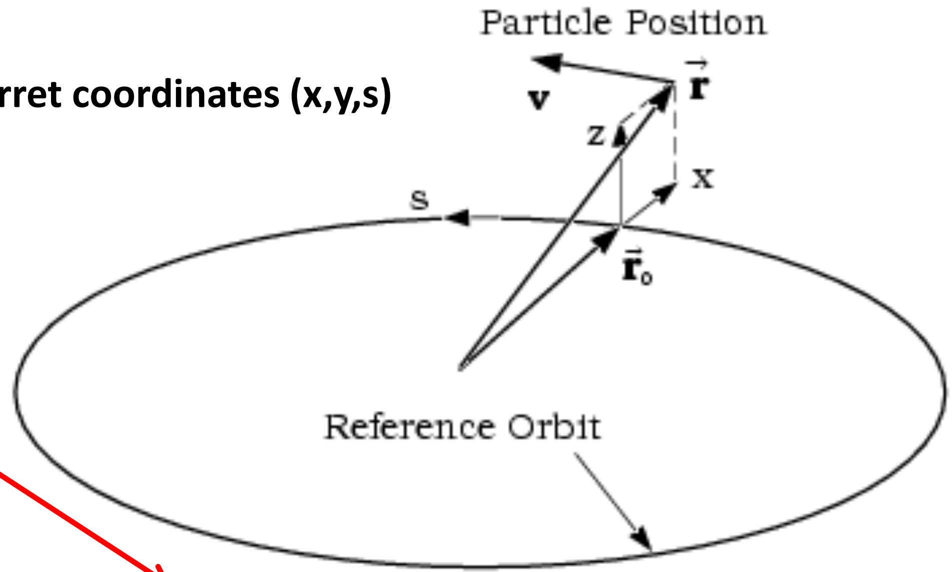
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

Review

Frenet-Serret coordinates (x,y,s)



Hill's equations (derivatives w.r.t. s)

$$x'' + K_x(s)x = \pm\frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

Natural focusing from dipoles (curvature)

Focusing from quadrupoles

Higher order magnet, usually field errors

$$\theta = \frac{s}{R} = \frac{\beta c t}{R}$$

Solution of Hill's equations X(s), X'(s) form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

$$|M(s,s_0)| = 1 \qquad\qquad |Trace(M(s,s_0))| \le 2$$

Stable solution conditions

Courant-Snyder parameterization

$$\mathrm{M}(s)=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}=I\cos\Phi+J\sin\Phi$$

$$I=\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix},\quad J=\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix},\quad J^2=-I,\ \text{ or }\ \beta\gamma=1+\alpha^2$$

Where α,β,γ,ϕ are functions of s and describes position dependent beam properties.

Focusing quadrupole:

$$M(s,s_0)=\begin{pmatrix}\cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 \\ -1/f & 1\end{pmatrix}$$

Defocusing quadrupole:

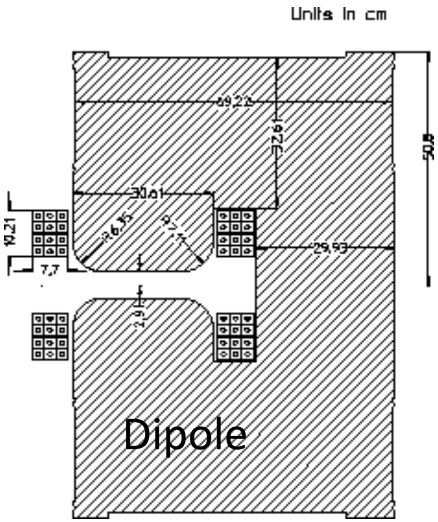
$$M(s,s_0)=\begin{pmatrix}\cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 \\ 1/f & 1\end{pmatrix}$$

Dipole: $K=1/\rho^2$

$$M(s,s_0)=\begin{pmatrix}\cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho}\end{pmatrix}\rightarrow\begin{pmatrix}1 & \ell \\ 0 & 1\end{pmatrix}$$

Drift space: K=0

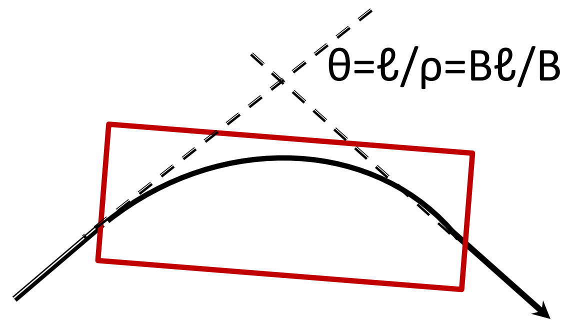
$$M(s,s_0)=\begin{pmatrix}1 & \ell \\ 0 & 1\end{pmatrix}$$



$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

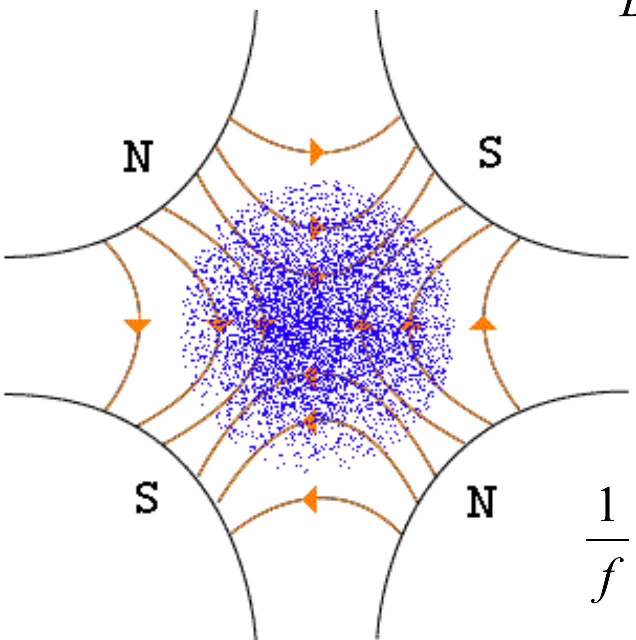
$$\gamma m \frac{\upsilon^2}{\rho} = q \upsilon B$$

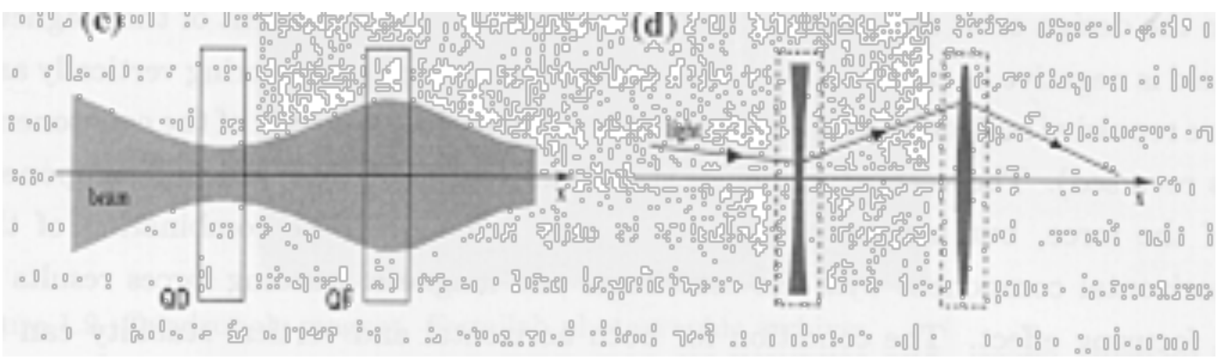
$$\rho = \frac{\gamma m v}{qB} = \frac{p}{qB}$$



$$B\rho[T-m] = \frac{p}{q} = \frac{A}{Z} \times 3.33564 \times p[GeV/c/u]$$

quadrupole





$$\frac{1}{f} = \mp \frac{B_1 \ell}{B\rho}$$

f>0, if focusing, f<0 if defocusing

For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**.

$$\vec{B} = B_x(x, y)\hat{x} + B_y(x, y)\hat{y}$$

$$B_x = -\frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial y} = -\frac{1}{h_s}\frac{\partial A_s}{\partial y}, B_y = \frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial x} = \frac{1}{h_s}\frac{\partial A_s}{\partial x}$$

For h_s=1 or ρ=∞, one obtains the multipole expansion:

$$B_y + jB_x = B_0\sum_n (b_n + ja_n)(x + jy)^n, \qquad A_s = \mathrm{Re}\left\{B_0\sum_n \frac{1}{n+1}(b_n + ja_n)(x + jy)^{n+1}\right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_y = B_0 b_0$, $B_x = B_0 a_0$,

b_1 : quad, a_1 : skew quad; $B_y = B_0 b_1 x$, $B_x = B_0 b_1 y$, $B_y = -B_0 a_1 y$, $B_x = B_0 a_1 x$,

b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_y + jB_x) = \mp\frac{1}{\rho}\sum_n (b_n + ja_n)(x + jy)^n$$

Floquet Theorem

$$X'' + K(s)X = 0 \qquad K(s) = K(s+L)$$

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s} \frac{1}{\beta} ds$$

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1 \sin\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s-s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

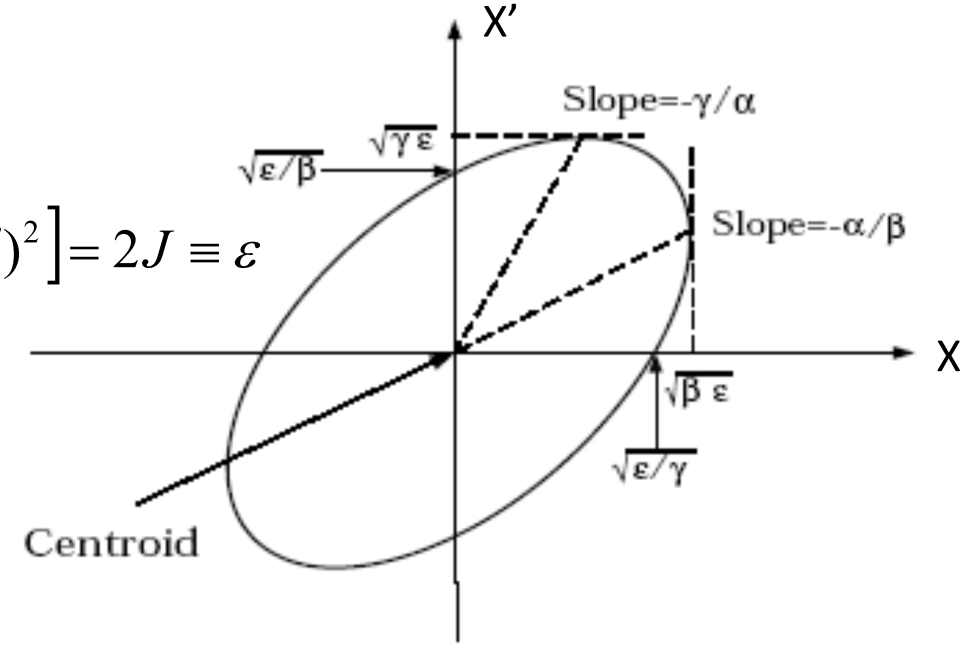
$$X = \sqrt{2\beta J}\cos\psi, \quad X' = -\sqrt{\frac{2J}{\beta}}(\sin\psi + \alpha\cos\psi)$$

$$P_X = \beta X' + \alpha X = -\sqrt{2\beta J}\sin\psi$$

(X,P_X) form a normalized phase space coordinates with $X^2+P_X^2=2\beta J$, here J is called **action**.

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



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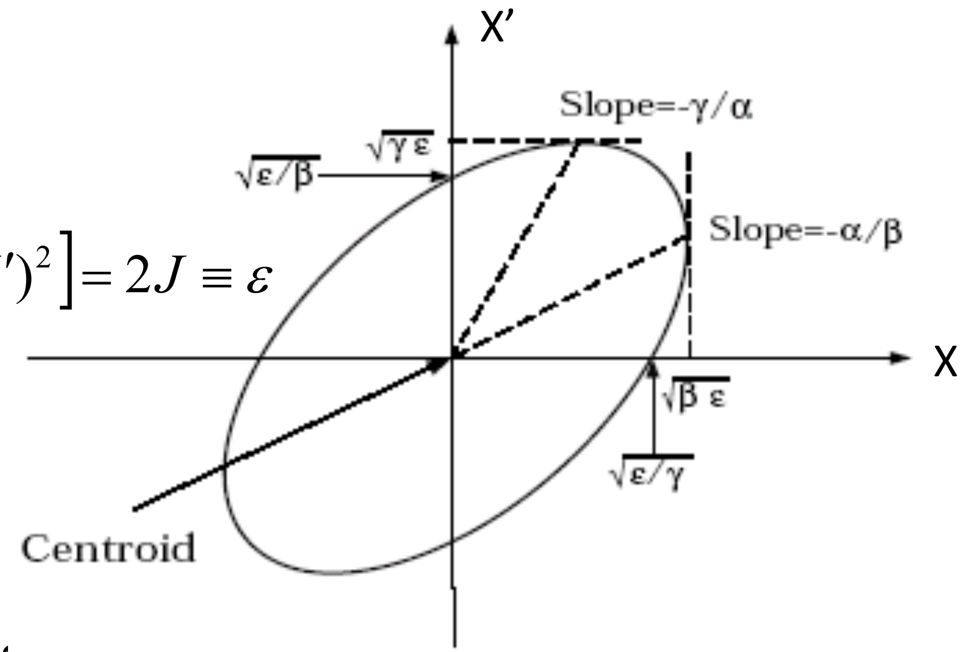


Emittance of a beam

$$\langle X \rangle = \int X \rho(X, X') dX dX', \quad \langle X' \rangle = \int X' \rho(X, X') dX dX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X, X') dX dX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X, X') dX dX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle) \rho(X, X') dX dX' = r \sigma_X \sigma_{X'}$$

$$\varepsilon_{rms} = \sqrt{\sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X \sigma_{X'} \sqrt{1 - r^2}$$

The rms emittance is invariant in linear transport:

$$\frac{d\varepsilon^2}{ds} = 0$$

normalized emittance $\varepsilon_n = \varepsilon\beta\gamma$ is invariant when beam energy is changed.

Adiabatic damping – beam emittance decreases with increasing beam momentum, i.e. $\varepsilon = \varepsilon_n / \beta\gamma$, which applies to beam emittance in **linacs**.

In storage rings, the beam emittance **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

The Gaussian distribution function

$$\rho(X, P_X) = \frac{1}{2\pi\sigma_X^2} e^{-(X^2 + P_X^2)/2\sigma_X^2}$$

$$\rho(\varepsilon) = \frac{1}{2\varepsilon_{rms}} e^{-\varepsilon/2\varepsilon_{rms}}$$

$\epsilon/\epsilon_{\rm rms}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

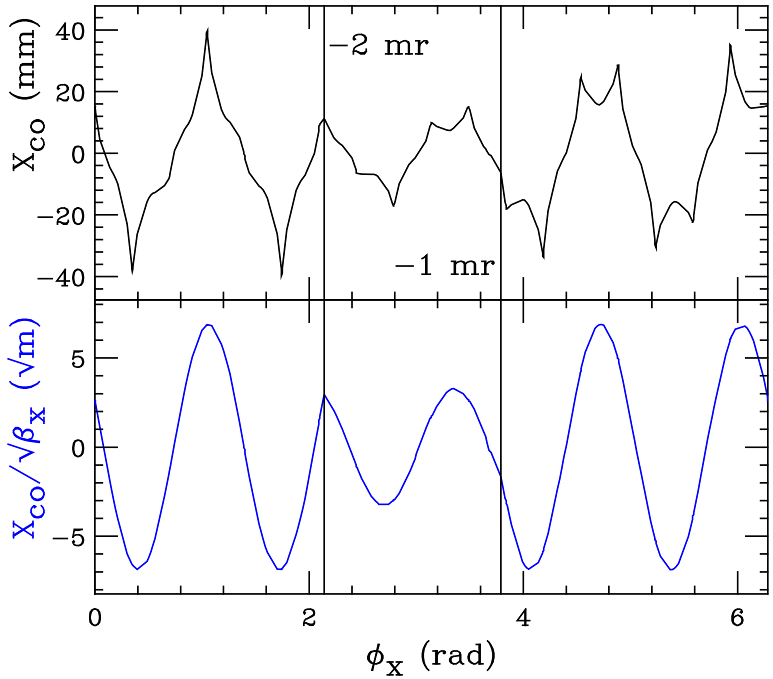
For a localized dipole field error: θ=ΔBℓ/Bρ

$$X'' + K_X(s)X = \theta\delta(s - s_0)$$

$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0{}' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

$$X_{co}(s) = G(s, s_0)\theta$$



$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

For a distributed dipole field error:

$$X_{co}(s) = \sqrt{\beta(s)} \sum_{k=-\infty}^{\infty} \frac{\nu^2 f_k}{\nu^2 - k^2} \; e^{jk\phi(s)}$$

Where the field error is expanded in Fourier series

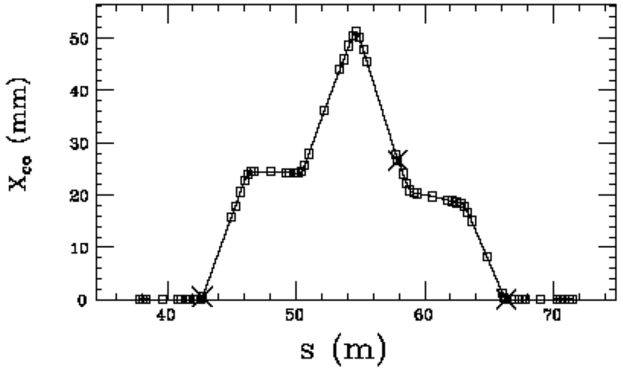
$$\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right] = \sum_{k=-\infty}^{\infty} f_k e^{jk\varphi}$$

$$f_k = \frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi = \frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$\text{Sensitivity factor} \equiv \frac{\left\langle \left(X_{co}(s)\right)^2 \right\rangle^{1/2}}{\theta_{rms}} \propto \sqrt{\beta(s)}$$

closed orbit bump: $X_{co}(s_f) = 0$, $X'_{co}(s_f) = 0$

$$\Delta x_{co}(s) = \left\{\sqrt{\beta_x(s_k)\beta_x(s)}\sin(\Delta\psi_x(s))\right\}\theta_k$$

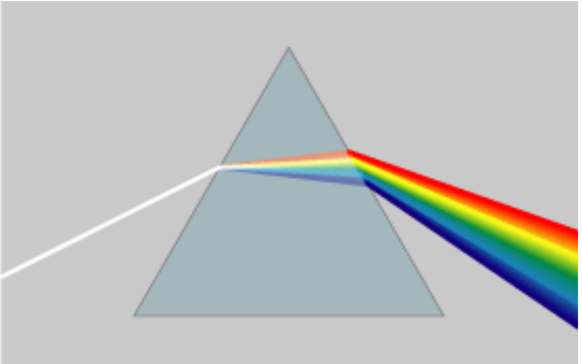


Orbit length change:

$$\Delta C = C - C_0 = \theta_0 \oint \frac{G_x(s, s_0)}{\rho} ds = D(s_0)\theta_0$$

$$\Delta C = \oint D(s_0)\frac{\Delta B_y(s_0)}{B\rho}ds_0$$

Off-momentum and dispersion



For different particle energy $$\delta = \frac{p - p_0}{p_0}$$

$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

Extend the matrix representation to 3 by 3

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole (K=0):

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathrm{L} & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

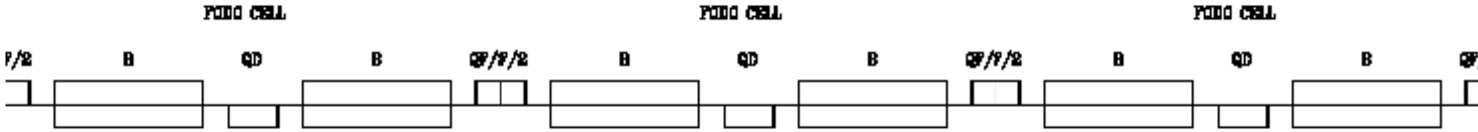
$$\theta \ll 1 \quad i.e. \quad L \ll \rho$$

For quadrupoles:

$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing
change K -> -K

FODO cell



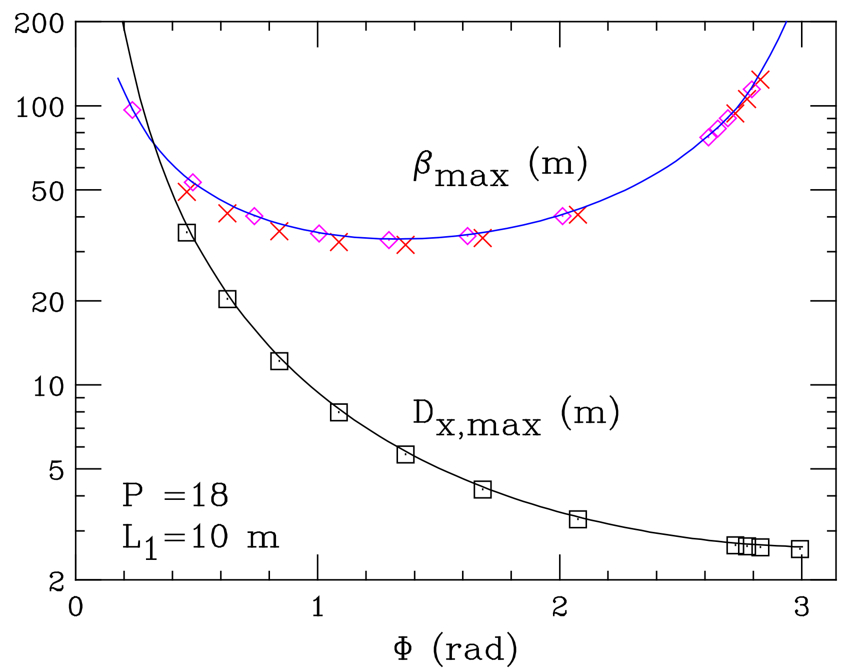
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

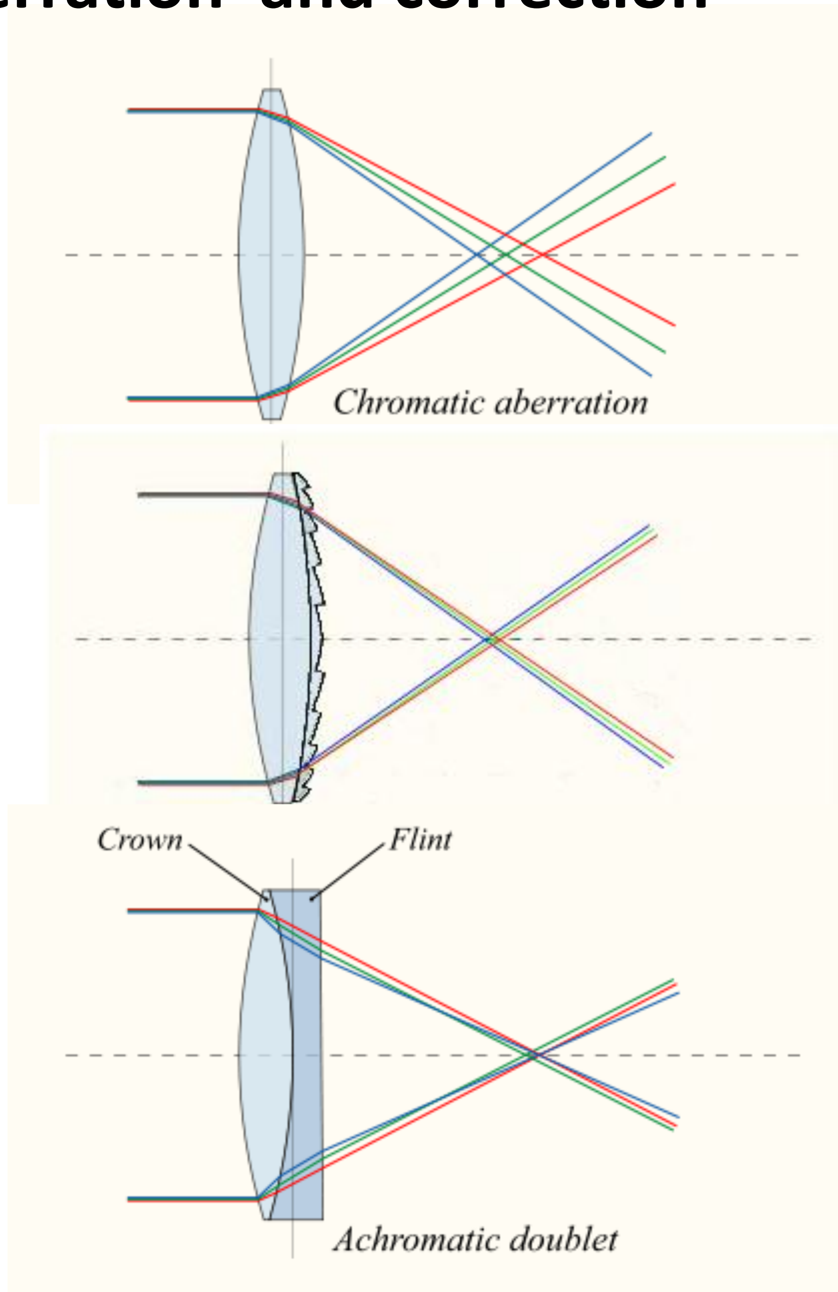
$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L(1+\frac{L}{2f}) & 2L\theta(1+\frac{L}{4f}) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta(1-\frac{L}{4f}-\frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

$$D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$

$$\beta_{\max} = \frac{2L_1(1+\frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$



Chromatic aberration and correction



Chromatic aberration in particle accelerators

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2, \qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

Inhomogeneous equation

$$p / p_0 = 1 + \delta$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, \quad K(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_y}{\partial x}$$

$$x = x_\beta + D\delta \qquad D'' + \left(K_x(s) + \Delta K_x\right)D = \frac{1}{\rho} + O(\delta)$$

$$x_\beta'' + \left(K_x(s) + \Delta K_x\right)x_\beta = 0, \quad y_\beta'' + \left(K_y(s) + \Delta K_y\right)x_\beta = 0$$

$$K_x(s) = \frac{1}{\rho^2} - K(s), \qquad \Delta K_x(s) = [-\frac{2}{\rho^2} + K(s)]\delta \approx -K_x(s)\delta,$$

$$K_y(s) = +K(s), \qquad \Delta K_y(s) = [-K(s)]\delta = -K_y(s)\delta$$

Note that the betatron motion for off momentum particle is perturbed by a chromatic term. The betatron tunes must avoid half-integer resonances. But, the quadrupole error is proportional to the designed quadrupole field. They are called systematic chromatic aberration. It is an important topic in accelerator physics.

Tune shift, or tune spread, due to chromatic aberration:

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x / d\delta$$

$$\Delta\nu_y = \left[-\tfrac{1}{4\pi}\oint \beta_y(s)K_y(s)ds\right]\delta \equiv C_y\delta, \quad C_y = d\nu_y / d\delta$$

The chromaticity induced by quadrupole field error is called natural chromaticity. For a simple FODO cell, we find

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \approx -\frac{1}{4\pi}\sum \frac{\beta_{xi}}{f_i}\ \delta$$

$$C_{X,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{1}{4\pi}N\left(\frac{\beta_{\max}}{f} - \frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

We define the specific chromaticity as $\quad \xi_x = C_x / \nu_x, \quad \xi_y = C_y / \nu_y$

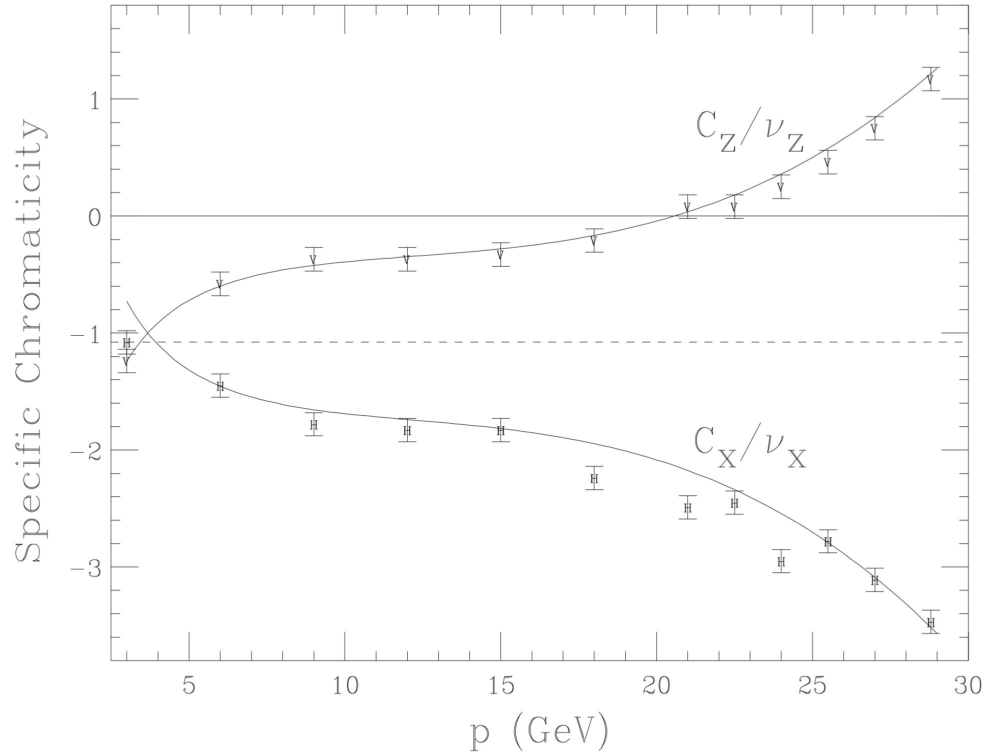
The **specific chromaticity is about −1 for FODO cells**, and can be as high as -4 for high luminosity colliders and high brightness electron storage rings.

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

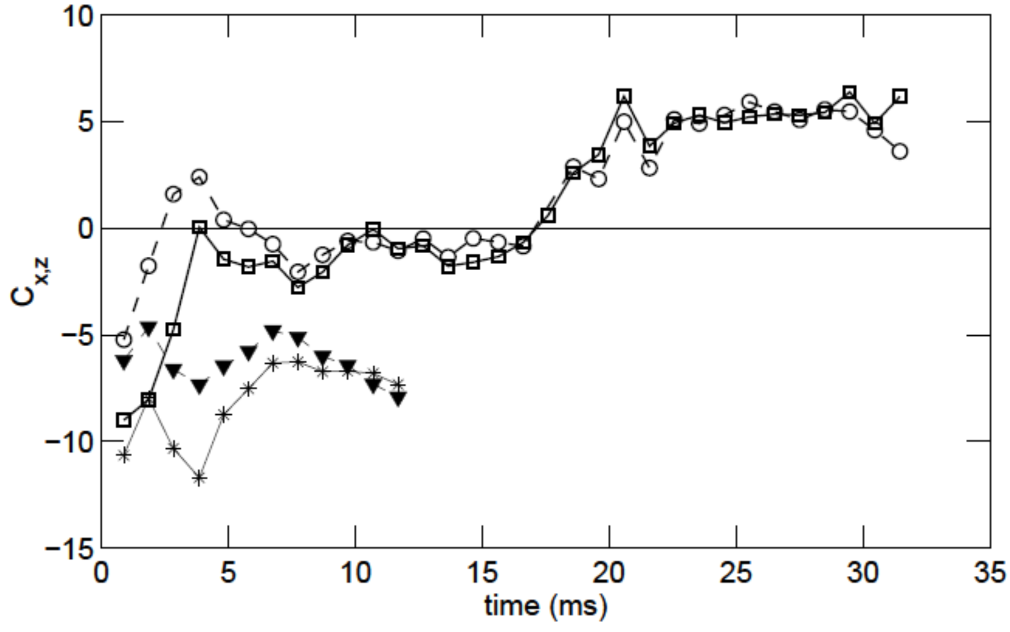
Examples:

BNL AGS (E. Blesser 1987): Chromaticities measured at the AGS.

$$C_{X,\text{nat}}^{\text{FODO}} = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$



Fermilab Booster (X. Huang, Ph.D. thesis, IU 2005): The measured horizontal chromaticity C_x when SEXTS is on (triangles) or off (stars), and the measured vertical chromaticity C_y when SEXTS is on (dash, circles) or off (squares). The error bar is estimated to be 0.5. The natural chromaticities are $C_{nat,y}=-7.1$ and $C_{nat,x}=-9.2$ for the entire cycle. The betatron tunes are 6.7(x) and 6.8(y) respectively.



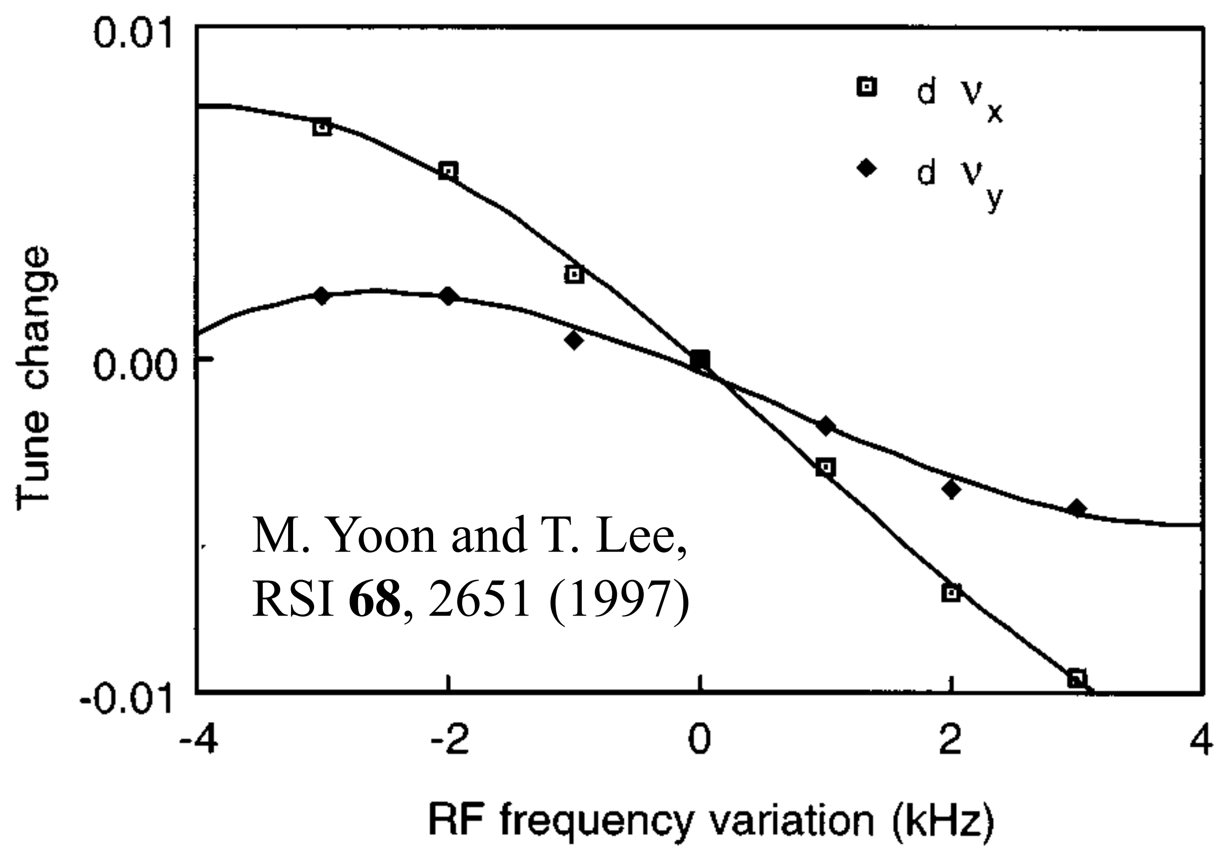
Chromaticity measurement:

The chromaticity can be measured by measuring the betatron tunes vs the rf frequency f, i.e.

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

$$\Delta f / f_0 = -\eta\delta,$$

$$C = \frac{d\nu}{dp/p} = -\eta f_{rf}\frac{d\nu}{df_{rf}}$$

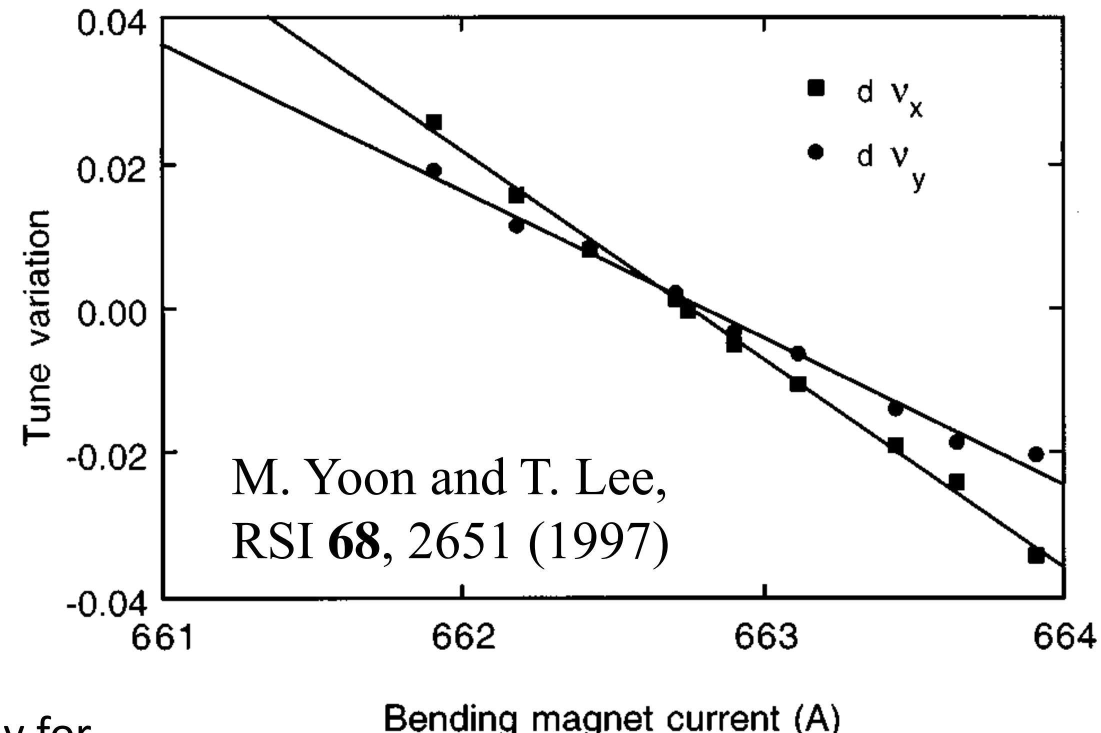


The chromaticites are C_x=+2.9, C_y=+1.4.

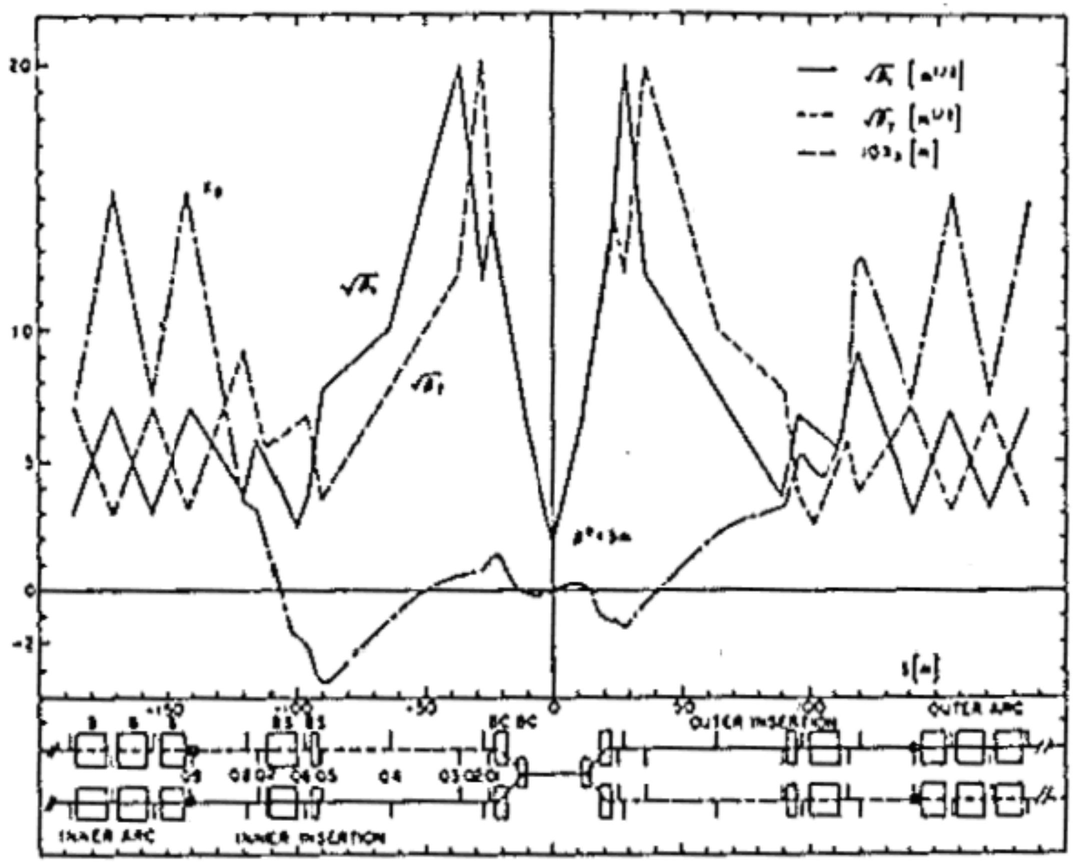
The **Natural chromaticity** can be obtained by measuring the tune variation vs the bending-magnet current at a **constant rf frequency**. Change of the bending-magnet current is equivalent to the change of the beam energy. Since the orbit is not changed, the effect of the sextupole magnets on the beam motion can be neglected. The Figure shows the horizontal and vertical tune vs the bending-magnet current in the PLS storage ring.

$$C = \frac{d\nu}{dp/p} = \frac{d\nu}{dB/B} = \frac{d\nu}{dI/I}$$

The data give C_x=−18.96, C_y=−13.42; vs theory: C_x=−23.36, C_y=−16.19.



Note that this method may not apply for combined function dipoles.

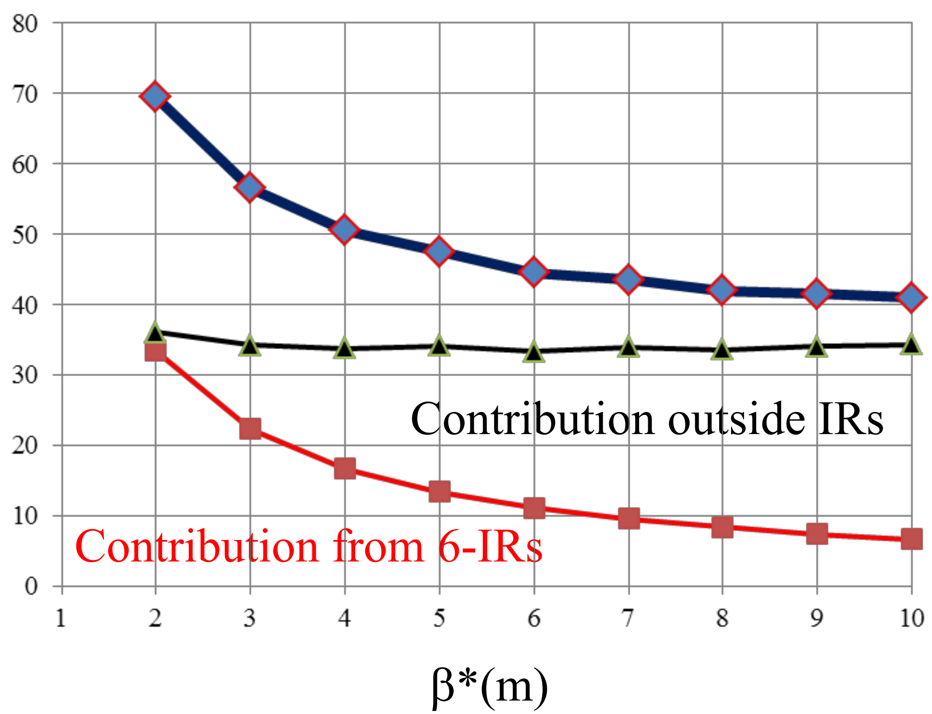


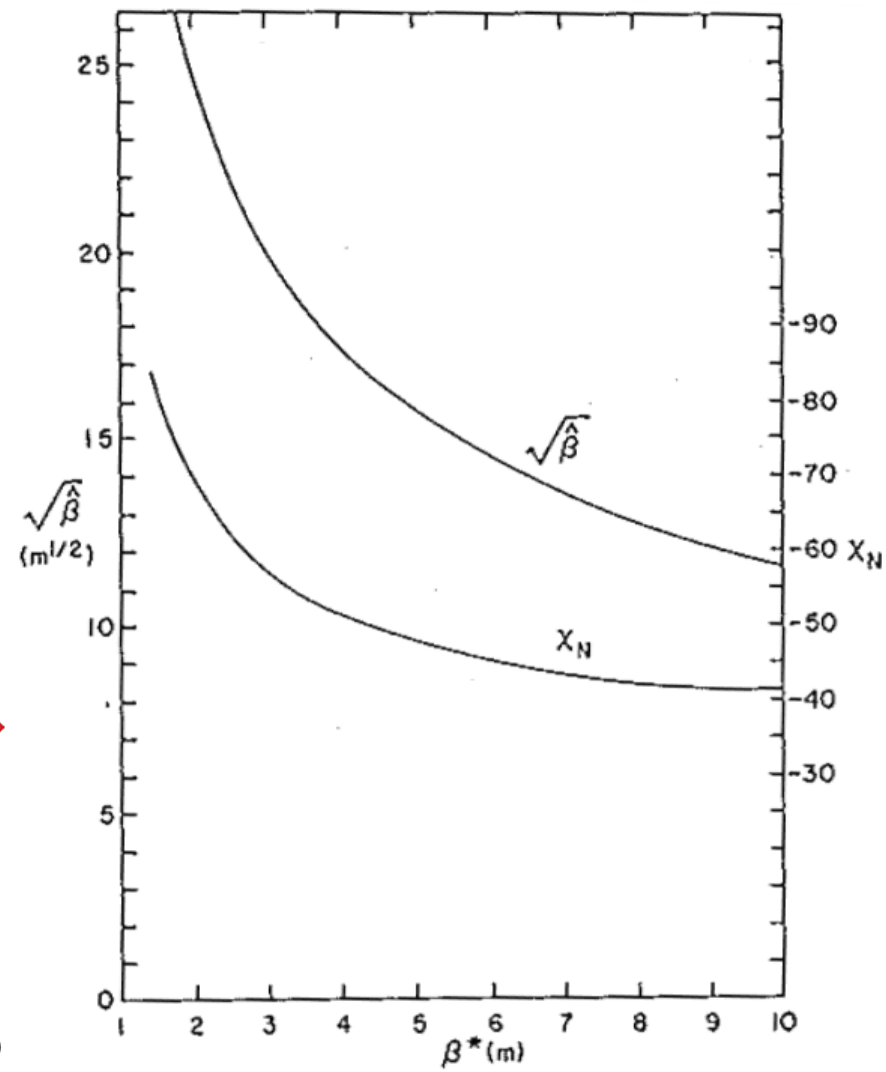
Contribution of low β triplets in an IR to the natural chromaticity is

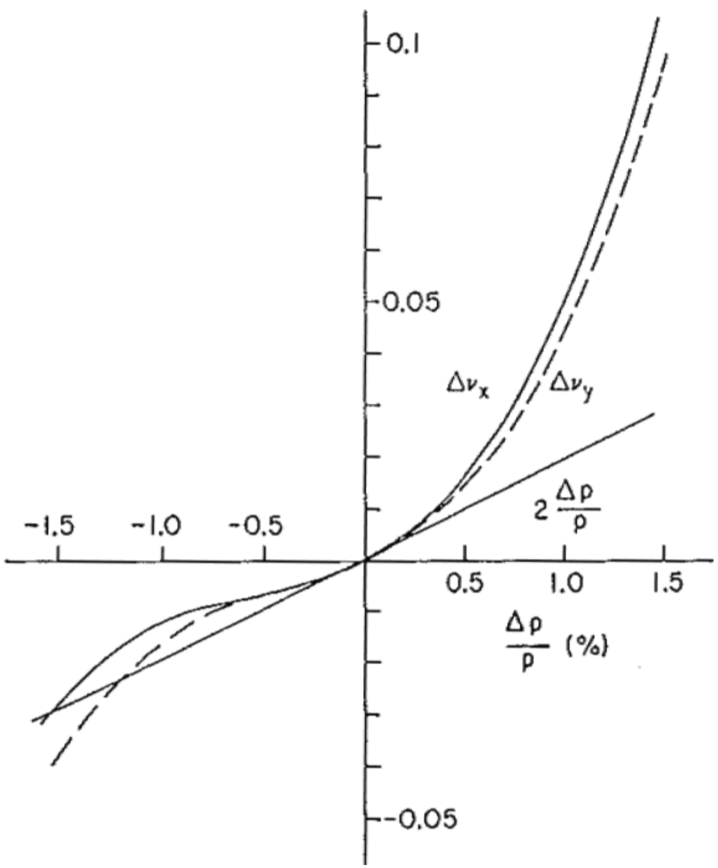
$$C_{total} = N_{IR} C_{IR} + C_{bare\,machine}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$

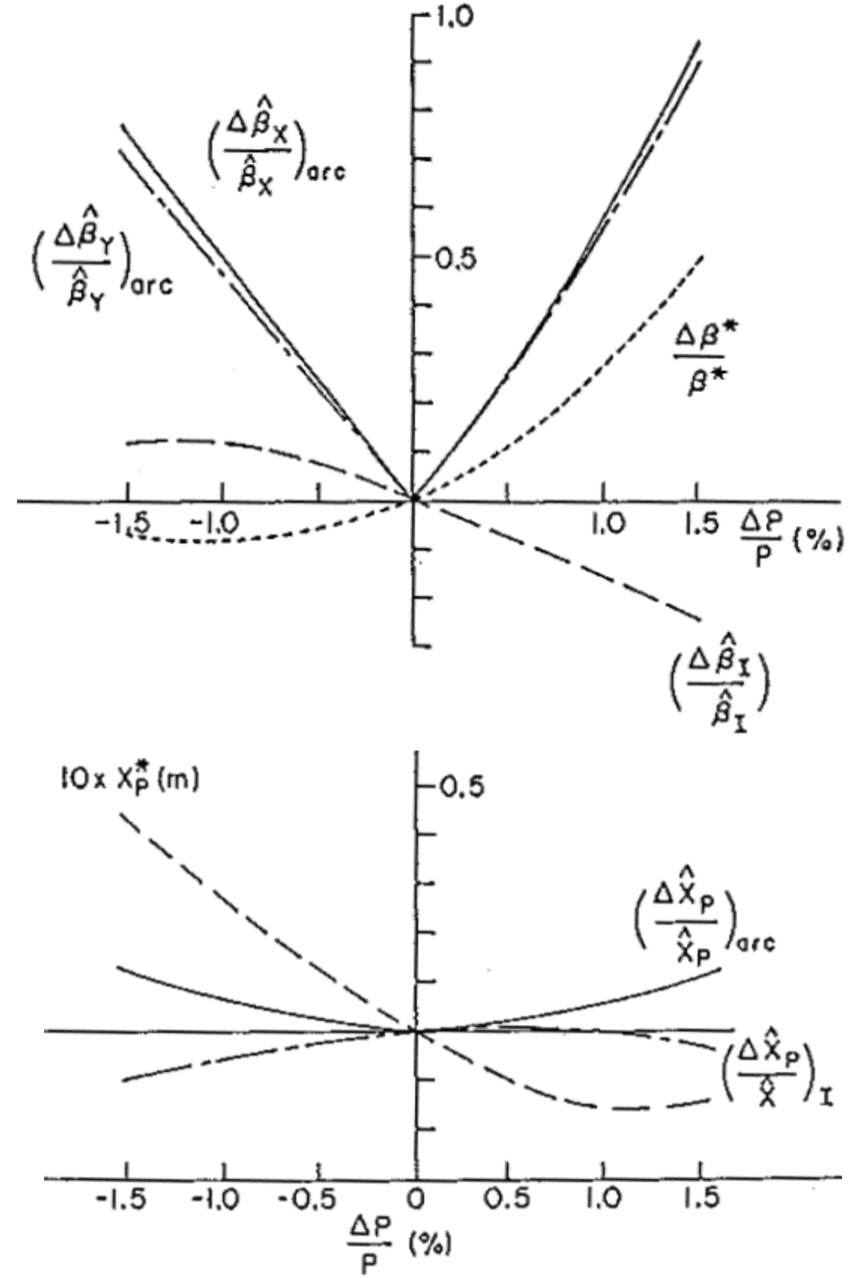
The total chromaticity is composed of contributions from the low β-quads and the rest of accelerators that is made of FODO cells. The decomposition to fit the data is **Δs≈35** m in RHIC.







ν vs Δp/p



β and D vs Δp/p

Chromaticity correction:

The chromaticity can cause tune spread to a beam with momentum spread $\Delta\nu=C\delta$. For a beam with C=-100, δ=0.005, $\Delta\nu$=0.5. The beam is not stable for most of the machine operation. Furthermore, there exists collective (head-tail) instabilities that requires positive chromaticity for stability! To correct chromaticity, we need to find magnetic field that provide stronger focusing for off-(higher)-momentum particles. We first try sextupole with

$$\Delta B_y + j\Delta B_x = B_0 b_2 (x + jy)^2, \qquad A_s = \tfrac{1}{3}\mathrm{Re}\left\{B_0 b_2 (x + jy)^3\right\}$$

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho} \qquad \begin{aligned} x &= x_\beta + D\delta \\ y &= y_\beta \end{aligned}$$

$$\Delta B_y = B_0 b_2 (x^2 - y^2) = B_0 b_2 (2x_\beta D\delta + D^2\delta^2 + x_\beta^2 - y_\beta^2)$$

$$\Delta B_x = B_0 b_2 2xy = B_0 b_2 2y_\beta D\delta + B_0 b_2 2x_\beta y_\beta$$

Let $K_2=-2B_0b_2/B\rho=-B_2/B\rho$, we obtain:

$$x_\beta'' + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y_\beta'' + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x''_\beta + (K_x(s) + K_2 D\delta) x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta) y_\beta = 0$$

$$x = x_\beta + D\delta$$

$$\Delta K_x(s) = K_2(s) D(s)\delta, \quad \Delta K_y(s) = -K_2(s) D(s)\delta$$

$$C_x = -\tfrac{1}{4\pi} \oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_y = -\tfrac{1}{4\pi} \oint \beta_y(s)[K_y(s) + K_2(s)D(s)]ds$$

- In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_y D_x$ are maximum.
- A large ratio of β_x/β_y for the focusing sextupole and a large ratio of β_y/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.
- The families of sextupoles should be arranged to minimize the systematic half-integer stopbands and the third-order betatron resonance strengths.

Revisit of half interger stopband intergral

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2\sin\Phi_0}\int_{\phi}^{\phi+2\pi} d\phi_1 \beta^2(\phi_1)k(\phi_1)\sin 2\nu_0(\pi+\phi-\phi_1)$$

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta(s)}{\beta(s)} + 4\nu_0^2\frac{\Delta\beta(s)}{\beta(s)} = -2\nu\beta^2 k(s)$$

$$\left[\nu_0\beta^2 k(s)\right] = \sum_{p=-\infty}^{\infty} J_p e^{jp\varphi},$$

$$J_p = \frac{1}{2\pi}\oint\left[\beta k(s)\right]e^{-jp\varphi}ds$$

Half integer stopband

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi}$$

What symmetry can do to stopbands?

Systematic chromatic half-integer stopband width

The effect of systematic chromatic gradient error on betatron amplitude modulation can be analyzed by using the chromatic half-integer stopband integrals

$$J_{p,x} = \frac{1}{2\pi}\oint \beta_x \Delta K_x e^{-jp\varphi_x} ds$$

$$J_{p,y} = \frac{1}{2\pi}\oint \beta_y \Delta K_y e^{-jp\varphi_y} ds$$

We consider a lattice made of P superperiods, where L is the length of a superperiod with K(s + L) = K(s), β(s + L) = β(s). Let C = PL be the circumference of the accelerator. The integral becomes

$$J_{p,X} = -\left\{\frac{\delta}{2\pi}\int_0^L \beta_X \Delta K_X e^{-jp\varphi} ds\right\}\left[1 + e^{-jp\frac{2\pi}{P}} + e^{-j2p\frac{2\pi}{P}} + \cdots + e^{-jp\frac{2\pi}{P}(P-1)}\right]$$

$$= -\left\{\frac{\delta}{2\pi}\int_0^L \beta_X \Delta K_X e^{-jp\varphi} ds\right\} P \quad when \quad p = 0\,(Mod\ P)$$

$$J_{p,X} = 0, \quad when \quad p \neq 0\,(Mod\ P)$$

When p = 0 (Mod P), the half-integer stopband integral increases by a factor of P, i.e. each superperiod contributes additively to the crhomatic stopband integral.

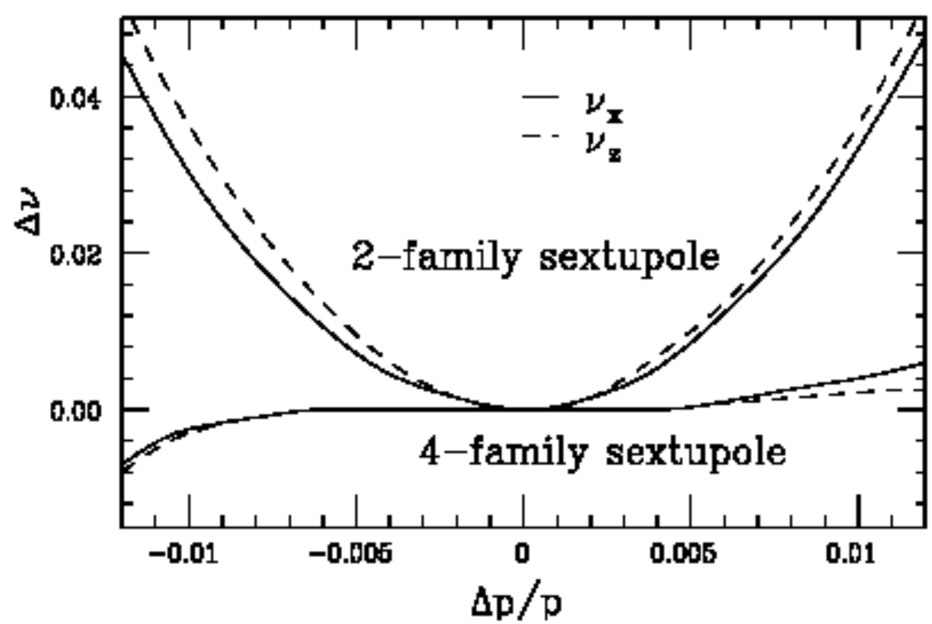
Effect of the chromatic stopbands on chromaticity

$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi} \approx -\frac{|J_p|\cos(p\phi)}{2(\nu_0-p/2)}$$

$$\Delta\nu_X = C_X^{(1)}\delta + C_X^{(2)}\delta^2 + \cdots$$

$$C_X{}^{(1)} = -\tfrac{1}{4\pi}\oint\beta_X(s)[K_X(s)-K_2(s)D(s)]ds$$

$$C_X^{(2)} = -C_X^{(1)} - \frac{|J_{p,X}|^2}{4(\nu_X-p/2)\delta^2}$$



Effect of sextupoles on the chromatic stopband integrals

First we evaluate the stopband integral due to the chromatic sextupoles. Let S_F and S_D be the integrated sextupole strengths at QF and QD of FODO cells in the arc. The *p-th* harmonic stopband integral from these chromatic sextupoles is

$$J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P\times p\varphi/2\nu)}{\sin(p\varphi/2\nu)}[\beta_F S_F D_F + \beta_D S_D D_D e^{-jp\varphi/2\nu}]e^{-j(N-1)p\varphi/2\nu}$$

To change the stopband integral without perturbing the first order chromaticities, we group the sextupoles in four families, i.e. (S_{F1}, S_{D1}, S_{F2}, S_{D2}). By asking

$$S_{F1} \to S_{F1} + (\Delta S)_F, \qquad S_{D1} \to S_{D1} + (\Delta S)_D,$$

$$S_{F2} \to S_{F2} - (\Delta S)_F \qquad S_{D2} \to S_{D2} - (\Delta S)_D$$

$C^{(1)}$ stays same while the change in stopband integral (2nd) gives

$$\Delta J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P\times(p\varphi/2\nu-\pi/2))}{\sin(p\varphi/2\nu-\pi/2)}[\beta_F(\Delta S)_F D_F + \beta_D(\Delta S)_D D_D e^{-jp\varphi/4\nu}]e^{-j(N-1)(p\varphi/2\nu-\pi/2)}$$

Under conditions $p \quad 2 \qquad \overline{2}$

$$\frac{\sin(P\times(p\varphi/2\nu-\pi/2))}{\sin(p\varphi/2\nu-\pi/2)} = P$$

$$C_X^{(1)} = -\frac{1}{4\pi}\oint \beta_X(s)[K_X(s) - K_2(s)D(s)]ds$$

$$C_X^{(2)} = -C_X^{(1)} - \frac{|J_{p,X}|^2}{4(\nu_X - p/2)\delta^2}$$

$$J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P \times p\varphi/2\nu)}{\sin(p\varphi/2\nu)}[\beta_F S_F D_F + \beta_D S_D D_D e^{-jp\varphi/2\nu}]e^{-j(N-1)p\varphi/2\nu}$$

$$S_{F1} \rightarrow S_{F1} + (\ \ S)_F, \qquad S_{D1} \rightarrow S_{D1} + (\Delta S)_D, \qquad p \approx 2\nu$$

$$S_{F2} \rightarrow S_{F2} \quad (\ \ S)_F \qquad S_{D2} \rightarrow S_{D2} - (\Delta S)_D \qquad \varphi \approx \frac{\pi}{2}$$

$$\Delta J_{p,sext} = \frac{\delta}{2\pi}P[\beta_F(\Delta S)_F D_F + \beta_D(\Delta S)_D D_D e^{-j\pi/4}]$$

Every FODO cell contributes additively to the chromatic stopband. The resulting stopband width is proportional to $(\Delta S)_F$ and $(\Delta S)_D$ parameters. By adjusting $(\Delta S)_F$ and $(\Delta S)_D$ parameters, the betabeat and the second order Chromaticity can be minimized.

Similarly, a six-family sextupole scheme works for 60 degree phase advance FODO lattice, where The six-family scheme (S_{F1}, S_{D1}, S_{F2}, S_{D2} , S_{F3}, S_{D3}) has two additional parameters.

Lattice Design Strategy

Based on our study of linear betatron motion, the lattice design of accelerator can be summarized as follows. The lattice is generally classified into three categories: low energy booster, collider lattice, and low-emittance lattice storage rings.

- The betatron tunes should be chosen to avoid systematic integer and half-integer stopbands and systematic low-order nonlinear resonances; otherwise, the stopband width should be corrected.
- The betatron amplitude function and the betatron phase advance between the kicker and the septum should be optimized to minimize the kicker angle and maximize the injection or extraction efficiency.
- Local orbit bumps can be used to alleviate the demand for a large kicker angle. Furthermore, the injection line and the synchrotron optics should be properly "matched" or "mismatched" to optimize the emittance control.
- To improve the slow extraction efficiency, the β value at the (wire) septum location should be optimized. The local vacuum pressure at the high-β value locations should be minimized to minimize the effect of beam gas scattering.

- The chromatic sextupoles should be located at high dispersion function locations. The focusing and defocusing sextupole families should be located in regions where $\beta_x \gg \beta_y$, and $\beta_x \ll \beta_y$ respectively in order to gain independent control of the chromaticities.
- It is advisable to avoid the transition energy for low to medium energy synchrotrons in order to minimize the beam dynamics problems during acceleration.

Besides these design issues, problems regarding the dynamical aperture, nonlinear betatron detuning, collective beam instabilities, rf system, vacuum requirement, beam lifetime, etc., should be addressed.

Symplectic integration

Outline

- Hamiltonian & symplecticness
- Numerical integrator and symplectic integration
- Application to accelerator beam dynamics
- Accuracy and integration order

Hamiltonian dynamics

In accelerator, particles' motion is predicted by Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad \text{or} \qquad \dot{q} = \nabla_p H(p,q), \quad \dot{p} = -\nabla_q H(p,q)$$

or it can be written in a compact form $\dot{z} = J\nabla_z H(z) \qquad z \equiv (p,q)$

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

The solution is a transformation mapping (flow)

$$(p,q) = A_{t,H}(p_0,q_0)$$

or for simplicity $z = A(z_0)$

in matrix representation, the map A is a 2n by 2n matrix.

Sympleсticness

A. Hamilton's equations predict the evolution of phase space.

B. Canonical transformation A preserves the form of Hamilton's equations.

C. Transformation A is canonical if and only if it satisfies the relation

$$A^T J A = J \qquad \det A = 1$$

and we call this transformation A symplectic

Proof. Hamilton's equation can be expressed as $\dot{x} = J \dfrac{\partial H}{\partial x}$

if we have transformation $y = y(x)$

$$\dot{y} = AJA^T \frac{\partial H}{\partial y} = J \frac{\partial H}{\partial y}$$ if $A^T J A = J$ i.e. symplectic

Preservation of area

Symplecticness is equivalent to the preservation of area.

In a 2d(d=1) space, the area of a parallelogram is defined as the magnitude of the wedge product $dp \wedge dq$

While for a transformation $z = A(z_0)$

we have

$$dp = \frac{\partial p}{\partial p_0} dp_0 + \frac{\partial p}{\partial q_0} dq_0, \quad dq = \frac{\partial q}{\partial p_0} dp_0 + \frac{\partial q}{\partial q_0} dq_0$$

$$dp \wedge dq = \frac{\partial p}{\partial p_0}\frac{\partial q}{\partial q_0} dp_0 \wedge dq_0 + \frac{\partial p}{\partial q_0}\frac{\partial q}{\partial p_0} dq_0 \wedge dp_0$$

wedge products are anticommutative $dp \wedge dq = -dq \wedge dp$

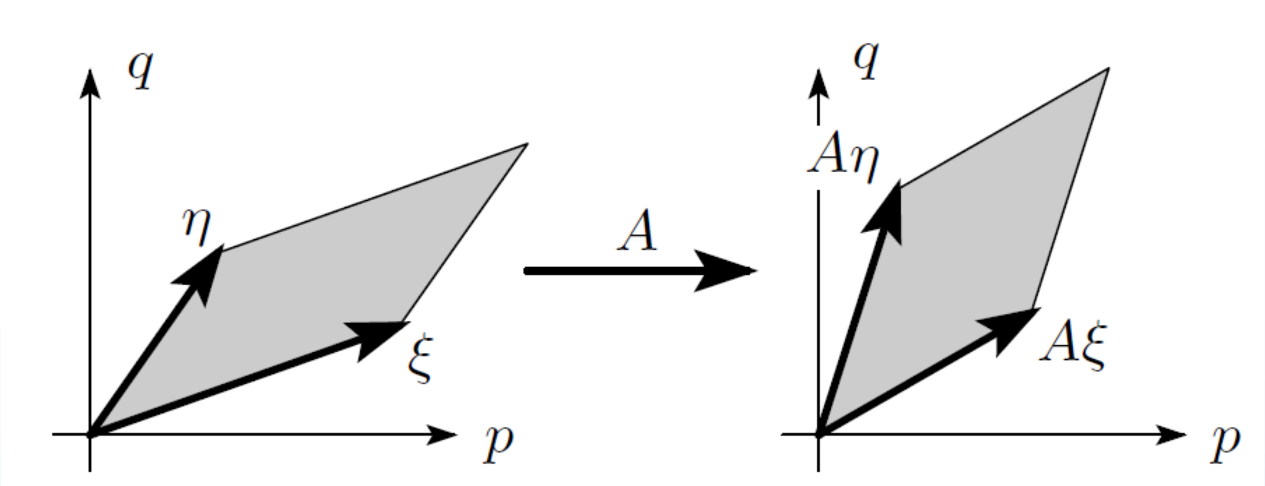
$$dp \wedge dq = \frac{\partial p}{\partial p_0}\frac{\partial q}{\partial q_0} dp_0 \wedge dq_0 - \frac{\partial p}{\partial q_0}\frac{\partial q}{\partial p_0} dp_0 \wedge dq_0 = \det A * dp_0 \wedge dq_0 = dp_0 \wedge dq_0$$

Preservation of area

The area of a parallelogram (with sides η and ξ) is given by $\eta^T J \xi$.

The area of a transformed parallelogram (with sides Aη and Aξ) is

$\eta^T A^T J A \xi = \eta^T J \xi$, if and only if A is symplectic



The symplecticness for a more general case (with d>1) can be written as

Conservation of volumn (Liouville's theorem)

Numerical integrators

A system with differential equations

$$\dot{x} = f(t,x) \qquad x = (p,q)$$

can usually be solved using integration method with infinitesimal integration steps Δt=h in each iteration. For Hamilton's equations,

<u>Euler(nonsymplectic)</u> $x_{n+1} = x_n + hJ\nabla H(x_n), \quad x_{n+1} = x_n + hJ\nabla H(x_{n+1})$

explicit implicit

<u>Euler(symplectic, 1st)</u> $p_{n+1} = p_n - h\nabla_q H(p_n, q_{n+1}), \quad q_{n+1} = q_n + h\nabla_q H(p_n, q_{n+1})$

<u>Implicit midpoint(symplectic, 2nd)</u> $x_{n+1} = x_n + hJ\nabla H(\frac{x_{n+1} + x_n}{2})$

Numerical integrators

Störmer-Verlet(symplectic,2nd)

$$p_{n+\frac{1}{2}} = p_n - \frac{h}{2}\nabla_q H(p_{n+\frac{1}{2}}, q_n)$$

$$q_{n+1} = q_n + \frac{h}{2}\left(\nabla_p H(p_{n+\frac{1}{2}}, q_n) + \nabla_p H(p_{n+\frac{1}{2}}, q_{n+1})\right)$$

$$p_{n+1} = p_{n+\frac{1}{2}} - \frac{h}{2}\nabla_q H(p_{n+\frac{1}{2}}, q_{n+1})$$

It is simply the symmetric composition (2nd order) of the two symplectic Euler methods with step size h/2.

For a 2nd order differential equation $\ddot{q} = -\nabla U(q), \quad H(p,q) = \frac{1}{2}p^T p + U(q)$

It can be simplified as $q_{n+1} - 2q_n + q_{n-1} = -h^2 \nabla U(q_n), \quad p_n = \frac{q_{n+1} - q_{n-1}}{2h}$

Runge-Kutta methods

s-stage Runge-Kutta

$$k_i = f(t + c_i h, x_n + h\sum_{j=1}^{s} a_{ij} k_j), \quad i = 1, \ldots, s$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i k_i$$

where $c_i = \sum_{j=1}^{s} a_{ij}, \quad \sum_{i=1}^{s} b_i = 1$. For a case where

$$s = 4, \quad c_1 = 0, \quad c_2 = c_3 = \frac{1}{2}, \quad c_4 = 1,$$

$$a_{21} = a_{32} = \frac{1}{2}, \quad a_{43} = 1$$

$$b_1 = b_4 = \frac{1}{6}, \quad b_2 = b_3 = \frac{2}{6}$$

it simplifies to the famous 4th order Runge-Kutta integrator.

Runge-Kutta methods

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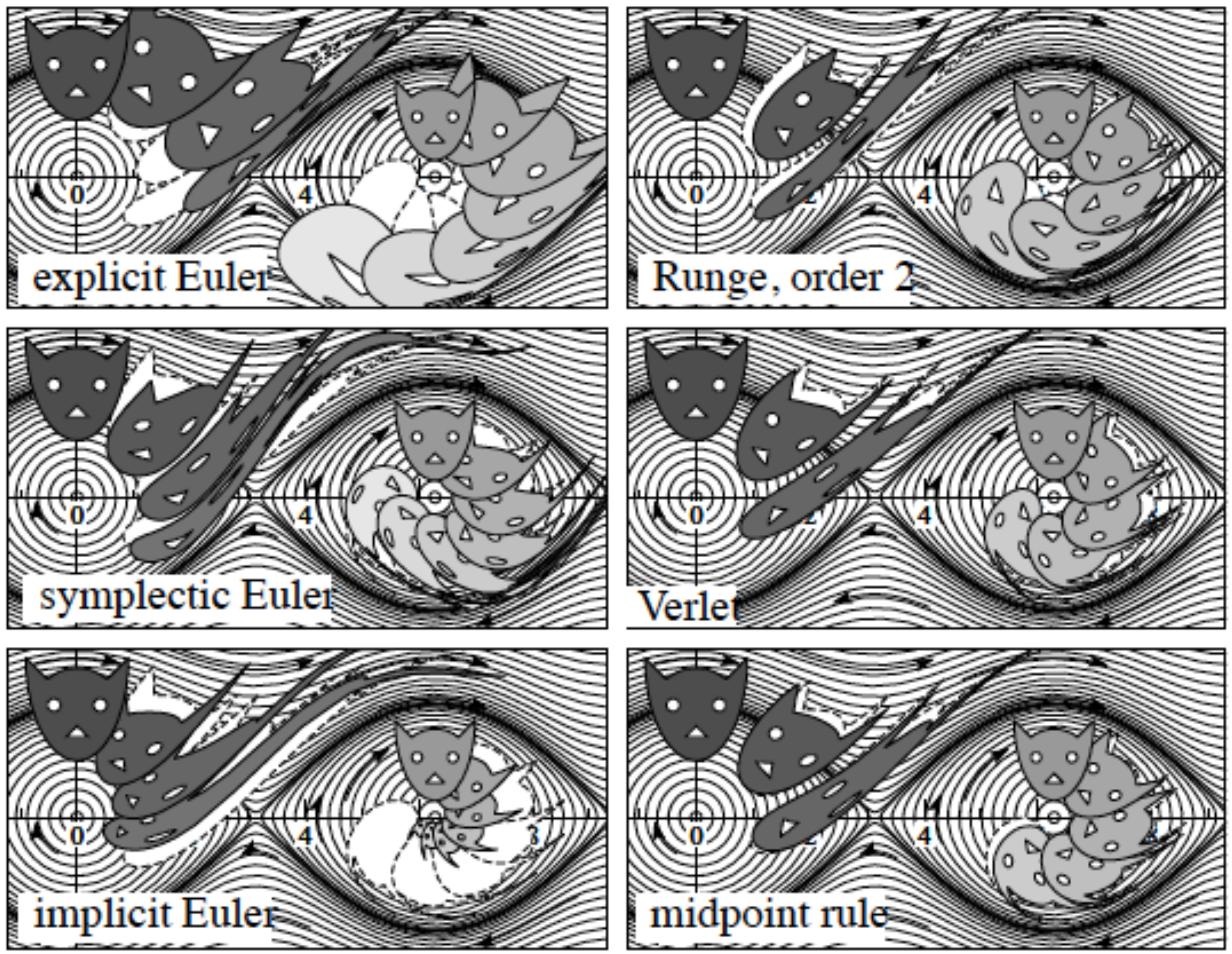


Symplectic mapping

In accelerator, we usually use transfer map to calculate lattice properties. For example, matrix for a quadrupole is

$$M = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{bmatrix}$$

$$k = \sqrt{K}$$

What a simulation code does is it slices the element into pieces and apply the kicks.

Symplectic mapping(cont'd)

Thus the transfer matrix becomes

$$M_{s\to s+\Delta s} = \begin{bmatrix} \cos k\Delta s & \frac{1}{k}\sin k\Delta s \\ -k\sin k\Delta s & \cos k\Delta s \end{bmatrix}$$

And then Taylor expansion gives

$$M_{s\to s+\Delta s} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \Delta s \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} + \Delta s^2 \begin{bmatrix} -\frac{k^2}{2} & 0 \\ 0 & -\frac{k^2}{2} \end{bmatrix} + \cdots$$

Truncation is required and up to 1st order

$$M_{s\to s+\Delta s} \approx \begin{bmatrix} 1 & \Delta s \\ -k^2\Delta s & 1 \end{bmatrix}$$

While the determinant of it is not unity– not symplectic.

Symplectic mapping(cont'd)

One trick to make the determinant 1 is to artificially add in one 2nd order term

$$M_{s\to s+\Delta s} \approx \begin{bmatrix} 1 & \Delta s \\ -k^2\Delta s & 1-k^2\Delta s^2 \end{bmatrix}$$

Which makes the transfer map not as accurate as if we simply keep it up to 2nd order

$$M_{s\to s+\Delta s} \approx \begin{bmatrix} 1-\frac{1}{2}k^2\Delta s^2 & \Delta s \\ -k^2\Delta s & 1-\frac{1}{2}k^2\Delta s^2 \end{bmatrix}$$

Which is not symplectic!

Symplecticity is not equal to accuracy!!

Symplectic mapping(cont'd)

1. Classical theories of numerical integration give information about how well different methods approximate the trajectories for fixed times as step sizes tend to zero. Dynamical systems theory asks questions about asymptotic, i.e. infinite time, behavior.
2. Geometric integrators are methods that exactly conserve qualitative properties associated to the solutions of the dynamical system under study.
3. The difference between symplectic integrators and other methods become most evident when performing long time integrations (or large step size).
4. Symplectic integrators do not usually preserve energy either, but the fluctuations in H from its original value remain small.

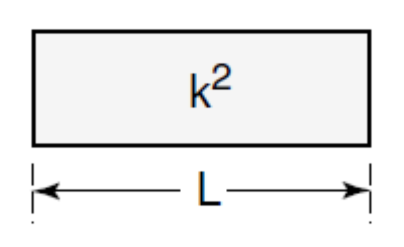
Symplectic mapping(cont'd)

One way of thinking is to use thin lens approximation, divide the quadrupole into drifts and thin lens which all have transfer matrices with unity determinant.

Transfer matrices for drift and sudden kick

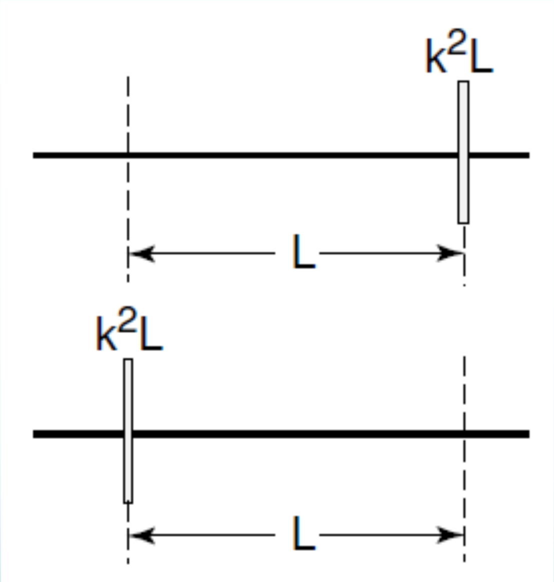
$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -k^2L & 1 \end{bmatrix}$$

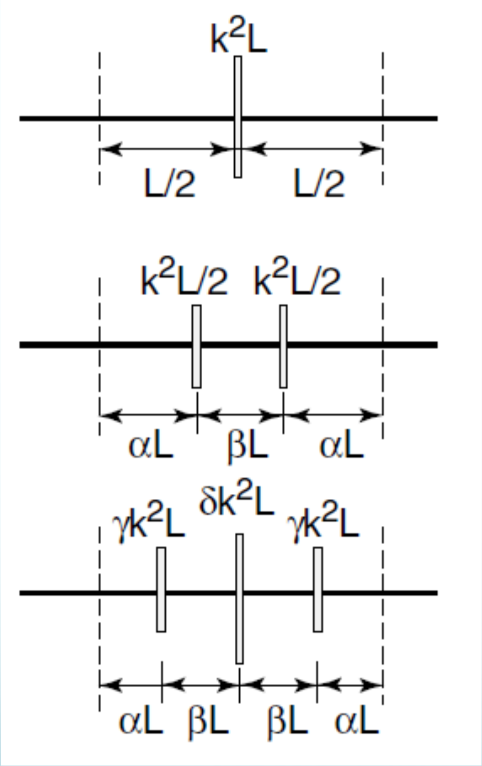
With a quadrupole at length L



Symplectic mapping(cont'd)

So we have various ways of dividing the quadrupole which result into different order of symplicticity.





Symplectic mapping(cont'd)

After splitting the magnets, we need to solve for the parameters(symplicticity is automatically preserved). Take the 2nd on the right as an example. Total transfer map is

$$M = \begin{bmatrix} 1 & \alpha L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}k^2 L & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}k^2 L & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha L \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{2}k^2L^2 + \frac{1}{4}\alpha\beta k^4 L^4 & L - \alpha(\alpha+\beta)k^2L^3 + \frac{1}{4}\alpha^2\beta k^4 L^5 \\ -k^2 L + \frac{1}{4}\beta k^4 L^3 & 1 - \frac{1}{2}k^2L^2 + \frac{1}{4}\alpha\beta k^4 L^4 \end{bmatrix}$$

Comparing with

$$M = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{bmatrix}$$

Symplectic mapping(cont'd)

Keeping them equal up to 4th order then gives

$$\alpha(\alpha+\beta) = \frac{1}{6}$$
$$\frac{1}{4}\beta = \frac{1}{6}$$
$$2\alpha + \beta = 1$$

Last one arises from geometry condition.

Unfortunately, this does not have a solution—symplicticify failure. But the 3rd one on the right has a solution

$$\beta = \frac{1-2^{1/3}}{2(2-2^{1/3})} \approx -0.1756$$

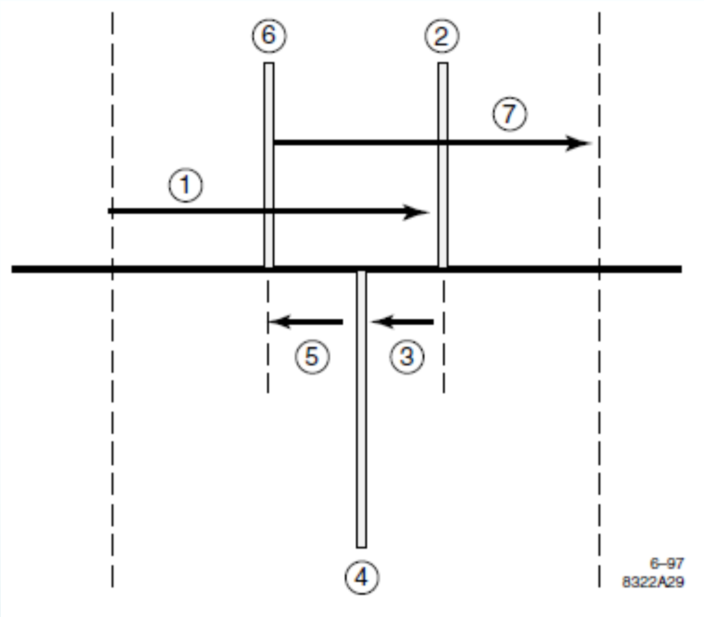
$$\alpha = \frac{1}{2} - \beta = \frac{1}{2(2-2^{1/3})} \approx 0.6756$$

$$\gamma = \frac{1}{24\beta^2} = \frac{1}{2-2^{1/3}} \approx 1.3512$$

$$\delta = 1 - 2\gamma = -\frac{2^{1/3}}{2-2^{1/3}} \approx -1.7024$$

Symplectic mapping(cont'd)

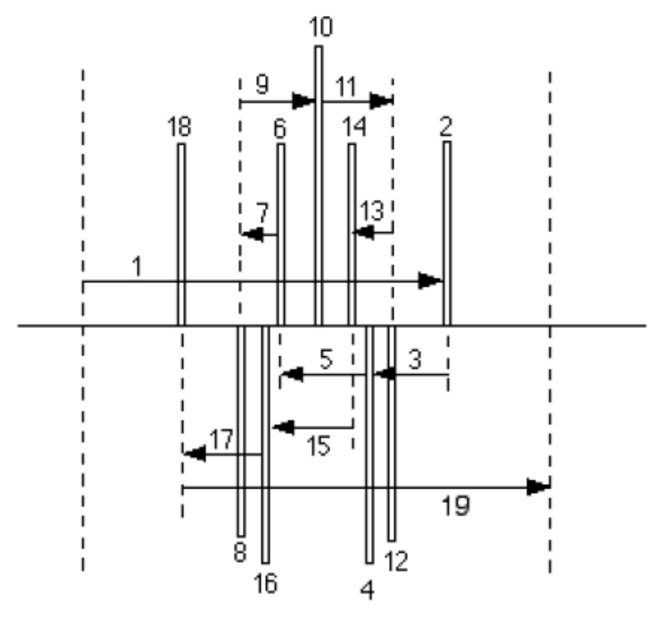
Notice that both β and δ are negative. This means we need to go through 7 steps for the symplectic integration shown as follows.



This results in a 4th order symplectic integration.

Symplectic mapping(cont'd)

Higher order of symplectic integration can be achieved simply by dividing the magnet into more pieces and solving much more complicated set of equations. A 6th order integration is done in 19 steps.



Accuracy vs order

Order does bring up complicity but does it provide higher accuracy? Considering the amplitude of phase space given by

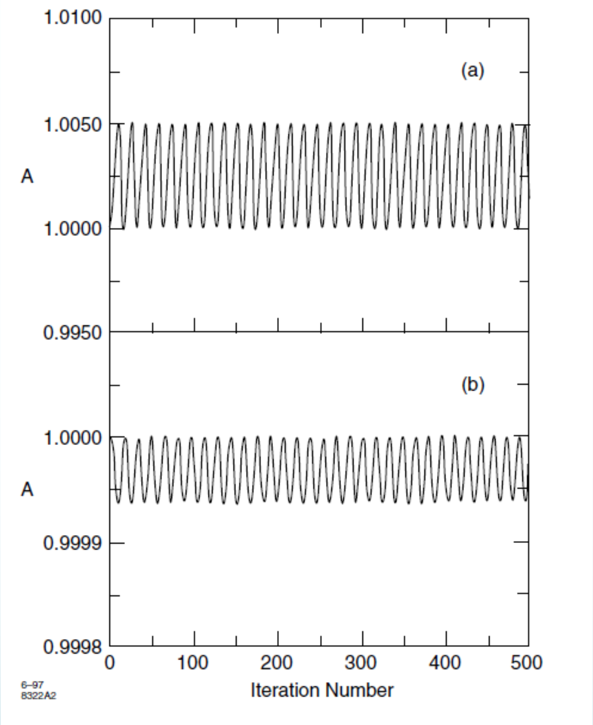
$$A = \sqrt{x^2 + (x'/k)^2}$$

With initial A to be normalized to 1.

Exact tracking should always A while if we use symplectic mapping it's not the case.

Accuracy vs order(cont'd)

Comparison of 2nd order and 4th order of symplectic integration is given as



With the top one as 2nd order and bottom one the 4th. Stability is always preserved but the accuracy is greatly improved by using higher order integration.

Notice that the deviation from 1 tells us the deviation from a pure circle– distortion. Higher order also improves the shape distortion introduced by this symplecticify process.

Accuracy vs order(cont'd)

A list shown all the integrators from 2nd order to 5th order is shown here with the error information and the model needed to achieve it.

Integrator	Model	Error
1st order	$(L)(SL)$	$\mathcal{O}(L^2)$
1st order	$(SL)(L)$	$\mathcal{O}(L^2)$
Ray tracing	$(\frac{L}{n})(\frac{SL}{n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^2}{n})$
2nd order(thin-lens)	$(\frac{L}{2})(SL)(\frac{L}{2})$	$\mathcal{O}(L^3)$
Ray tracing	$(\frac{L}{2n})(\frac{SL}{n})(\frac{L}{2n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^3}{n^2})$
4th order	$(\alpha L)(\gamma SL)(\beta L)(\delta SL)(\beta L)(\gamma SL)(\alpha L)$	$\mathcal{O}(L^5)$
Ray tracing	$(\frac{\alpha L}{n})(\frac{\gamma SL}{n})(\frac{\beta L}{n})(\frac{\delta SL}{n})(\frac{\beta L}{n})(\frac{\gamma SL}{n})(\frac{\alpha L}{n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^5}{n^4})$

Notice that simple repetition doesn't improve order.

Have to change way of slicing.

4th order Runge-Kutta is not symplectic

Considering DE $x'' = f(x, x', s)$, with given initial x & x'. A 4th order Runge-Kutta solves it at x=L

$$
\begin{aligned}
x(L) &\approx x(0) + Lx'(0) + \frac{1}{6}L(t_1 + t_2 + t_3) \\
x'(L) &\approx x'(0) + \frac{1}{6}(t_1 + 2t_2 + 2t_3 + t_4)
\end{aligned}
$$

With

$$
\begin{aligned}
t_1 &= Lf[x(0), x'(0), 0] \\
t_2 &= Lf[x(0) + \frac{1}{2}Lx'(0), x'(0) + \frac{1}{2}t_1, \frac{1}{2}L] \\
t_3 &= Lf[x(0) + \frac{1}{2}Lx'(0) + \frac{1}{4}Lt_1, x'(0) + \frac{1}{2}t_2, \frac{1}{2}L] \\
t_4 &= Lf[x(0) + Lx'(0) + \frac{1}{2}Lt_2, x'(0) + t_3, L]
\end{aligned}
$$

4^{th} order Runge-Kutta is not symplectic

For a quadrupole, it gives

$$x(L) \approx x(0)\left[1-\frac{1}{2}k^2L^2+\frac{1}{24}k^4L^4\right]+\frac{1}{k}x'(0)\left[kL-\frac{1}{6}k^3L^3\right]$$
$$x'(L) \approx -kx(0)\left[kL-\frac{1}{6}k^3L^3\right]+x'(0)\left[1-\frac{1}{2}k^2L^2+\frac{1}{24}k^4L^4\right]$$

with sextupole, it becomes

$$\begin{aligned} x(L) &\approx x_0+x_0'L+\frac{1}{2}Sx_0^2L^2+\frac{1}{3}Sx_0x_0'L^3+\frac{S}{12}(x_0'^2+Sx_0^3)L^4 \\ &+\frac{1}{24}S^2x_0^2x_0'L^5+\frac{1}{96}S^3x_0^4L^6 \\ x'(L) &\approx x_0'+Sx_0^2L+Sx_0x_0'L^2+\frac{S}{3}(x_0'^2+Sx_0^3)L^3+\frac{5}{12}S^2x_0^2x_0'L^4 \\ &+S^2x_0(\frac{5}{24}x_0'^2+\frac{1}{16}Sx_0^3)L^5+\frac{1}{12}S^2x_0'(\frac{1}{2}x_0'^2+x_0^3)L^6 \\ &+\frac{1}{16}S^3x_0^2x_0'^2L^7+\frac{1}{48}S^3x_0x_0'^3L^8+\frac{1}{384}S^3x_0'^4L^9 \end{aligned}$$

4^{th} order Runge-Kutta is not symplectic

For quadrupole, the determinant is

$$1-\frac{k^6L^6}{72}+\frac{k^8L^8}{576}$$

For sextupole, the determinant is

$$\begin{aligned}1-\frac{1}{72}(2x_0'^2-9Sx_0^3)S^2L^6+\frac{7}{36}x_0^2x_0'S^3L^7+\frac{1}{144}x_0(7x_0'^2+15Sx_0^3)S^3L^8\\+\frac{1}{288}x_0'(-x_0'^2+46Sx_0^3)S^3L^9+\frac{1}{576}x_p^2(45x_0'^2+16Sx_0^3)S^4L^{10}\\+\frac{1}{288}x_0x_o'(4x_o'^2+13Sx_0^3)S^4L^{11}+\frac{1}{576}x_0^3(15x_0'^2+2Sx_0^3)S^5L^{12}\\+\frac{1}{576}x_0^2x_0'(4x_0'^2+3Sx_0^3)S^5L^{13}+\frac{1}{1152}x_0x_0'^2(x_0'^2+3Sx_0^3)S^5L^{14}\\+\frac{1}{2304}x_0^3x_0'^3S^6L^{15}\end{aligned}$$

Both of them are not 1– not symplectic!!