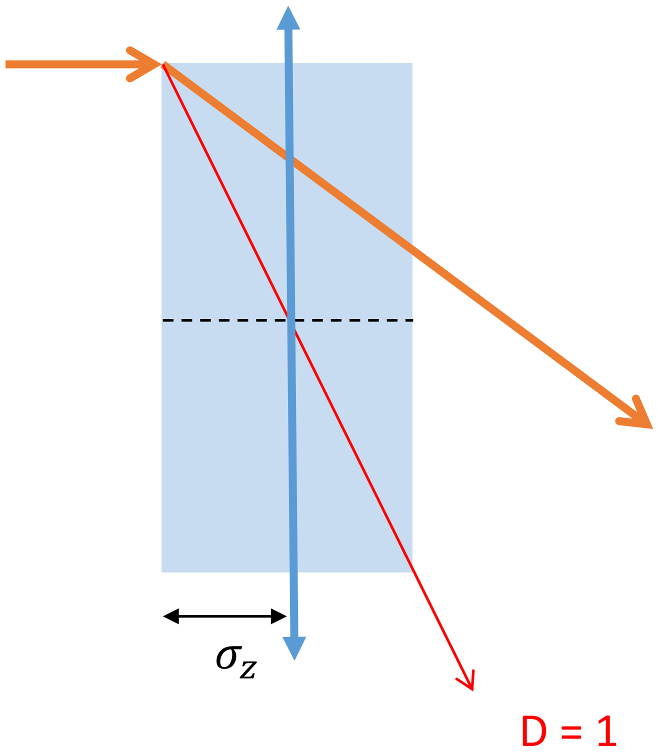
In general, $\sigma_x \neq \sigma_y$

$$\frac{1}{f_{x,y}} = \frac{2Nr_e}{\gamma \sigma_{x,y}(\sigma_x + \sigma_y)}$$

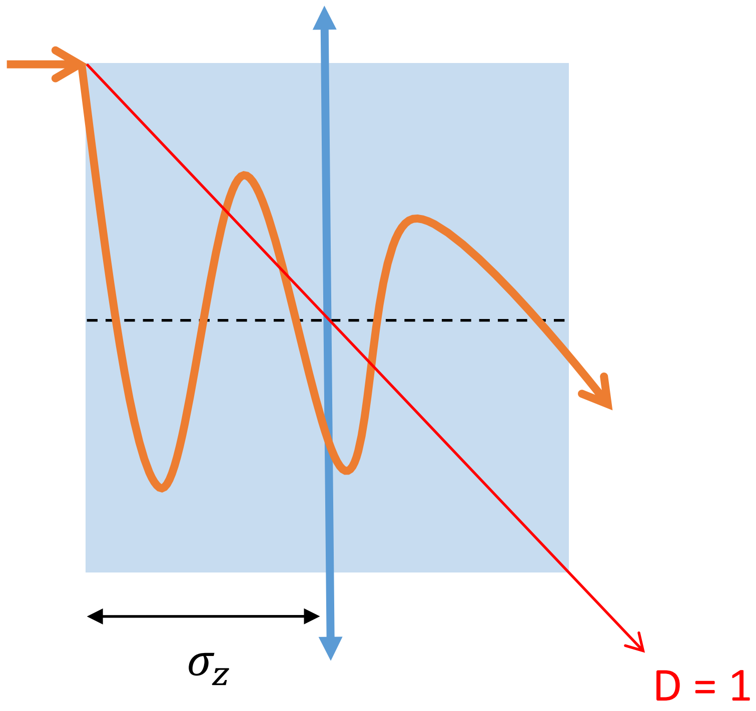
Beam A Frame

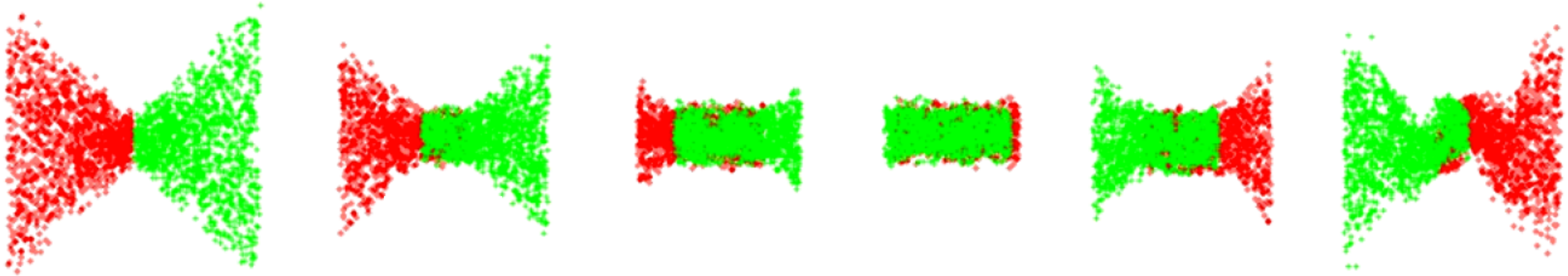
We can define a disruption parameter $D = \frac{\sigma_z}{f} = \frac{N r_e \sigma_z}{\gamma \sigma^2}$

For D < 1



For D >> 1





- Collision of two long beam
- Both beam experiences a focusing force
- Beams envelope are oscillating when they met

(*graph cited from “Beam-Beam Effects in Particle Colliders”, Mauro Pivi, US Particle Accelerator School, 2011)

Beam-Beam Tune Shift

$$M = \begin{bmatrix} \cos 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*} \sin 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix}$$

For D < 1

Beam–beam effect like a angular kick,
therefore the one turn mapping M_T:

$$K = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix} \quad M_T = \begin{bmatrix} \cos 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*} \sin 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$$

$$M_T = \begin{bmatrix} \cos 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*} \sin 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\pi Q_y - \frac{\beta^*}{f} \sin 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*} \sin 2\pi Q_y - \frac{1}{f} \cos 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix}$$

However, in general, beam-beam effect can also being treated as a perturbative tune shift in one turn:

$$M_P = \begin{bmatrix} \cos 2\pi(Q_y + \xi_y) & \beta^* \sin 2\pi \cos 2\pi(Q_y + \xi_y) \\ \frac{-1}{\beta^*} \sin 2\pi \cos 2\pi(Q_y + \xi_y) & \cos 2\pi(Q_y + \xi_y) \end{bmatrix}$$

$$\because \quad Tr(M_T) = Tr(M_P)$$

$$2\cos 2\pi Q_y - \frac{\beta^*}{f}\sin 2\pi Q_y = 2\cos 2\pi(Q_y + \xi_y)$$

$$2\cos 2\pi Q_y - \frac{\beta^*}{f}\sin 2\pi Q_y = 2\cos 2\pi Q_y \cos 2\pi\xi_y - 2\sin 2\pi Q_y \sin 2\pi\xi_y$$

$$2\cos 2\pi Q_y - \frac{\beta^*}{f}\sin 2\pi Q_y \approx 2\cos 2\pi Q_y - 4\pi\xi_y \sin 2\pi Q_y$$

$$\therefore \boxed{\xi_y = \frac{\beta^*}{4\pi f}}$$

In order for the beam to be stable :

$$|Tr(M_T)| \leq 2$$

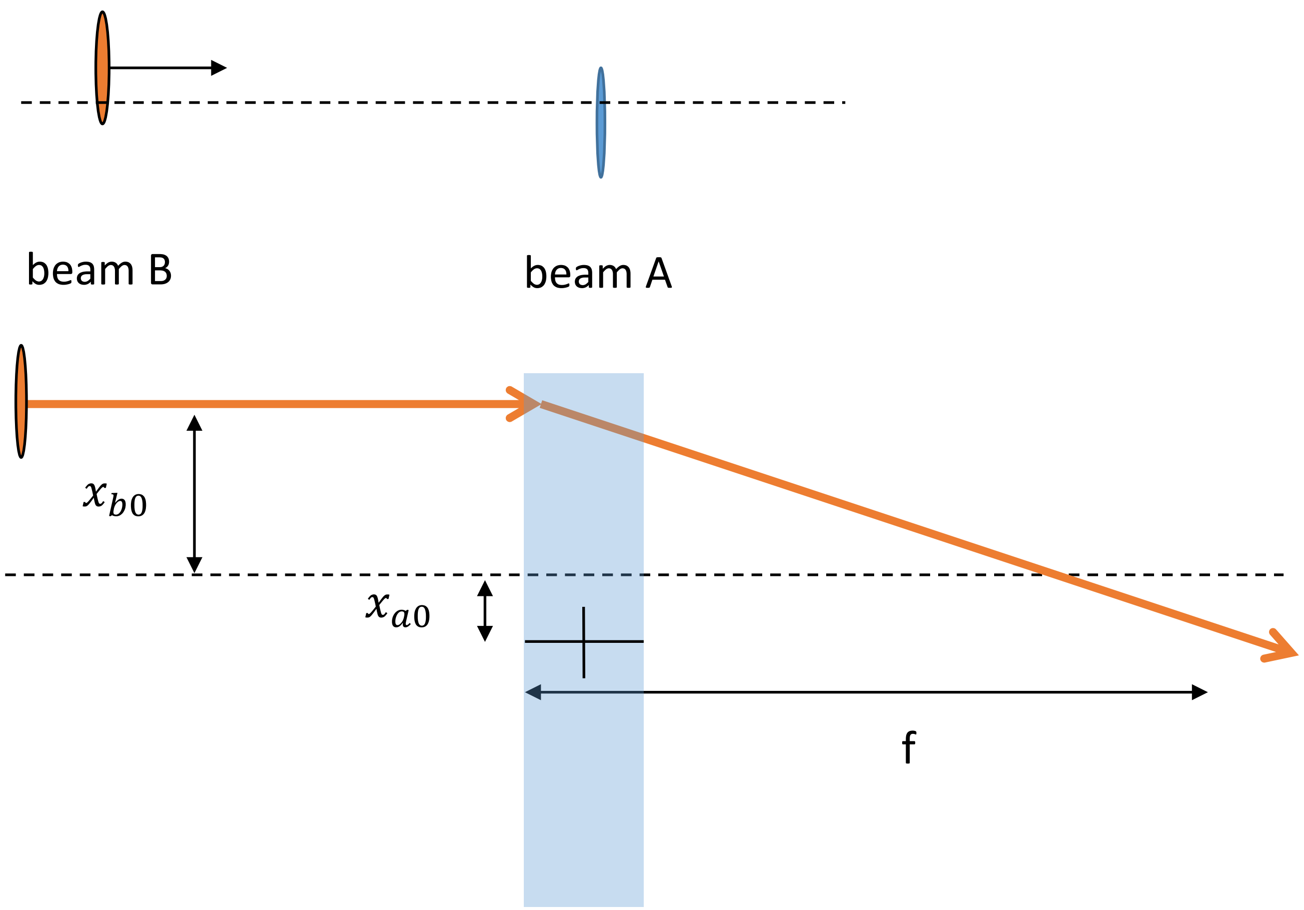
$$-2 \leq 2\cos 2\pi Q_y - 4\pi\xi_y \sin 2\pi Q_y \leq 2$$

$$0 \leq 2\cos^2 \pi Q_y - 4\pi\xi_y \sin\pi Q_y cos\, \pi Q_y \leq 2$$

$$\frac{-1}{2\pi} \tan \pi Q_y \leq \xi_y \leq \frac{1}{2\pi} \cot \pi Q_y$$

In general, the two beam centroids can have a offset

The angular kick matrix for the beam centroids:



$$\therefore\ x_{b1} = x_{b0}$$

$$\therefore\ x'_{b1} = -\frac{x_{b0} - x_{a0}}{f_a} + x'_{b0} - x'_{a0}$$

$$\begin{bmatrix} x_{a1} \\ x'_{a1} \\ x_{b1} \\ x'_{b1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/f_b & 1 & 1/f_b & -1 \\ 0 & 0 & 1 & 0 \\ 1/f_a & -1 & -1/f_a & 1 \end{bmatrix} \begin{bmatrix} x_{a0} \\ x'_{a0} \\ x_{b0} \\ x'_{b0} \end{bmatrix}$$

However, there is no coupling in the

One turn mapping

$$\begin{bmatrix} x_{a1} \\ x'_{a1} \\ x_{b1} \\ x'_{b1} \end{bmatrix} = \begin{bmatrix} \begin{matrix} \cos 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*}\sin 2\pi Q_y & \cos 2\pi Q_y \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \cos 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*}\sin 2\pi Q_y & \cos 2\pi Q_y \end{matrix} \end{bmatrix} \begin{bmatrix} x_{a0} \\ x'_{a0} \\ x_{b0} \\ x'_{b0} \end{bmatrix}$$

$$M_{T4X4} = \begin{bmatrix} \cos 2\pi Q_y - \frac{\beta^*}{f_b}\sin 2\pi Q_y & \beta^* \sin 2\pi Q_y & \frac{\beta^*}{f_b}\sin 2\pi Q_y & -\beta^* \sin 2\pi Q_y \\ \frac{-1}{\beta^*}\sin 2\pi Q_y - \frac{1}{f_b}\cos 2\pi Q_y & \cos 2\pi Q_y & \frac{1}{f_b}\cos 2\pi Q_y & -\cos 2\pi Q_y \\ \frac{\beta^*}{f_a}\sin 2\pi Q_y & -\beta^* \sin 2\pi Q_y & \cos 2\pi Q_y - \frac{\beta^*}{f_a}\sin 2\pi Q_y & \beta^* \sin 2\pi Q_y \\ \frac{1}{f_a}\cos 2\pi Q_y & -\cos 2\pi Q_y & \frac{-1}{\beta^*}\sin 2\pi Q_y - \frac{1}{f_a}\cos 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix}$$

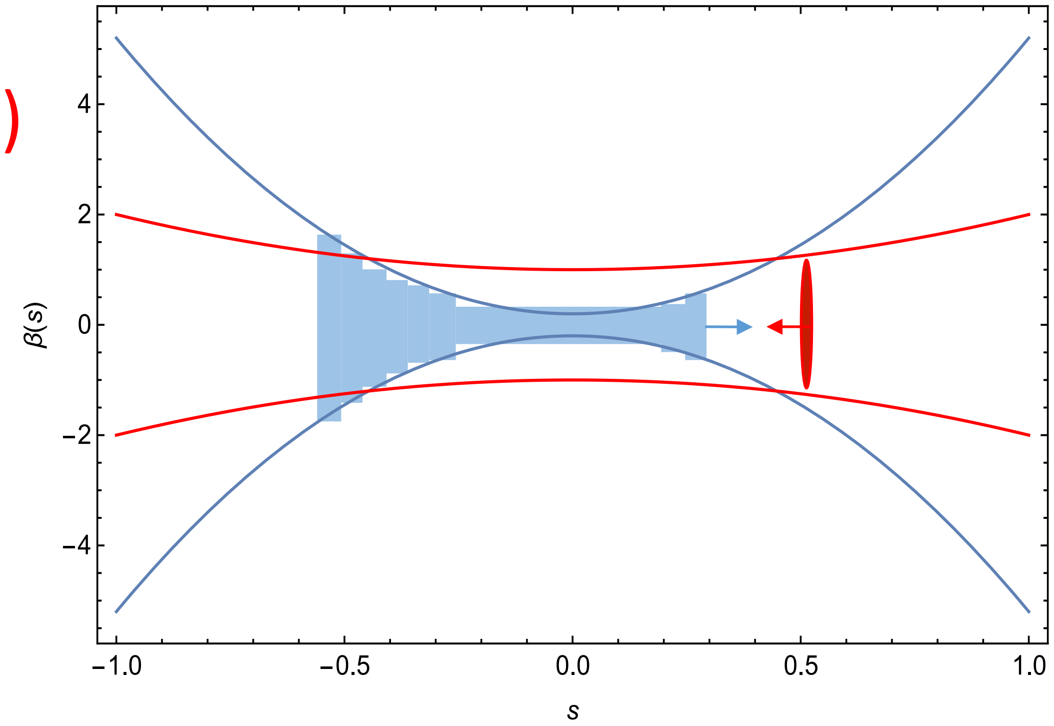
Hour-Glass Effect

- Let's consider a "long" beam
- Due to the longitudinal beam size,
  IP will happen in a series of different $\beta^*(s)$
- Without beam-beam effect

$$\sigma(s) = \sqrt{\varepsilon\beta(s)} = \sqrt{\varepsilon(\beta^* + \frac{s^2}{\beta^*})}$$

Collision happened in different beam size

Beta function of two beams



(Red->electrons, Blue-> Protons)

Consider a short electron beam collide with a long Proton beam

$$\xi_p(s) = \frac{\beta_p(s)}{4\pi f_e}$$

$$= \frac{N_e r_P \beta_P(s)}{4\pi\sigma_e(s)^2\gamma_p}$$

$$\xi_e = \int \frac{\beta_e(s)}{4\pi} d\,\frac{1}{f_e}$$

$$= \int \frac{N_P \lambda(s) r_e \beta_e(s)}{4\pi\sigma_P(s)^2\gamma_e} ds$$

From above: $\frac{1}{f} = \frac{N r_e}{\gamma\sigma^2}$

Where $\lambda(s)$ is the normalized proton density distribution that the electrons met. $dN_P(s) = N_P\lambda(s)\,ds$

As luminosity formula defined as:

$$L = \frac{N_e N_p \nu h}{4\pi\sigma^2}$$

Therefore luminosity formula will change to this form

$$L = \int \frac{N_e N_p \lambda(s) \nu h}{2\pi[\sigma_p(s)^2 + \sigma_e(s)^2]} ds$$

Only the center part of proton beam collide in minimum β. Luminosity was reduced!

However, Hour-Glass effect start to be important when $\sigma_z \geq \beta^*$

Conclusion

- Beam-Beam effect is a 2D force
- For a Gaussian beam, it is like a quadrupole error.
- For a short beam, it act as focusing/defocusing thin lens
- Beam-beam effect creates a extra tune shift
- Beam radius will be also affected, which can cause luminosity change.
- The Longitudinal beam size causes Hour-glass effect , which can also cause luminosity change.

References

1. “Beam-Beam Interaction Study in ERL Based ERHIC”, Yue Hao, Ph.D. Thesis, Department of Physics, Indiana University September, 2008
2. “Beam-Beam Effects in Particle Colliders”, Mauro Pivi, US Particle Accelerator School, 2011