CeC Theory

Developing Theoretical Tools for CeC

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The role of theoretical tools

- Make predictions for an ideal/simplified system:
  - Validate numerical simulations;
  - Improve our understandings of the physical processes involved in CeC and if possible, obtain scaling laws;
- Theoretical tools are often inadequate for accurate prediction of a realistic CeC system (requires numerical simulation).

Overall Structure of CeC Prediction

A. Prediction of the single pass kicks received by an ion in the cooling section

Ion's initial condition	Kicks due to CeC
x, x', y, y', t, E	dx', dy', dE
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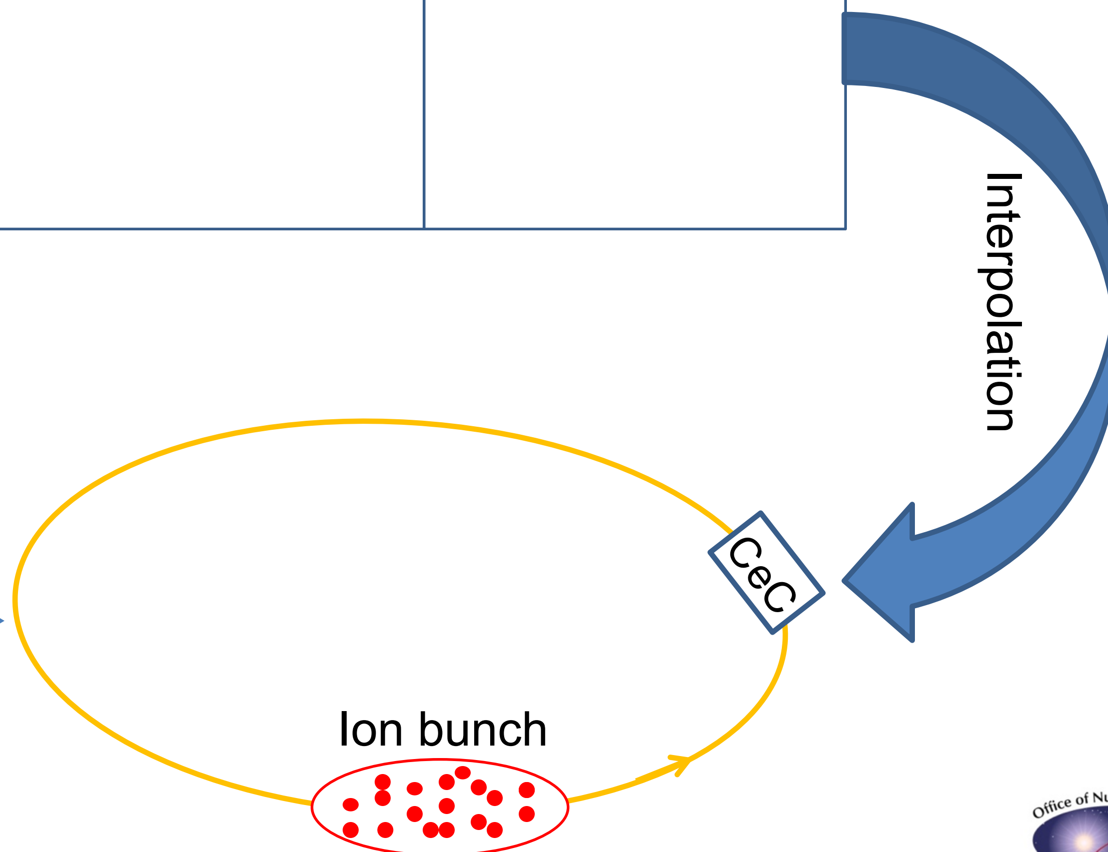
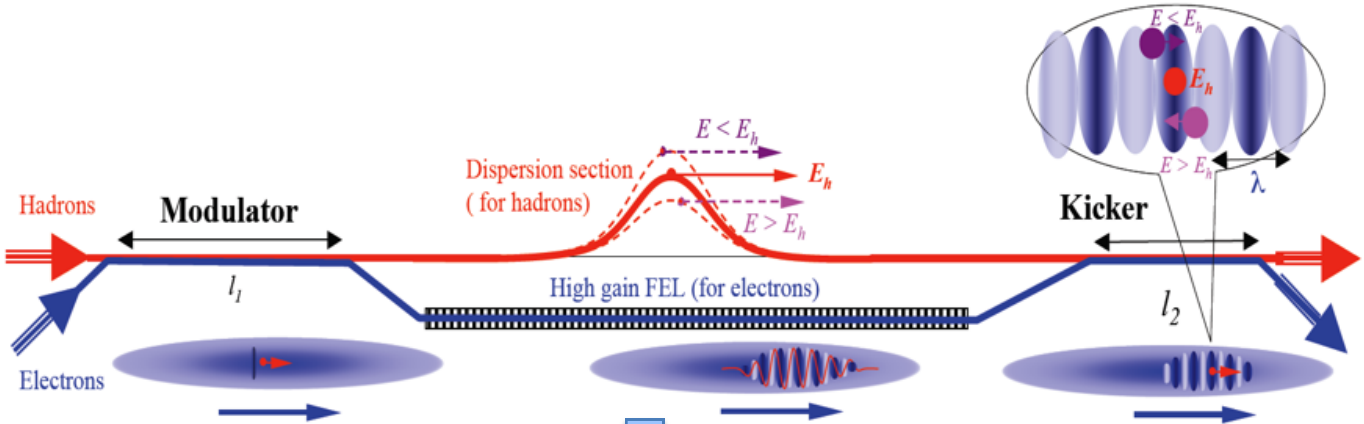


B. Long term prediction for circulating ions

Outline

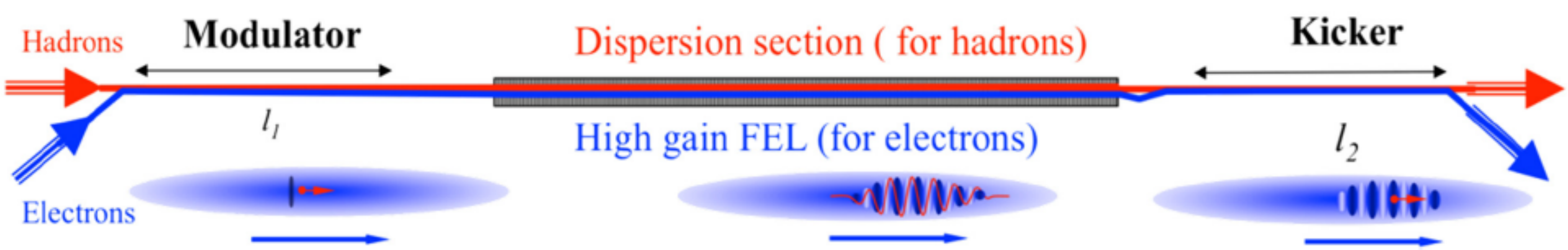
- Prediction of cooling force in a single pass
  - FEL-based CeC system
  - PCA-based CeC system
  - Chicane-based CeC system / MBEC
- Prediction of Au beam evolution in the presence of cooling
  - Analytical approach: solving 1-D Fokker-Planck equation
  - Comparison with macro-particle tracking
- Future work and challenges

FEL-based CeC

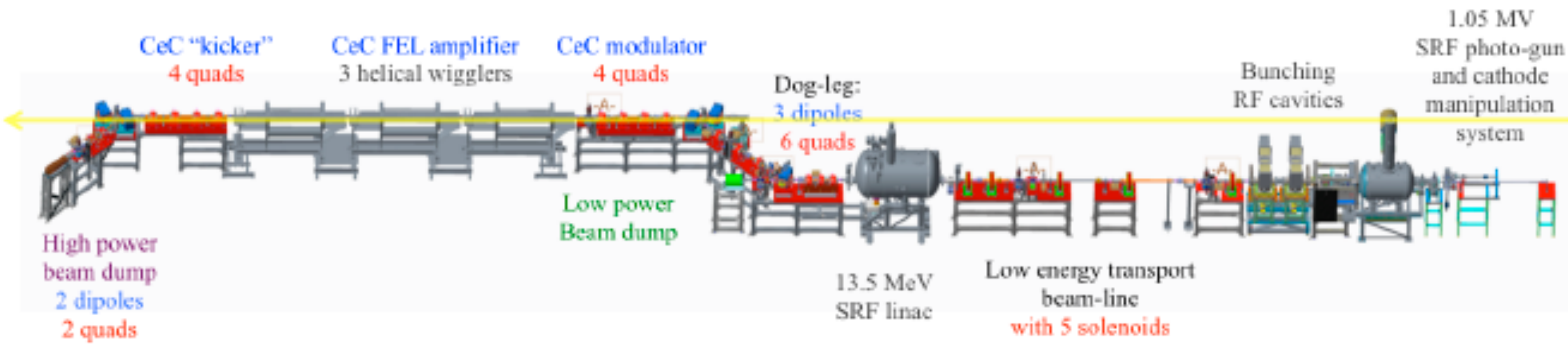


The economical version is made possible by:
1. Use relatively weak undulator field to reduce the delay of electrons;
2.Wave-packet moves faster than electrons.

Our Proof-of-Principle is an economic version of CeC, where electrons and hadrons are co-propagating along the entire CeC system



Parameters for the CeC PoP experiment



Electron beam parameters* (γ=28.6)	
Peak current*	40 A
Bunch length, (full length with uniform profile)	25~30 ps
RMS emittance, normalized	3~5 μm
RMS beam width at modulator/kicker	700 μm
RMS beam width at FEL amplifier	235 μm
RMS energy spread	1e-3

System parameters*	
Length of modulator/kicker	3 m
Undulator period	4 cm
Total number of undulator period (3 sections)	188
Undulator parameter, a_w	0.5
FEL optical wavelength	30.5 μm
Pierce parameter, ρ	0.012

* The designed peak current was changed to 75A to compensate FEL gain reduction due to longitudinal space charge

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Analytical Tools for the Modulation Process I

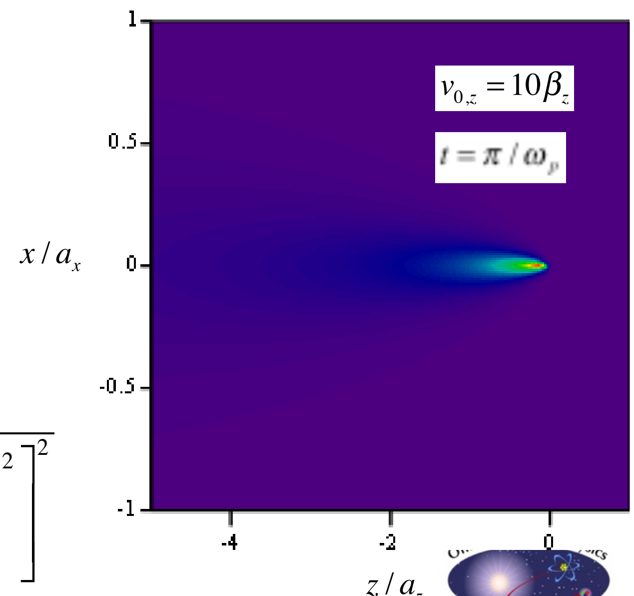
- Cold uniform electron beam (© V.N. Litvinenko)
  - Density modulation: $q = -Ze\cdot(1-\cos\varphi_1)$ $\quad \varphi_1 = \omega_p l_1 / c\gamma_0$
  - Energy modulation ($\varphi_1 << 1$): $\left\langle \frac{\delta E}{E} \right\rangle \cong -2Z \frac{r_e}{a^2} \cdot \frac{L_{pol}}{\gamma} \cdot \left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2/\gamma^2 + z^2}} \right)$
- Warm uniform electron beam with k-2 velocity distribution

$$f_0(\vec{v}) = \frac{1}{\pi^2 \beta_x \beta_y \beta_z} \left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2} \right)^{-2}$$

G. Wang and M. Blaskiewicz, Phys Rev E 78, 026413 (2008)

  - Density modulation:

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 a_x a_y a_z} \int_0^{\omega_p t} \frac{\tau \sin\tau \cdot d\tau}{\left[\tau^2 + \left(\frac{x}{a_x} + \frac{v_{0,x}}{\beta_x}\tau \right)^2 + \left(\frac{y}{a_y} + \frac{v_{0,y}}{\beta_y}\tau \right)^2 + \left(\frac{z}{a_z} + \frac{v_{0,z}}{\beta_z}\tau \right)^2 \right]^2}$$



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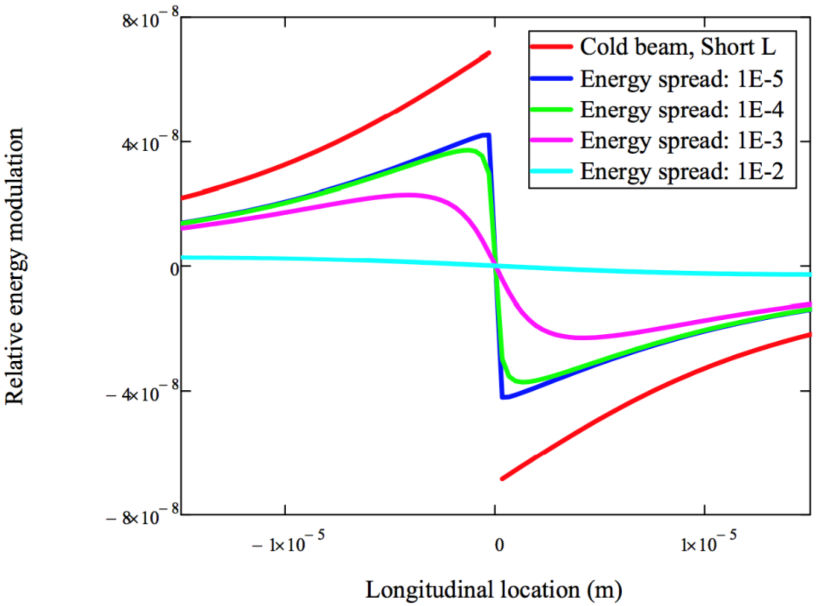
Analytical Tools for the Modulation Process II

– Energy modulation:

$$\left\langle \frac{\delta E}{E_0} \right\rangle = \frac{\langle v_z \rangle}{c} = -\frac{1}{e n_0 \pi a^2 c} I_d \left(\gamma_0 z_l , \frac{L_{\text{mod}}}{\beta_0 \gamma_0 c} \right)$$

where $I_d(z,t)$ is an 1-D integral with finite integration limits (see backup slides).



It reduces to the previously derived cold beam result at the corresponding limits:

$$\bar{\beta} = 0 \qquad v_{0,z} = 0 \qquad L_{\text{mod}} << \beta_0 \gamma_0 c / \omega_p$$

$$\left\langle \frac{\delta E}{E} \right\rangle \approx -2 Z_i \frac{r_e}{a^2} \frac{L_{\text{mod}}}{\gamma} \cdot \left[\frac{z_l}{|z_l|} - \frac{z_l}{\sqrt{z_l^2 + a^2/\gamma^2}} \right].$$

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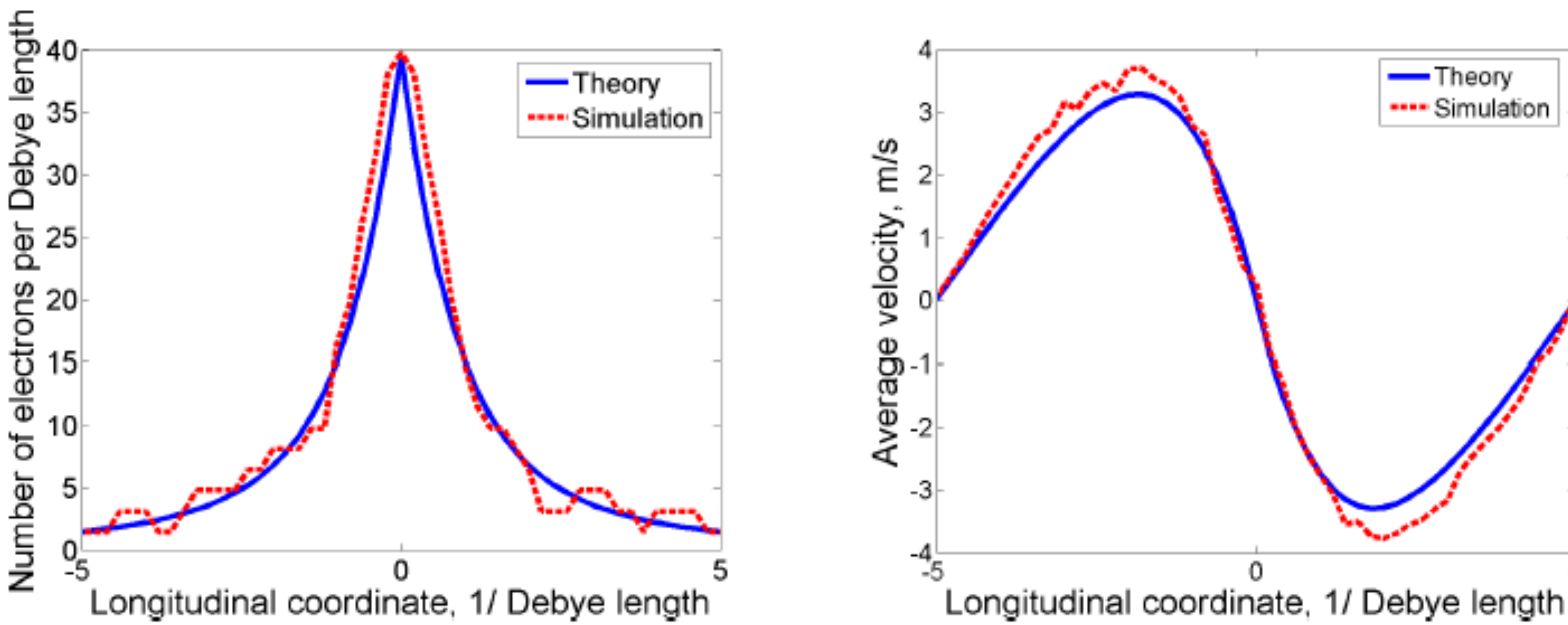
Validating Numerical Simulations with Theoretical Prediction

- Simulations based on perturbative trajectory approach (© J. Ma, with code SPACE)
  - Benchmarked with theory for uniform warm beam



(a) Longitudinal density

(b) Longitudinal velocity

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Analytical Prediction for FEL Amplifier

- For the CeC PoP parameters, the FEL amplifier works in the diffraction dominated regime

$$\eta_d = \frac{l_{1D}}{2k_{opt}\sigma_x^2} = 6.6 >> 1 \qquad l_{1D} = \frac{1}{2\sqrt{3}\rho k_u} = 0.15m \qquad \sigma_{x,FEL} = 0.235mm$$

and hence we can't rely on 1-D theory to predict the performance of the amplifier.

- There are formulae to make corrections to the 1-D gain length for the general case developed by Ming Xie,

$$l_{3D} = l_{1D}\left(1 + \Lambda_{3D}\right) = 0.363m$$

and using this number we get (for 40A peak current and 5 mm.mrad emittance)

$$G = \frac{1}{3}\delta_\omega \exp\left(\frac{l_{FEL}}{2l_{3D}}\right) = 74.6$$

$$\rho = \frac{1}{2}\left(\frac{2\lambda_u j_0}{\gamma k_{opt} I_A} \cdot \frac{a_w^2}{1 + a_w^2}\right)^{1/3} = 0.012 \qquad \delta_\omega = \sqrt{\frac{3\sqrt{3}\rho}{k_u l_{FEL}}} = 0.0073$$

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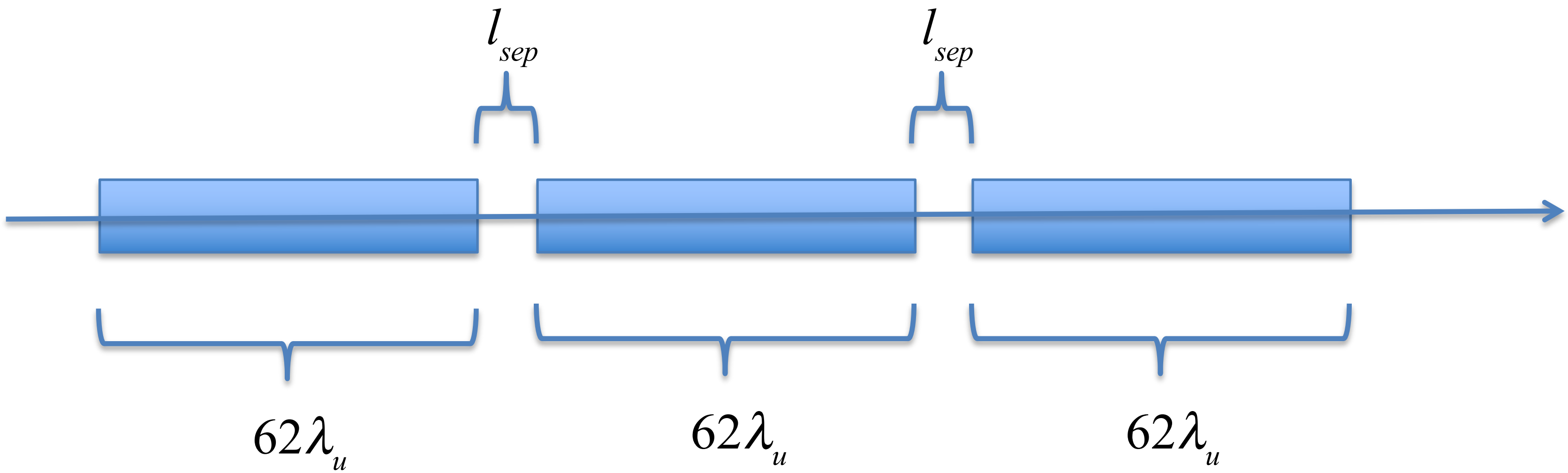
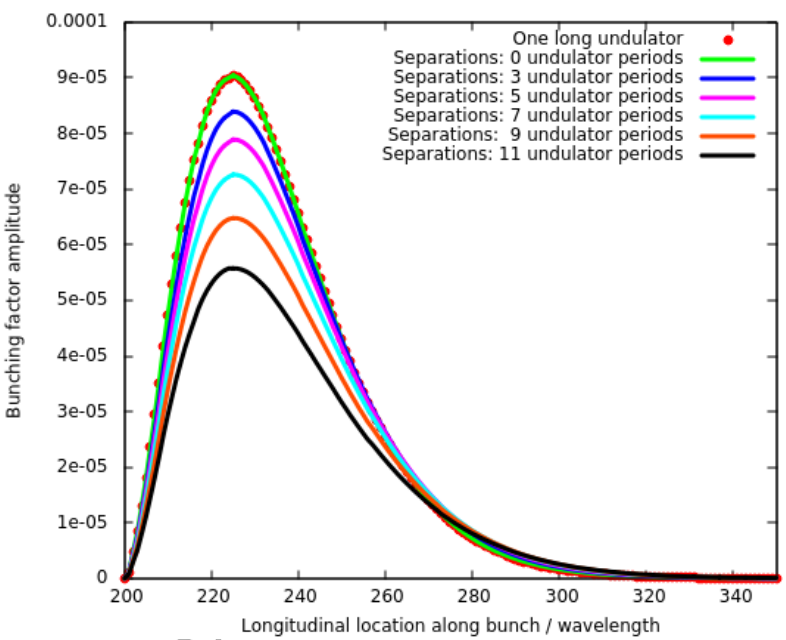
Effects due to multi-subsections





$$\sigma_r = \sqrt{\frac{2\lambda_1 L_G}{4\pi}} = \sqrt{\frac{30\times 10^{-6}\times 0.4}{2\pi}} = 1.38mm$$ The radiation width

$$L_R = \frac{\pi\sigma_r^2}{\lambda_1} = \frac{\pi\times 1.38^2\times 10^{-6}m^2}{30\times 10^{-6}m} = 20cm$$ Rayleigh length

$$R = \frac{A(s=0)}{A(s=44cm)} = \frac{1}{1+\frac{44^2}{20^2}} = 17\%$$ Only 17% of the radiation power propagates to the next subsection

- Our undulator consists of 3 sub-sections and more than 80% of radiation power is lost in the 44cm gap between two successive subsections.

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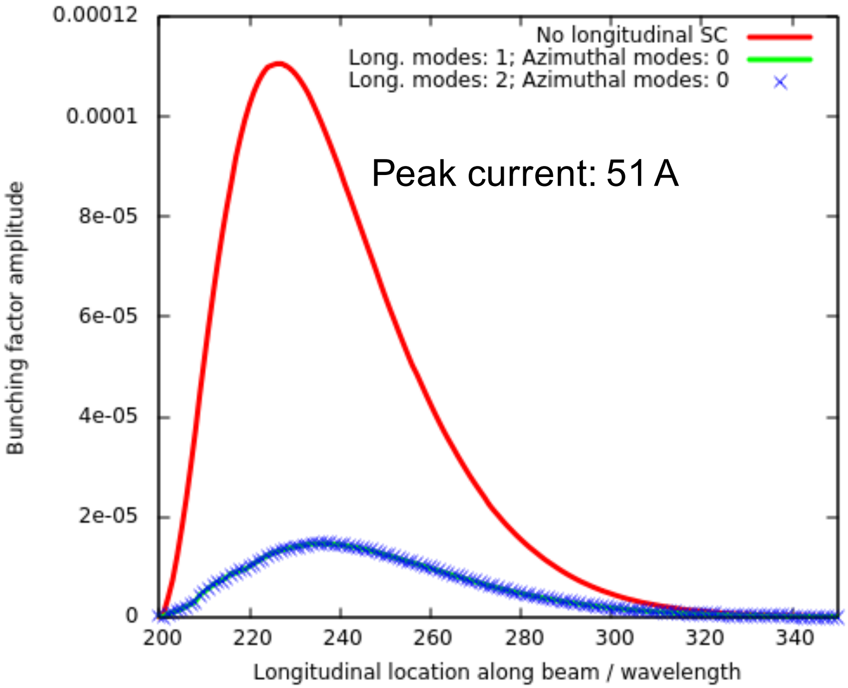
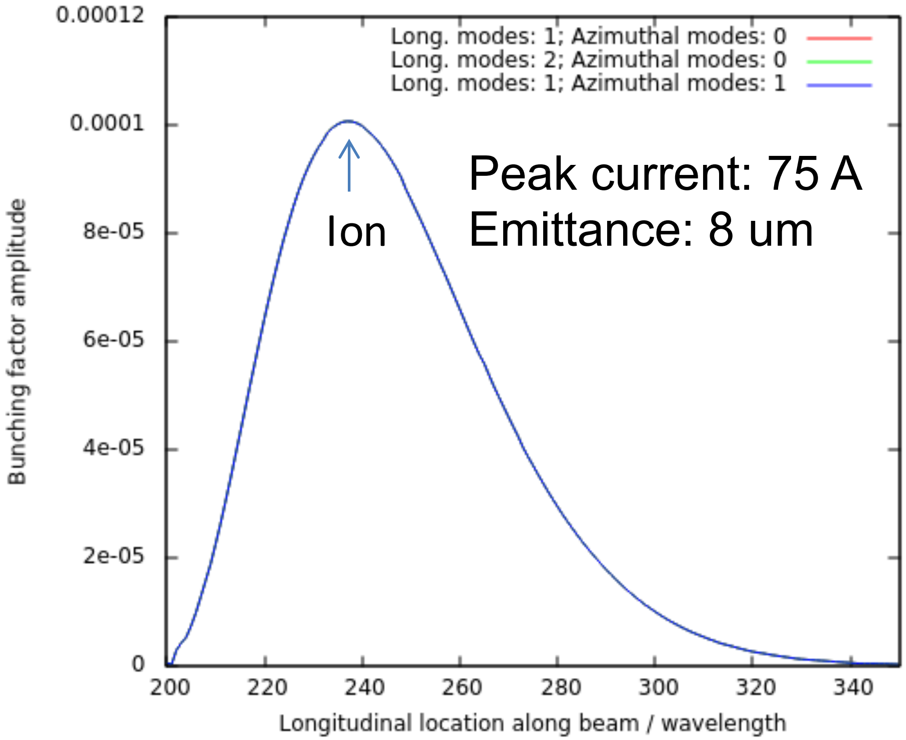
Effects due to Longitudinal Space Charge





- Longitudinal space charge has more pronounced effects on the FEL gain than what to be expected from theory.

FEL Saturation

Using the saturating criteria that radiation power at saturation is $\rho\gamma mc^2 I / e$

$$\sqrt{\frac{l_G}{l_{sat}}}\exp\left(\frac{l_{sat}}{l_G}\right) = \frac{3}{2\sqrt{\pi}}\frac{\lambda_{opt}}{\rho r_e}\frac{I}{I_A} \Rightarrow l_{sat} = 22.8 l_G \Rightarrow l_{sat} = 22.8 l_{3D} = 8.3m$$

$$G_{sat} = \frac{1}{3}\delta_{\omega,sat}\exp\left(\frac{l_{sat}}{2l_{3D}}\right) = 330$$

Using the criteria $\left|\delta\hat{n}/n_0\right|_{\max} < 1 \Rightarrow$

$$G_{sat} = \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}} = \sqrt{N_{e\lambda}\delta_{\omega,sat}} = 523$$

* There is a factor of $(2\pi)^{\frac{-1}{4}} \approx 0.632$ between the two estimates.

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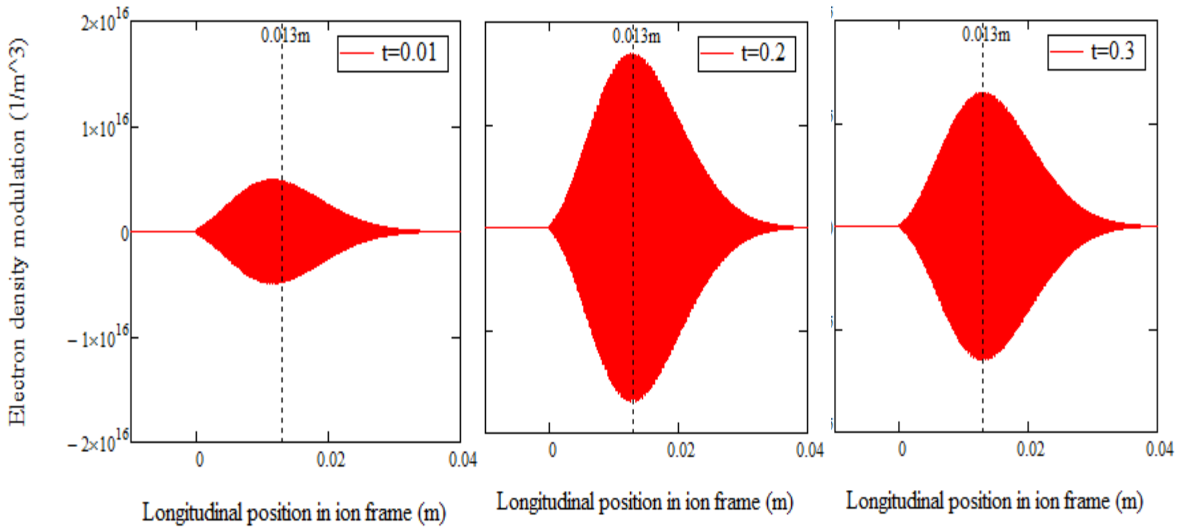
Analytical Tools for Kicker

- Dynamic equation in Kicker is very similar to that in the modulator except the initial modulation in 6D phase space dominates the process. For 1D FEL output with the certain assumptions for the transverse distribution, the following analytical formula for density modulation can be derived

$$\tilde{n}_1(k_z,t) = -Z_i\tilde{\Lambda}_{drive}(k_z)e^{ik_z v_{0z}t}e^{-|k_z|\sigma_{vz}t}\left[\cos(\omega_p t) + \frac{(\lambda_1 + i\hat{C})\beta_z c\gamma_z\Gamma - ik_z v_{0z} + |k_z|\sigma_{vz}}{\omega_p}\sin(\omega_p t)\right]$$



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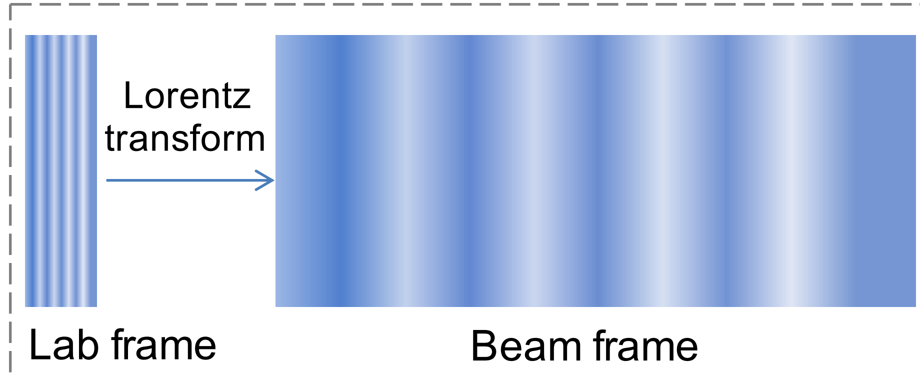
Field Reduction due to Finite Transverse Size

To estimate the reduction of electric field due to finite transverse size, we solve Poisson equation for charge distribution of the form:

$$\rho(\vec{r}) = \rho_o(r)\cdot\cos(k_{cm}z)$$

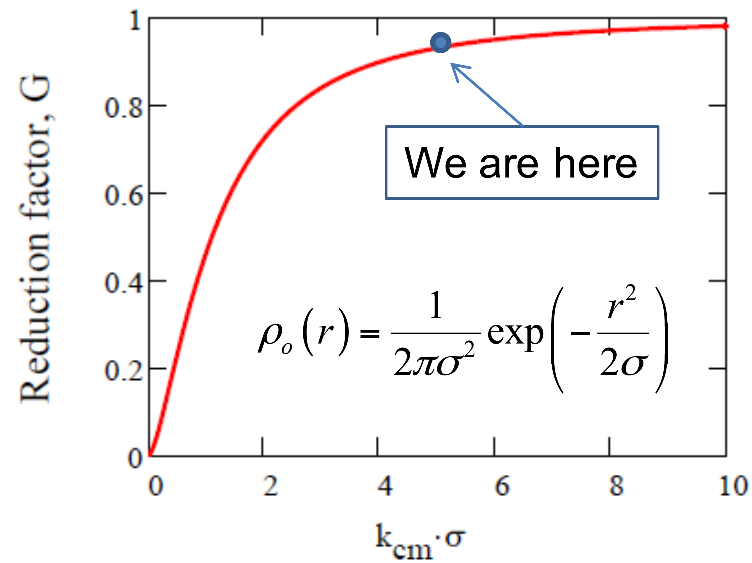
and get the on-axis longitudinal electric field



$$E_z(r=0) = -\frac{1}{k_{cm}\varepsilon_0}\sin(k_{cm}z)\int_0^\infty \eta K_0(\eta)\cdot\rho_o\left(\frac{\eta}{k_{cm}}\right)d\eta$$

For Gaussian transverse density distribution, we obtain

$$E_z(r=0) = -\frac{\sin(k_{cm}z)}{2\pi k_{cm}\varepsilon_0\sigma^2}G(k_{cm}\sigma)$$



$$G(q) = \int_0^\infty \eta K_0(\eta)\cdot\exp\left(-\frac{\eta^2}{2q^2}\right)d\eta = \frac{1}{q^2}\int_0^\infty \eta\left[1-\eta K_1(\eta)\right]\cdot\exp\left(-\frac{\eta^2}{2q^2}\right)d\eta$$

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*The results reproduces what previously derived by G. Stupakov.

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Single-Pass Kick Received by an ion in FEL-based CeC section

Energy kicks from CeC is $\Delta E_j = \Delta E_{coh,j} + \Delta E_{inc,j}$

Coherent kick induced by the ion itself $\Delta E_{coh,j} \equiv -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right)$

Incoherent kick induced by the neighbor ions (using the Gaussian profile as obtained by quadratic expansion of FEL eigenvalues)

$$\Delta E_{inc,j} \equiv -Z_i e E_p l \sum_{i \neq j} \exp\left[-\frac{\left(z_j - z_i\right)^2}{2\sigma_{z,rms}^2}\right] \sin\left(k_0\left(D\delta_j + z_j - z_i\right) - k_2^2\left(z_j - z_i\right)^2\right)$$

z_i: longitudinal location of the i^{th} ion; $\sigma_{z,rms}$: RMS width of the wave-packet; D : R_{56}.

Since there is no correlation between any successive incoherent kicks, one can use a random kick to represent the incoherent kicks

$$\Delta E_{j,N} \approx -Z_i e E_p l_1 \sin\left(k_0 D \cdot \delta_j\right) + \sqrt{\frac{\left\langle \Delta E_{inc,j}^2 \right\rangle}{\left\langle X^2 \right\rangle}} \cdot X_{j,N}$$

For a random number uniformly distributed between -1 and 1

$$\left\langle X^2 \right\rangle = \frac{1}{2}\int_{-1}^{1} X^2 dX = \frac{1}{3}$$

Simulation tools for predicting the influences of CeC to a circulating ion beam II

- Assuming the ion density does not vary significantly over the width of the wave-packet

$$\left\langle \Delta E_{inc,j}^{2} \right\rangle = \frac{\left(Z_i e E_p l_1\right)^2}{2} \int_{-\infty}^{\infty} \rho_{ion}\left(z_i\right) e^{-\frac{\left(z_i - z_i\right)^2}{\sigma_{z,rms}^2}} dz_i \approx \frac{\left(Z_i e E_p l_1\right)^2}{2} \sqrt{\pi} \rho_{ion}\left(z_j\right) \sigma_{z,rms}$$

- The one-turn energy kick due to CeC is

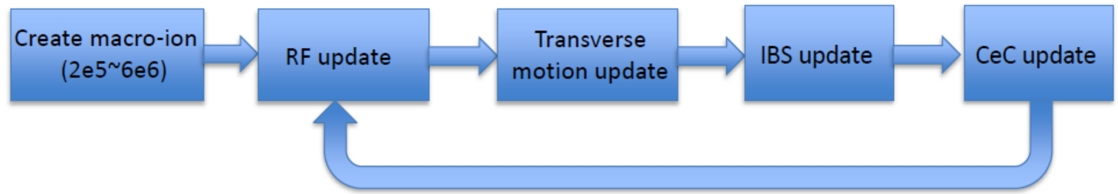
$$\Delta E_{j,N} \approx -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right) + Z_i e E_p l \sqrt{\frac{3}{2} \sqrt{\pi} \rho_{ion}\left(z_j\right) \sigma_{z,rms}} \cdot X_{j,N} + \Delta E^{e}{}_{j,N}$$

Diffusive kick induced by neighbor electrons, i.e. electrons' shot noise

$$\Delta E_{j,N}^{e} \approx e E_p l \sqrt{\frac{3}{2} \sqrt{\pi} \rho_e\left(z_j\right) \sigma_{z,rms}} \cdot Y_{j,N}$$

Applying Single-pass Kick to Predict Ion Beam Evolution with Cooling

1. Macro-particle tracking
2. Solving Fockker-Planck equation

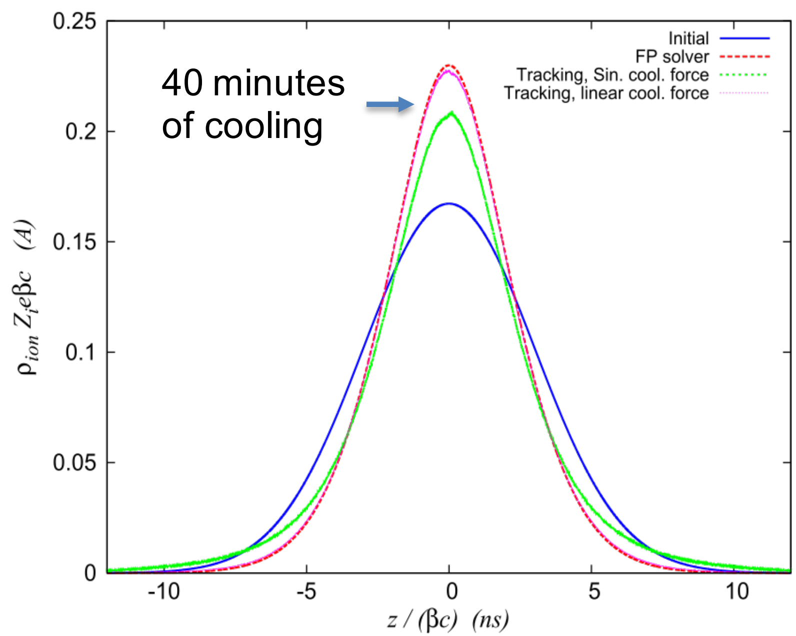


Electron beam parameters		Ion beam parameters	
Peak current, A	75	Charge number, Z_{ion}	79
Full bunch length, ps	25	Bunch intensity	10^8
Norm. emittance, RMS, μm	5	Bunch length, RMS, ns	3.06
Relative energy spread, RMS	10^{-3}	Relative energy spread, RMS	3.35×10^{-4}
Beam energy, γ	28.66	RF frequency, MHz	28

Table 1: Beam parameters for the proof of CeC principle experiment

Gain of FEL amplification	80	FEL wavelength, μm	30.5
Peak correcting field, V/m	36	R_{56}, cm	1.2
Kicker length, m	3	Coherent length, σ_w, mm	0.54
Coherent kick amplitude, g_γ	4.657×10^{-8}	Local cooling time (T_0), s	3.185
CeC diffusive kick from neighbor ions, d_{ion}	1.163×10^{-5}	CeC diffusive kick from electrons, d_e	2.038×10^{-5}
IBS diffusive kick at bunch center, $d_{IBS}(0)$	1.886×10^{-6}		

Table 2: CeC system parameters of the proof of CeC principle experiment



PCA-based CeC

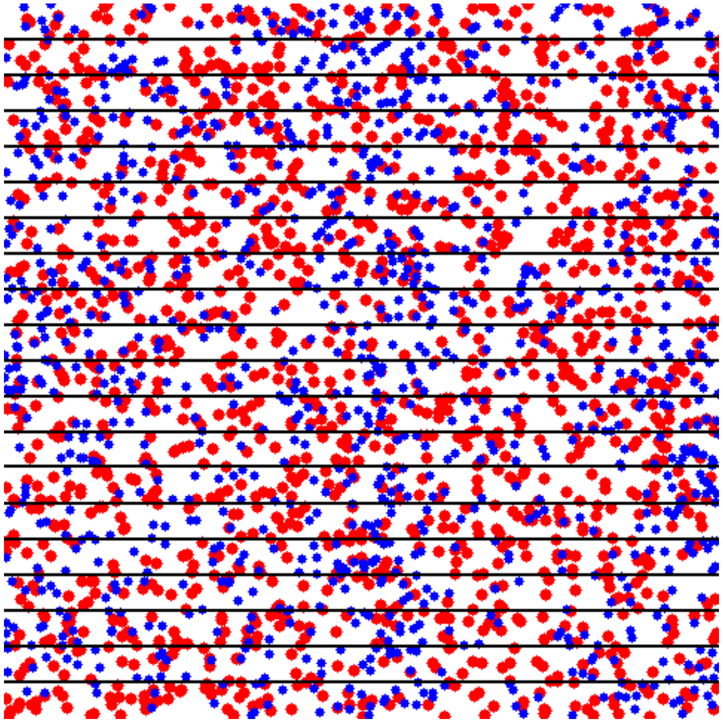
- Plasma-Cascade Instability
  - Longitudinal plasma oscillation

$$\frac{d^2\tilde{n}}{dt^2} + \omega_p^2 \tilde{n} = 0;$$

$$\omega_p^2 = \frac{4\pi n_o e^2}{m};$$

Plasma oscillation also exist in single-species plasma like electron beam.

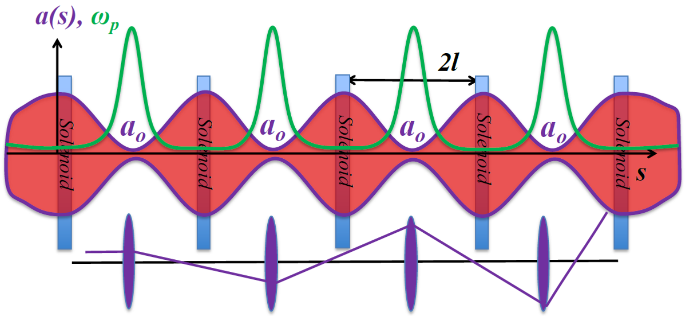
Longitudinal plasma oscillation in a neutral plasma (from the internet, only for illustration)



Plasma-Cascade Instability

Longitudinal plasma oscillation with periodically varying plasma frequency

$$\frac{d^2\tilde{n}}{dt^2} + \omega_p^2(t)\tilde{n} = 0;$$



$$\hat{n}'' + 2k_{sc}^2\hat{a}(\hat{s})^{-2}\hat{n} = 0$$

$$\hat{a}'' = k_{sc}^2\hat{a}^{-1} + k_\beta^2\hat{a}^{-3}$$

$$\hat{a} = \frac{a}{a_o};\ \hat{s} = \frac{s}{l} \in \{-1,1\};$$

$$k_{sc} = \sqrt{\frac{2}{\beta_o^3\gamma_o^3}\frac{I_o}{I_A}\frac{l^2}{a_o^2}};\quad k_\beta = \frac{\varepsilon l}{a_o^2}$$

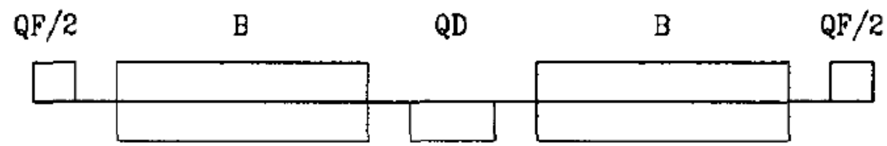
Stability condition $\begin{pmatrix}\hat{n}\\ \hat{n}'\end{pmatrix}_{s=-l} = M_{total}\begin{pmatrix}\hat{n}\\ \hat{n}'\end{pmatrix}_{s=l}$

$$|\lambda| = \left|(M_{total})_{1,1} \pm \sqrt{(M_{total})_{1,1}^{\ 2} - 1}\right| \le 1$$

$$(M_{total})_{1,1} = 2m_{11}m_{22}\big|_{\hat{s}=1} - 1 \qquad \begin{pmatrix}\hat{n}(\hat{s})\\ \hat{n}'(\hat{s})\end{pmatrix} = \begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}\hat{n}(0)\\ \hat{n}'(0)\end{pmatrix}$$

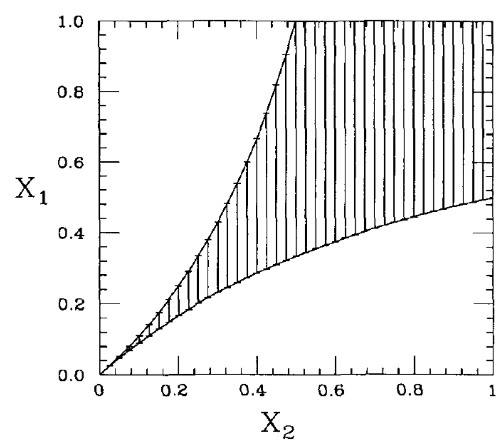
Betatron motion in a FODO cell

$$y'' + K_y(s)y = 0,$$



$$\mathbf{M} = \begin{pmatrix}1 & 0\\ -\frac{1}{2f_1} & 1\end{pmatrix}\begin{pmatrix}1 & L_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f_2} & 1\end{pmatrix}\begin{pmatrix}1 & L_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{2f_1} & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 + \frac{L_1}{f_2} - \frac{L_1}{f_1} - \frac{L_1^2}{2f_1f_2} & 2L_1(1 + \frac{L_1}{2f_2})\\ \frac{1}{f_2} - \frac{1}{f_1} - \frac{L_1}{f_1f_2} + \frac{L_1}{2f_1^2} + \frac{L_1^2}{4f_1^2f_2}) & 1 + \frac{L_1}{f_2} - \frac{L_1}{f_1} - \frac{L_1^2}{2f^2}\end{pmatrix},$$



$$X_1 = L_1/2f_1$$

$$X_2 = L_1/2f_2$$

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Analytical solution for emittance dominated beam

If we assume $k_\beta >> k_{sc}$, the envelope equation become $\hat{a}'' = k_\beta^2 \hat{a}^{-3}$ and its solution for initial condition of $\hat{a}(0) = 1$ and $\hat{a}'(0) = 0$ is

$$\hat{a}^2 = k_\beta^2 \hat{s}^2 + 1$$

and the equation of longitudinal density perturbation becomes

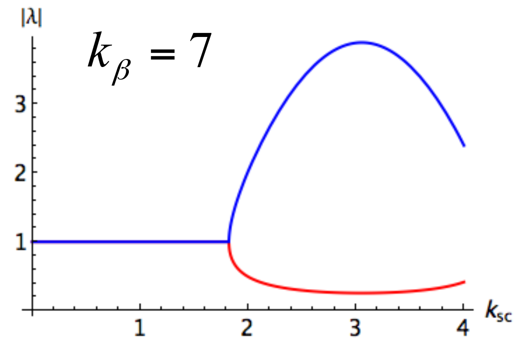
$$\hat{n}'' + \frac{2k_{sc}^2}{k_\beta^2 \hat{s}^2 + 1} \hat{n} = 0$$

which can be solved in terms of Hypergeometric function:

$$\hat{n}(\hat{s}) = c_1 \cdot F(\alpha_1, \beta_1; \gamma_1; x_1) + c_2 \hat{s} \cdot F\left(\alpha_1 + \frac{1}{2}, \beta_1 + \frac{1}{2}; \gamma_1 + 1; x_1\right)$$

The eigenvalues for the transfer matrix of one cell is

$$\lambda = (M_{total})_{1,1} \pm \sqrt{(M_{total})_{1,1}^{\;\;2} - 1}$$



$$(M_{total})_{1,1} = 2F\left(\alpha_1, \beta_1; \frac{1}{2}; -k_\beta^2\right)\left\{F\left(\alpha_1 + \frac{1}{2}, \beta_1 + \frac{1}{2}; \frac{3}{2}; -k_\beta^2\right) - \frac{2}{3}k_{sc}^2 F\left(\alpha_1 + \frac{3}{2}, \beta_1 + \frac{3}{2}; \frac{5}{2}; -k_\beta^2\right)\right\} - 1$$

Gain of Plasma-Cascade Instability

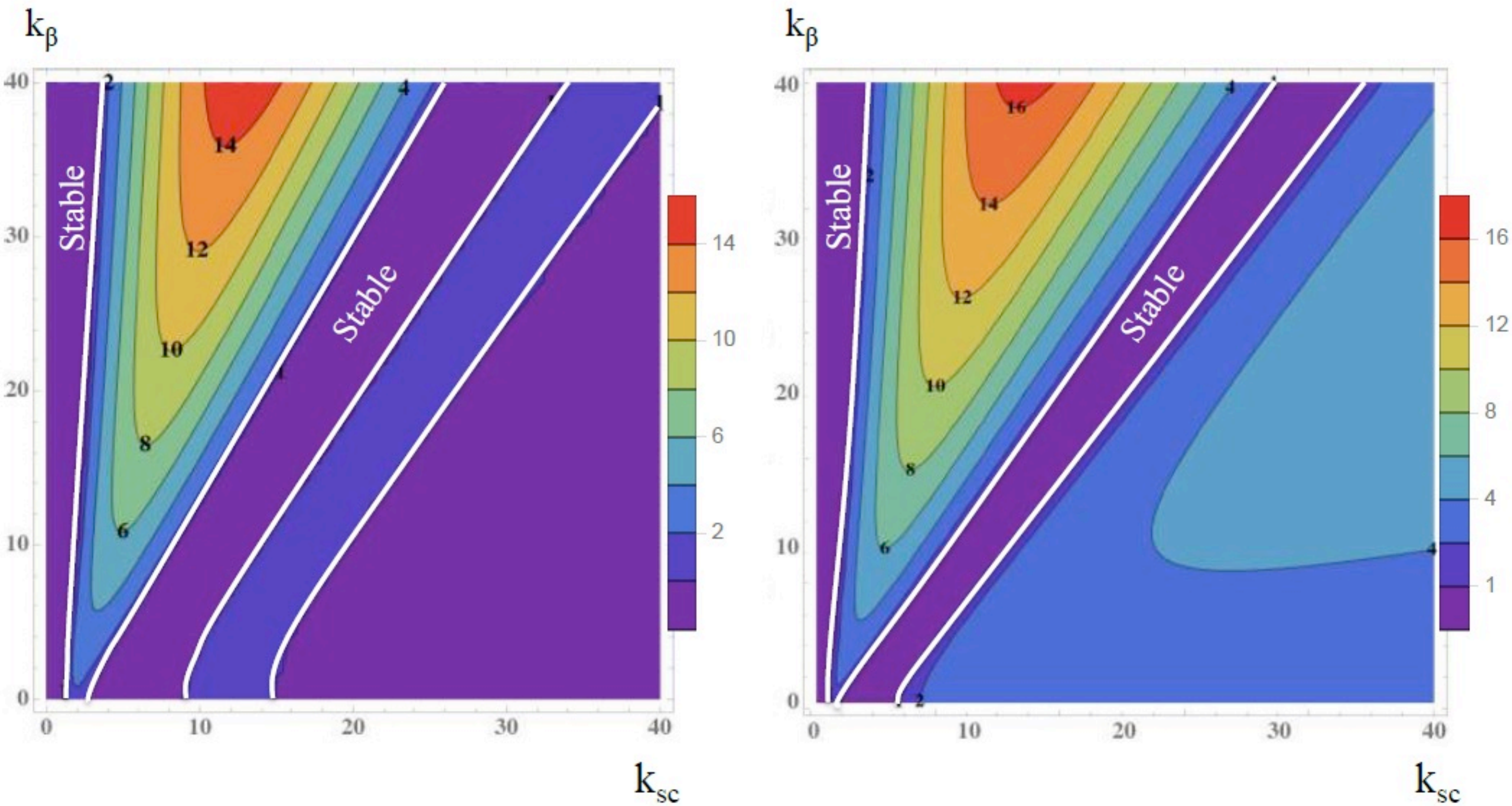
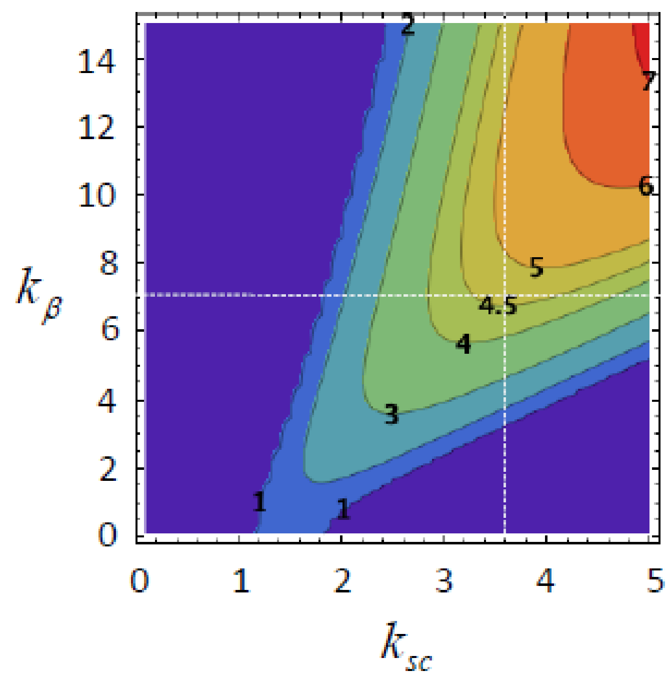


Figure 7. Contour plots of $\lambda = \max\left(\left|\mathrm{Re}\,\lambda_1\right|,\left|\mathrm{Re}\,\lambda_2\right|\right)$, the absolute value of maximum growth rate per cell, using (a) an analytical solution (67) for an approximate beam envelope $\hat{a}(\hat{s})$ (as in eq. (33), and (b) exact numerical solution of the problem using code described in Appendix A. Purple area highlighted by white lines indicates areas of stable oscillation $\left|\lambda_{1,2}\right| = 1$. Outside these areas oscillations are growing exponentially.

Estimate Cooling Force for PCA-based CeC: Parameters

Energy, γ	**28.5**
Electron beam peak current, A	100
Bunch length, ns	0.015
Bunch charge, nC	1.5
Modulator length, m	3
Amplifier length, m	8 (4 sections)
Beam width at modulator, mm	0.94
Amplifier gain (Cold, infinitely wide), g_{amp}	200
RMS energy spread	1e-4
KV envelope norm. emittance, μm	8
Minimal beam width at PCA, mm	0.2

$k_{sc} = 3.6 \qquad k_{\beta} = 7$



$|\lambda| \approx 4.5 \qquad g_{amp} \equiv \frac{1}{2}|\lambda|^4 \approx 200$

Estimate Cooling Force for PCA-based CeC: Line density perturbation

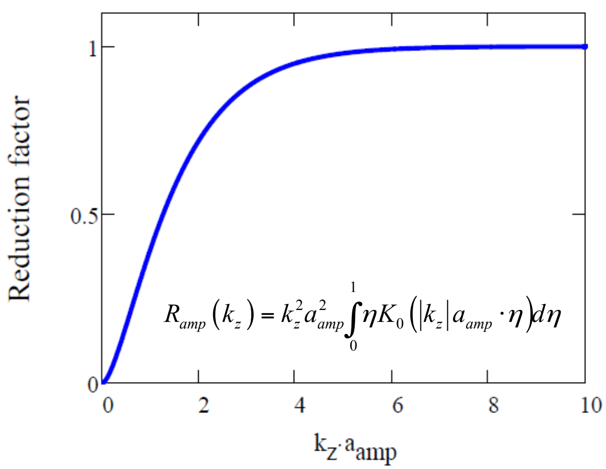
Line density perturbation at the exit of the modulator:

$$\tilde{\rho}_1(k_z) = \frac{Z_i e}{1+\bar{\lambda}_z(k_z)^2}\left[1-e^{\bar{\lambda}_z(k_z)\psi_{\mathrm{m}}}\left(\cos\psi_{\mathrm{m}} - \bar{\lambda}_z(k_z)\sin\psi_{\mathrm{m}}\right)\right]$$

$$\rho_{1z}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_z z}\tilde{\rho}_1(k_z)\,dk_z = \frac{Z_i e}{\pi a_z}\int_0^{\psi_m}\frac{\tau\sin\tau}{\left(\frac{z}{a_z}+\frac{v_{0z}}{\beta_z}\tau\right)^2+\tau^2}$$

Line density perturbation at the exit of the Plasma-Cascade Amplifier:

$$\tilde{\rho}_2(k_z) = g_{amp}R_{amp}(k_z)\exp\left(-|k_z|\beta_z\frac{L_{amp}}{\gamma c}\right)\tilde{\rho}_1(k_z)$$

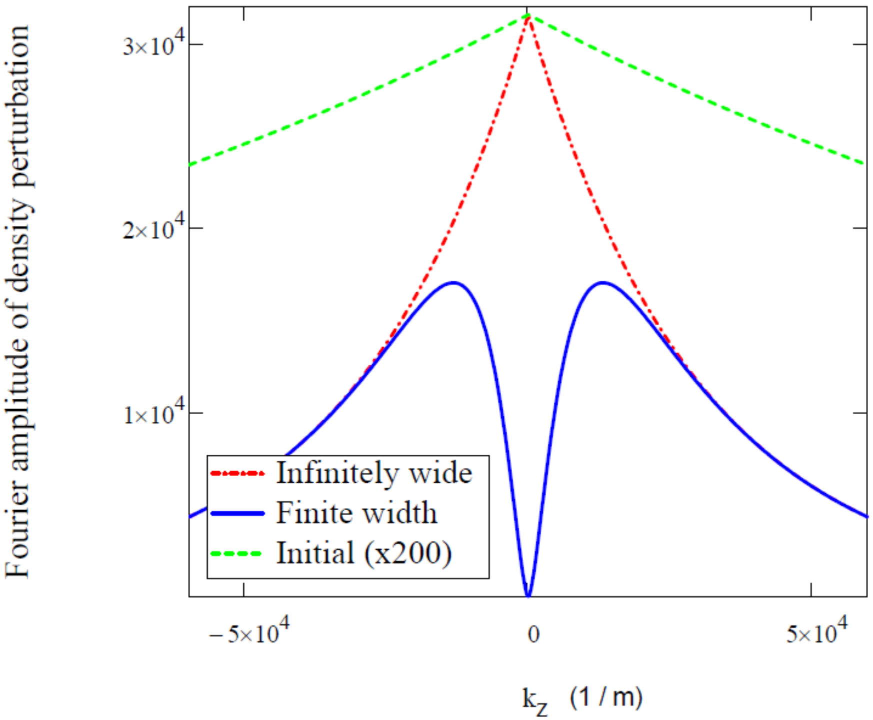


Gain reduction factor due to finite transverse beam size

Landau damping factor for Lorentzian energy distribution

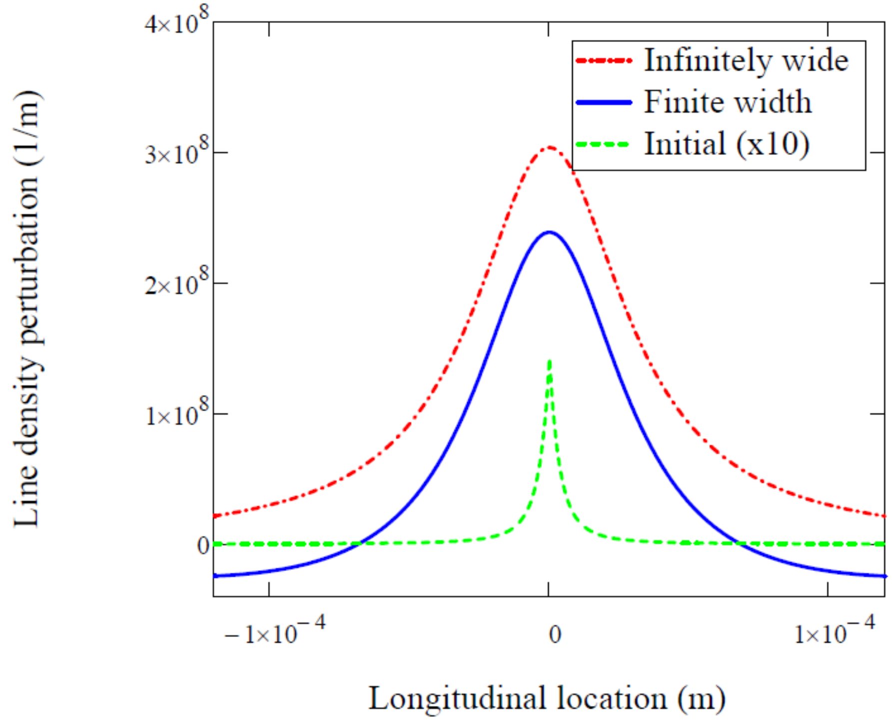
Estimate Cooling Force for PCA-based CeC: Line density perturbation

Line density perturbation in wave-number domain



$$\tilde{\rho}_2(k_z) = g_{amp} R_{amp}(k_z) \exp\left(-|k_z| \beta_z \frac{L_{amp}}{\gamma c}\right) \tilde{\rho}_1(k_z)$$

Line density perturbation in spatial domain



$$\rho_2(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\rho}_2(k_z) e^{ik_z z} dk_z$$

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Estimate Cooling Force for PCA-based CeC: Longitudinal Electric Field in the Kicker Section

The electric potential induced by the line density perturbation is determined by the following equations

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\varphi(r,z)\right)\right]+\frac{\partial^2}{\partial z^2}\varphi(r,z)=\frac{1}{\varepsilon_0}\rho_2(z)f_\perp(r)$$

If we take the transverse distribution of the electrons as

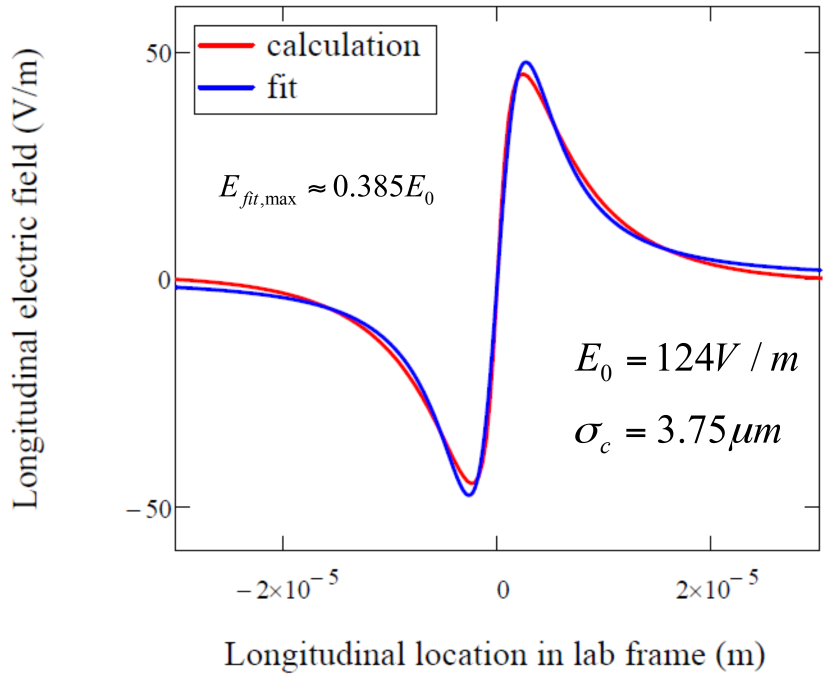
$$f_\perp(r)=\frac{1}{\pi a^2}H(a-r)$$

The electric field can be solved as

$$E_z(r,z)=-\frac{\partial\varphi}{\partial z}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_z(r,k_z)e^{ik_z z}dk_z$$

$$\tilde{E}_z(r)=-ik_z\frac{\tilde{\rho}_2(k_z)}{\pi\varepsilon_0}$$

$$\times\left[I_0(k_z r)\int_{r/a}^{1}\eta K_0(k_z a\cdot\eta)d\eta+K_0(k_z r)\int_{0}^{r/a}\eta I_0(k_z a\cdot\eta)d\eta\right]$$



For easy implementation into ion tracking code, we use the fitting formula:

$$E_{fit}(z)=E_0\cdot\frac{z}{\sigma_c}\left[1+\frac{z^2}{\sigma_c^2}\right]^{-3/2}$$

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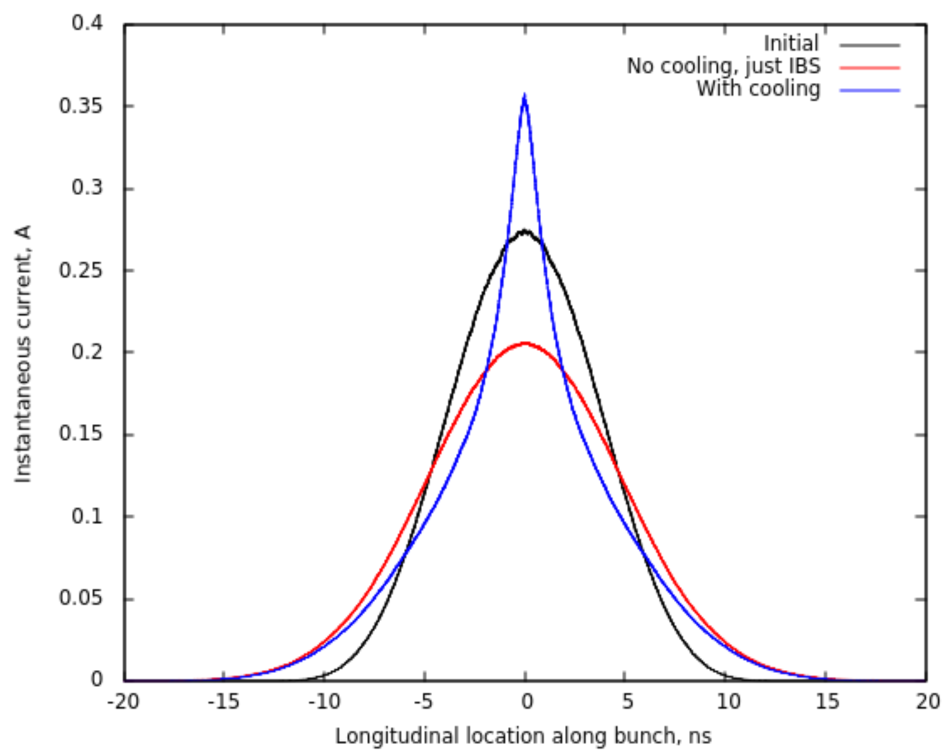
Single-pass Kick and Tracking Results for PCA-based CeC

$$\Delta\gamma_{j,N} = -g_\gamma \frac{\left(D\cdot\delta_{j,N}\right)}{\sigma_c}\left[1+\frac{\left(D\cdot\delta_{j,N}\right)^2}{\sigma_c^2}\right]^{-3/2} + g_\gamma\sqrt{\frac{3\pi}{8}\rho_{ion}\left(z_{j,N}\right)\sigma_c}\cdot X_{j,N} + \frac{g_\gamma}{Z_i}\sqrt{\frac{3\pi}{8}\rho_e\left(z_{j,N}\right)\sigma_c}\cdot Y_{j,N}$$

$$g_\gamma = Z_i e E_0 L_k \Big/ \left(A_i m_u c^2\right)$$

Parameters used in ion tracking	
E_0^*	62 V/m
σ_c	3.75 μm
Z_{ion}	79
Ion bunch intensity	2E8
D (R_{56})	1.2 cm

*E_0 is reduced by a factor of 2 to account for reduced cooling for ions with large betatron amplitude



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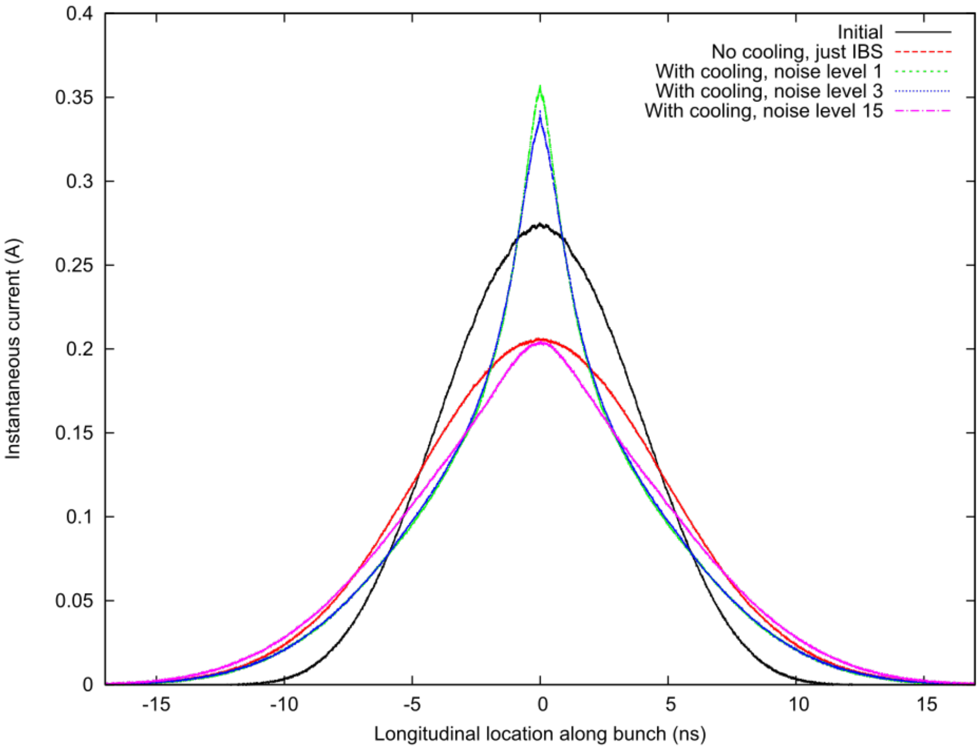
Tolerance of PCA-based CeC on the noise of electron beam

$$\Delta\gamma_{j,N} = -g_\gamma \frac{\left(D\cdot\delta_{j,N}\right)}{\sigma_c}\left[1+\frac{\left(D\cdot\delta_{j,N}\right)^2}{\sigma_c^2}\right]^{-3/2} + g_\gamma\sqrt{\frac{3\pi}{8}\rho_{ion}\left(z_{j,N}\right)\sigma_c}\cdot X_{j,N} + R_{NL}\cdot\frac{g_\gamma}{Z_i}\sqrt{\frac{3\pi}{8}\rho_e\left(z_{j,N}\right)\sigma_c}\cdot Y_{j,N}$$

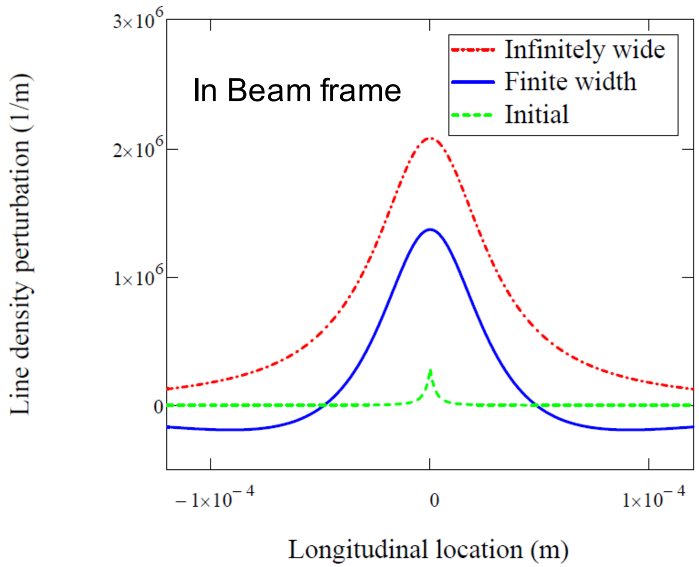
$$R_{NL} = \sqrt{\frac{\text{Actual noise power in the electron beam}}{\text{Shot noise power for uncorrelated electrons}}}$$

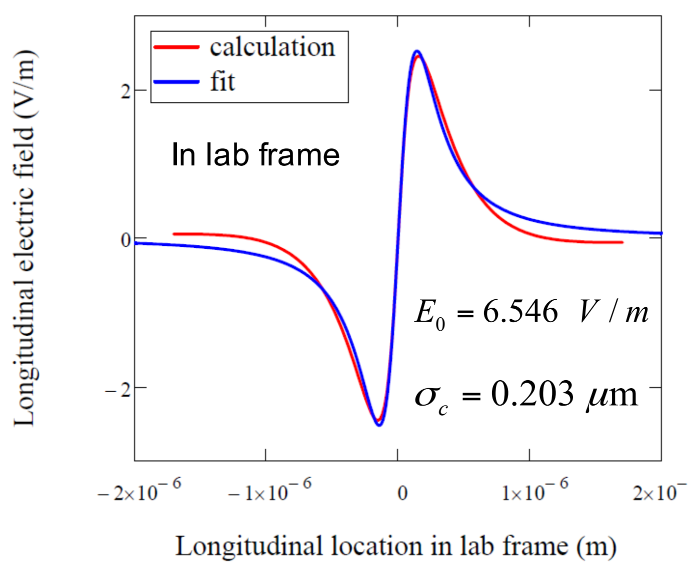


- According to the simulation, the noise power in the electron beam should not exceed 200 times of the shot noise power.

Some Preliminary Estimates for eRHIC

Energy, γ	293.09
Electron beam peak current, A	200
Bunch length, ns	0.05
Bunch charge, nC	10
Modulator (and kicker) length, m	20
Amplifier length, m	80 (4 sections)
Beam width at modulator, mm	0.26
Amplifier gain	100
RMS energy spread	1e-4
Minimal beam width at PCA, mm	0.1
Theoretical estimates of local cooling time	
Maximal local cooling time for ions at $r << a_0$ and $\Delta E / E << \sigma_\delta$	9.4 seconds
Local cooling time for ions at $r = a_0$ and $\Delta E / E = \sigma_\delta$	22.9 seconds





$$E_{fit}(z) = E_0 \frac{z}{\sigma_c} \cdot \left(1 + \frac{z^2}{\sigma_c^2}\right)^{-3/2}$$

$$E_{fit,\max} \approx 0.385 E_0$$

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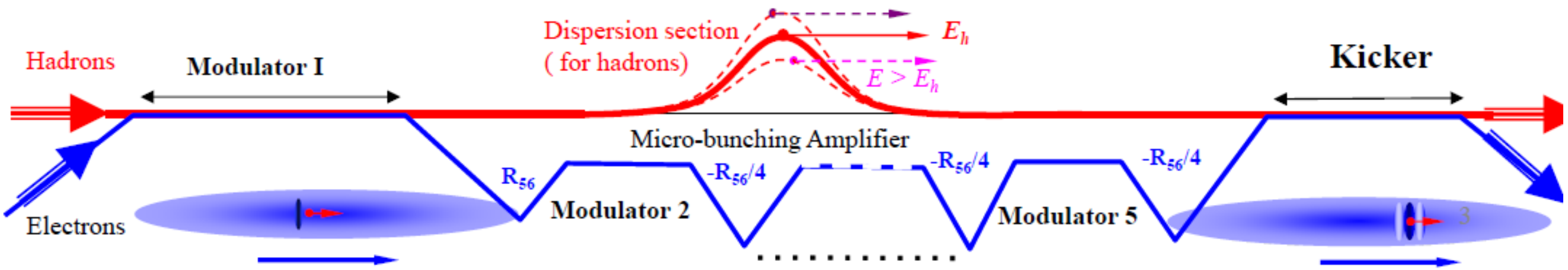
MBEC (Chicane-based CeC)



Enhanced bunching: single stage – VL, FEL2007

Micro-bunching: MB Amplifier, Single & Multi-stage, D. Ratner, PRL, 2013

Cooling rate for microbunched electron cooling without amplification, G. Stupakov, PRAB, 2018

Microbunched electron cooling with amplification cascade, G. Stupakov and P. Baxevanis, PRAB, 2019

Analysis of MBEC: Single Ion Approach I

Consider the following initial distribution of electrons (Spatially Beercan and Gaussian energy distribution)

$$f_1(r,\vec{p}_\perp,x_1,p_1) = n_o \cdot \theta(r-a)\cdot\theta(z-l_z)\cdot \frac{e^{-\frac{p_1^2}{2\sigma_{p_1}^2}}}{\sqrt{2\pi}\sigma_{p_1}}\cdot g(\vec{p}_\perp) \qquad p \equiv \frac{\delta\gamma}{\gamma_o}$$

After the modulation (kicked by a single ion) and the buncher, the distribution function is

$$f_2\left(\vec{r}_2,\vec{p}_2\right) = f_1(\vec{r}_1\left(\vec{r}_2,\vec{p}_2\right),\vec{p}_1\left(\vec{r}_2,\vec{p}_2\right));$$

$$p_2 = p_1 - \frac{Zr_eL_{\text{mod}}}{\gamma_o^3}\frac{x_1}{\left(r^2/\gamma_o^2+x_1^2\right)^{3/2}}; \qquad x_2 = x_1 + D\left(p_1 - \frac{Zr_eL_{\text{mod}}}{\gamma_o^3}\frac{x_1}{\left(r^2/\gamma_o^2+x_1^2\right)^{3/2}}\right)$$

The line density modulation at the exit of the buncher is thus

$$\begin{aligned}\rho(z) &= 2\pi\int f_2(r,\vec{p}_\perp,z,p_2)rdrd\vec{p}_\perp^{\,2}dp_2 \\ &= 2\pi\int \delta(z-x_2(r,x_1,p_1))f_1(r,\vec{p}_\perp,x_1,p_1)rdrd\vec{p}_\perp^{\,2}dp_1dx_1\end{aligned}$$

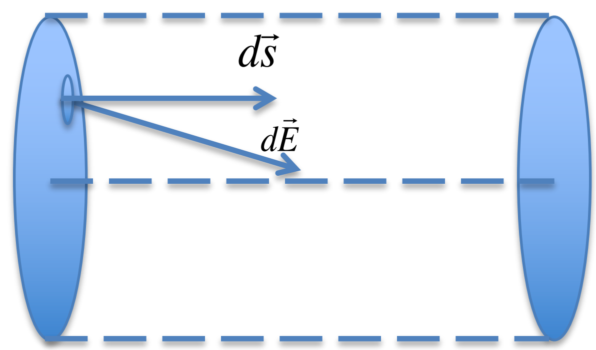
Analysis of MBEC: Single Ion Approach II

For the specific form of initial distribution, the integral can be reduced to

$$\rho(z) = 2\pi n_o \sigma_\gamma{}^2 D^2 \cdot \Psi_{u2}\left(\frac{\gamma_o z}{\sigma_\gamma |D| \sqrt{2}}, Z\frac{r_e L_{\text{mod}}}{2\sqrt{2}\sigma_\gamma{}^3 D|D|}, \frac{a}{\sqrt{2}\sigma_\gamma |D|} \right)$$

$$\Psi_{u2}(\zeta,\Xi,\tilde{a}) = \int_0^\infty q\,dq \left\{ \frac{Erf\left(q\left(1-\Xi q^{-3}\right)+\zeta\right) + Erf\left(q\left(1-\Xi q^{-3}\right)-\zeta\right)}{1-\Xi q^{-3}} - \frac{Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)+\zeta\right) + Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)-\zeta\right)}{1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}} \right\}$$

The electric field due to the density modulation can be calculated as follows (disc charge model)



$$E_{z,disc}(0,0,z_{lab}) = \frac{e\sigma z_{lab}}{2\varepsilon_0}\left[\frac{1}{\sqrt{z_{lab}^2 + a^2/\gamma_0^2}} - \frac{1}{|z_{lab}|} \right]$$

$$E_z(0,0,z) = \frac{1}{\pi a^2}\int_{-\infty}^{\infty} dz_1 \tilde{\rho}(z_1)\frac{1}{\sigma}E_{z,disc}(0,0,z-z_1) = \frac{\sqrt{2} e n_o \sigma_\gamma{}^3 D^2 |D|}{\varepsilon_0 a^2 \gamma_o}\int_{-\infty}^{\infty}\Psi_{u2}(\zeta_1,\Xi,\tilde{a})\cdot\left[\frac{\tilde{z}-\zeta_1}{\sqrt{(\tilde{z}-\zeta_1)^2+\tilde{a}^2}} - \frac{\tilde{z}-\zeta_1}{|\tilde{z}-\zeta_1|} \right] d\zeta_1$$

Analysis of MBEC: Single Ion Approach III

The following shows an example of our analysis:

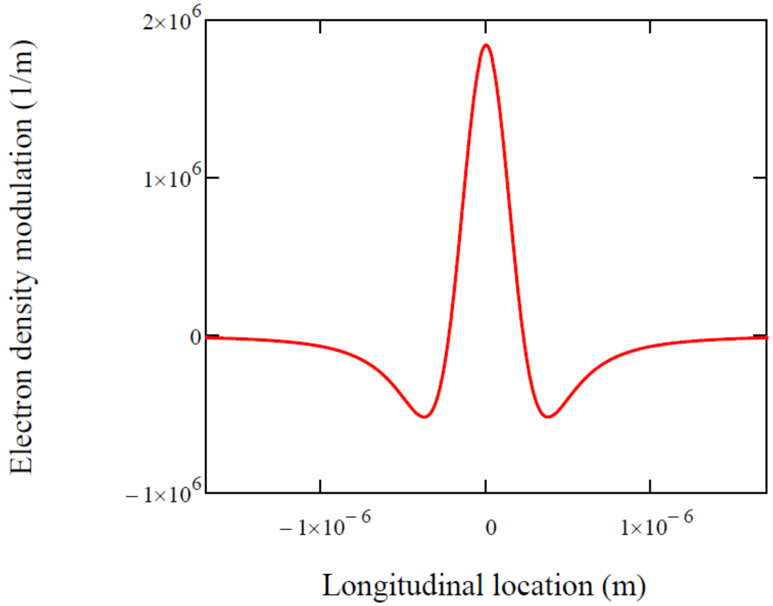
$$a = 0.1mm \quad I_{peak} = 1A \quad L_{kick} = 10m \quad \sigma_{\delta,ion} = 6 \cdot 10^{-4}$$

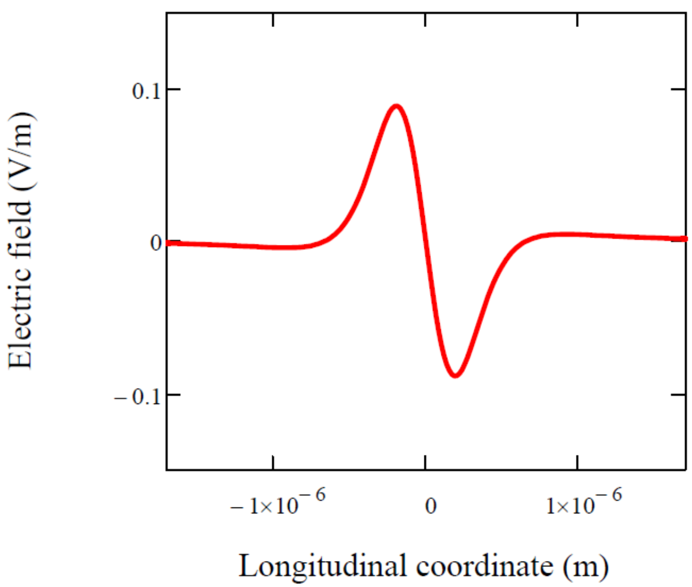
$$L_{\mathrm{mod}} = 10m \quad \sigma_{\delta,e} = 10^{-4} \quad D = 1.5mm \quad E_p = 275GeV$$

Maximal reduction of energy error in one pass: $\Delta E_{correction} = eE_{peak}L_{kick}$

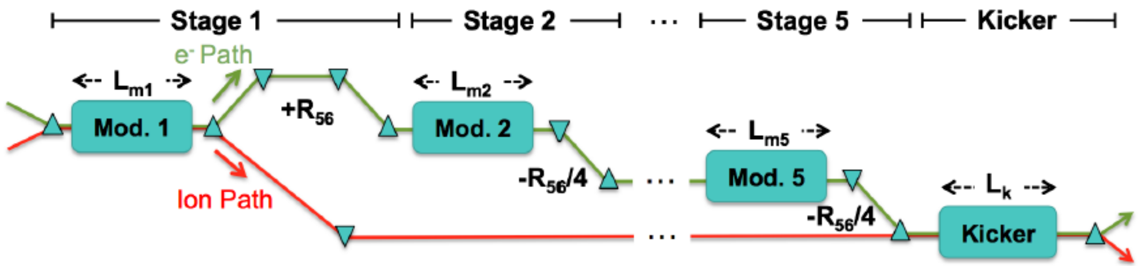
Local cooling time: $T_{cool} = T_{rev}\dfrac{\sigma_{\delta,ion}E_p}{\Delta E_{correction}} \approx 40\,\mathrm{min}$

*This is the cooling time for energy spread and the cooling time for emittance should be a factor of 2 shorter.





Approach Used in PRL 111, 084802 (2013) by D. Ratner



© D. Ratner

$$A_1 \equiv -2cqL_m/a^2 I_A$$

$$A_2 \equiv -qIL_k/2\epsilon_0 c\pi a^2,$$

$L_m >> \beta$

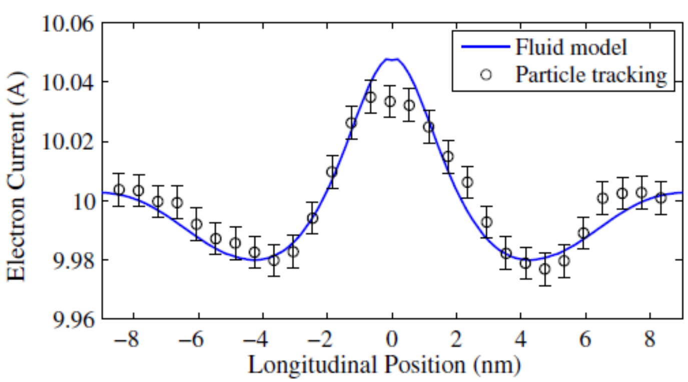
$$M(z) = \frac{-2cqL_m}{\gamma a^2 I_A}\left[\frac{z}{|z|} - \frac{\gamma z}{\sqrt{a^2 + \gamma^2 z^2}}\right],$$

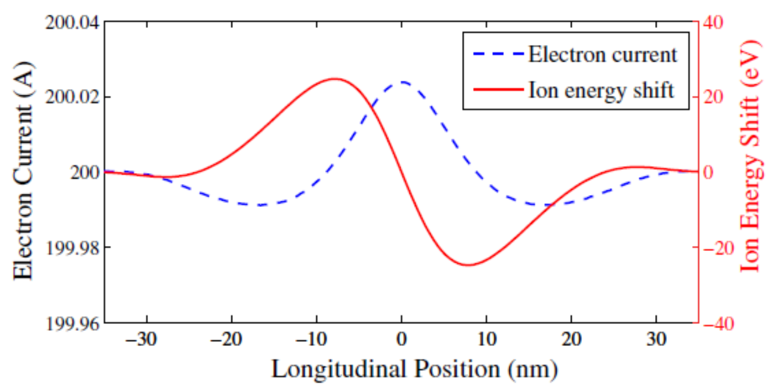
$$\langle W_1(z_I)\rangle = \frac{|R_{56}A_1|A_2}{\gamma}\int_{-\infty}^{\infty} d\xi \frac{e^{-\xi^2/2\sigma_\xi^2}}{\sqrt{2\pi}\sigma_\xi}\int_{-\infty}^{\infty} d\zeta \times \left(t(\zeta,\xi) - \frac{[\zeta_I - \tilde{\zeta}(\zeta,\xi)]}{\sqrt{\alpha^2 + [\zeta_I - \tilde{\zeta}(\zeta,\xi)]^2}}\right),$$

'Disc' model

$L_m << \beta$

$$M(r,z) = \frac{-cqL_m}{\gamma I_A}\frac{\gamma z}{[r^2 + \gamma^2 z^2]^{3/2}}.$$





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Analysis in Frequency Domain by G. Stupakov and P. Baxevanis

(PRAB 21 (2018), 114402)

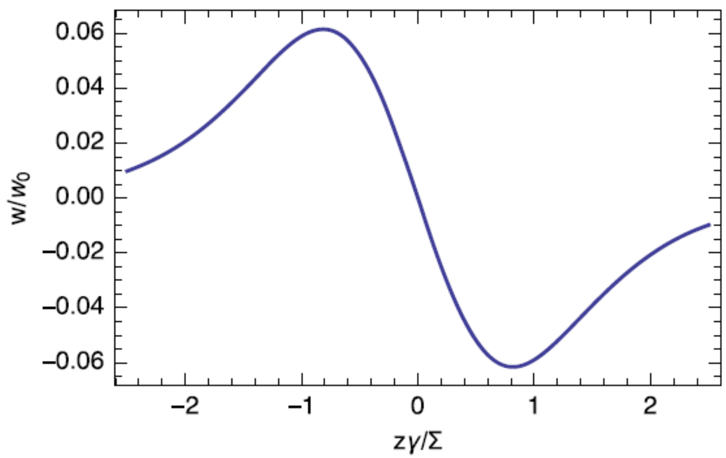
Single pass energy kick

$$\Delta\eta^{(h)}(z) = -\frac{r_h c}{2\pi\gamma}\int_{-\infty}^{\infty} dk \mathcal{Z}(k)\delta\hat{n}_k^{(M)} e^{ikz},$$

$$\varkappa = \frac{k\Sigma}{\gamma}, \qquad q_e = \frac{R_e \sigma_e \gamma}{\Sigma},$$

Impedance due to MBEC

$$\mathcal{Z} = -\frac{2in_{0e}L_k}{c\Sigma\gamma} g(k)\zeta(k)H\left(\frac{k\Sigma}{\gamma}\right)$$
$$= -\frac{4iI_e L_m L_k}{c\Sigma^2\gamma^3 I_A \sigma_e} q_e \varkappa e^{-\varkappa^2 q_e^2/2} H^2(\varkappa),$$



Cooling rate in 1/number of turns

$$N_c^{-1} = \frac{4I_e r_h L_m L_k}{\pi\Sigma^3\gamma^3 I_A \sigma_e \sigma_h}\left[q_e \mathrm{Re}\int_0^{\infty} d\varkappa \varkappa e^{-\varkappa^2 q_e^2/2} H^2(\varkappa) R(\varkappa)\right],$$

$$t_c^{-1} = \left(\int_{-\infty}^{\infty} d\eta \eta^2 \frac{\partial F_h}{\partial t}\right)\left(\int_{-\infty}^{\infty} d\eta \eta^2 F_h\right)^{-1}$$

Cooling rate is maximized at $q_e = q_h = 0.6$

$$N_c^{-1} = 0.10 \frac{1}{\gamma^3 \sigma_h \sigma_e}\frac{I_e}{I_A}\frac{r_h L_m L_k}{\Sigma^3}.$$

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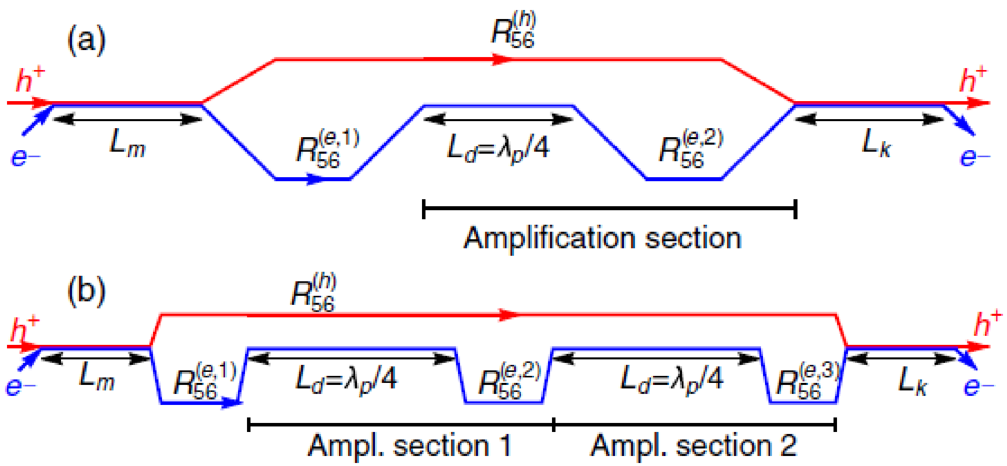
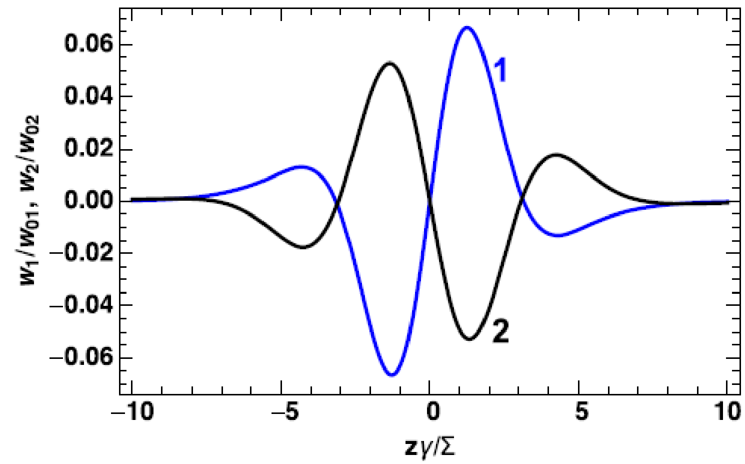
Analysis in Frequency Domain by G. Stupakov and P. Baxevanis

(PRAB 22 (2019), 034401)

- With multiple amplification stages





With one amplification stage:

$$N_c^{-1} = -\frac{4\sqrt{2}}{\pi} A \frac{I_e r_h L_m L_k}{\Sigma^3 \gamma^3 I_A \sigma_e \sigma_h} \operatorname{sgn}(q_h q_1 q_2) I_1(q_h, q_1, q_2, r, l), \qquad N_c^{-1} = 0.075 \frac{I_e^{3/2} r_h L_m L_k}{\Sigma^3 \gamma^{7/2} I_A^{3/2} \sigma_e^2 \sigma_h}.$$

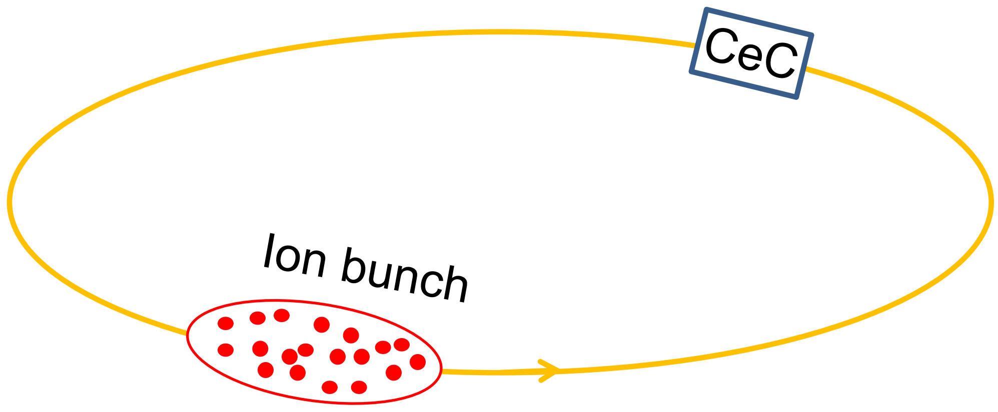
With two amplification stage:

$$N_c^{-1} = \frac{8}{\pi} A^2 \frac{I_e r_h L_m L_k}{\Sigma^3 \gamma^3 I_A \sigma_e \sigma_h} I_2(q, r, l), \qquad N_c^{-1} = 0.066 \frac{I_e^2 r_h L_m L_k}{\Sigma^3 \gamma^4 I_A^2 \sigma_e^3 \sigma_h}.$$

Roughly speaking, every amplification stage increase the cooling rate by a factor of A.

$$r = \frac{\varkappa_p}{\varkappa} = \frac{\Sigma_p}{\Sigma}, \qquad l = r\frac{\Omega L_d}{c} \qquad \boxed{A = \frac{1}{\sigma_e}\sqrt{\frac{I_e}{\gamma I_A}},}$$

Circulating Ion Beam Evolution in the Presence of CeC



We take two approach to predict the evolution of the ion bunch in the presence of CeC:

- Solving 1-D Fokker-Planck equation (analytical tool);
  - Very fast (a few minutes on a pc)
  - With limitations (currently work with linear cooling force, no beam losses from RF bucket, static diffusion coefficient…)

- Macro-particle tracking (simulation tool).
  - Time consuming (a few hours on a pc)
  - More realistic and versatile

Analytical tools for predicting the influences of CeC to a circulating ion beam I

- Evolution of the longitudinal phase space density of the ion bunch, after averaging over the synchrotron oscillation phase, follows the 1-D Fokker-Planck equation

$$\frac{\partial}{\partial t}F(I,t)-\frac{\partial}{\partial I}\left(\zeta(I)\cdot I\cdot F(I,t)\right)-\frac{\partial}{\partial I}\left(I\cdot D(I)\cdot\frac{\partial F(I,t)}{\partial I}\right)=0$$

- In the limit of $D(I)=0$, an analytical solution can be derived for the following form of cooling profile and initial condition,

$$\zeta(I)=\zeta_0\frac{I_e}{I+I_e} \qquad F_0(I)=\exp\left(-\frac{I}{I_{ion}}\right)$$

$P_{\log}(x)$ is called product logarithm function and can be directly evaluated in Mathematica.

as

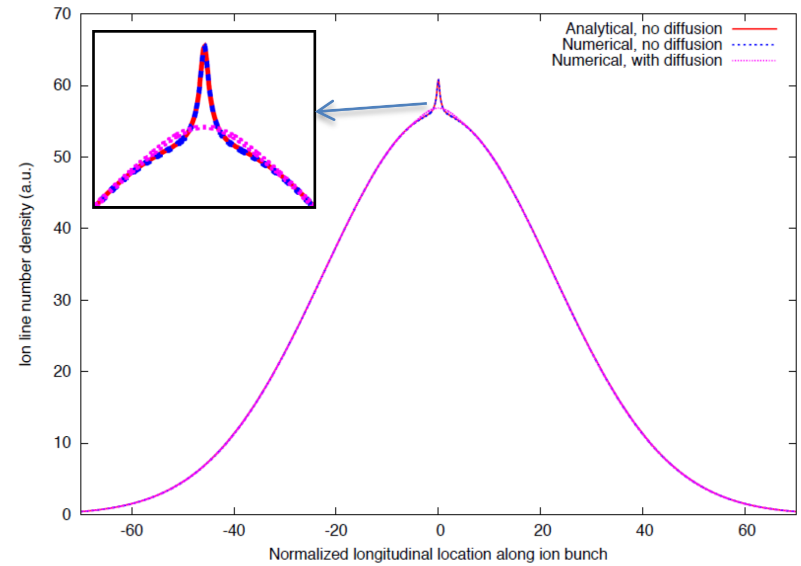
$$F(I,t)=\left(1+\frac{I_e}{I}\right)\frac{P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t+\frac{I}{I_e}\right)\right)\exp\left(\frac{-I_e}{I_{ion}}P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t+\frac{I}{I_e}\right)\right)\right)}{1+P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t+\frac{I}{I_e}\right)\right)}$$

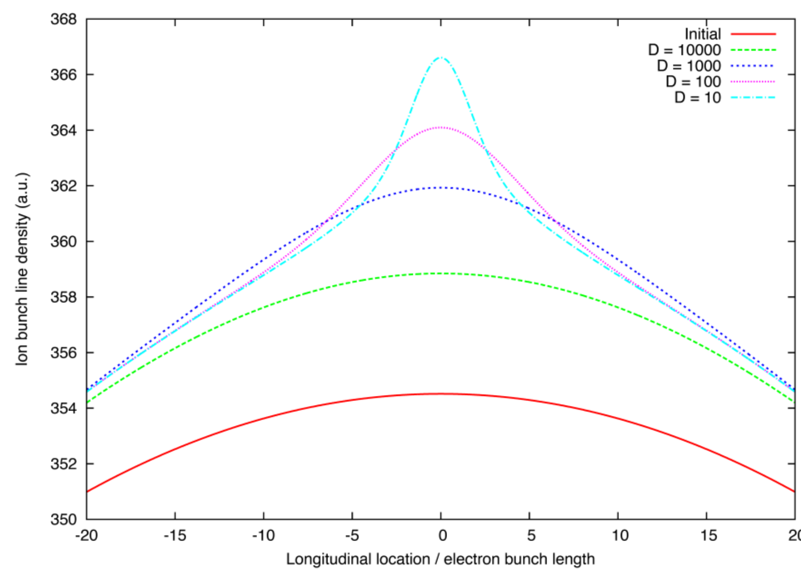
Analytical tools for predicting the influences of CeC to a circulating ion beam II

- The longitudinal line density of ion bunch is given by

$$\rho_{ion}\left(t,z\right)=\int_{-\infty}^{\infty}F\left(z^2+\delta^2,t\right)d\delta$$

- For $D\left(I\right)\neq 0$, the 1-D Fokker-Planck equation can be solved numerically with arbitrary form of cooling rate and initial ion distribution.
- The analytical studies reveals the fact that the central blips due to local cooling tends to be smeared out by diffusive kicks from IBS and more significantly, from incoherent kicks induced by neighbor ions.





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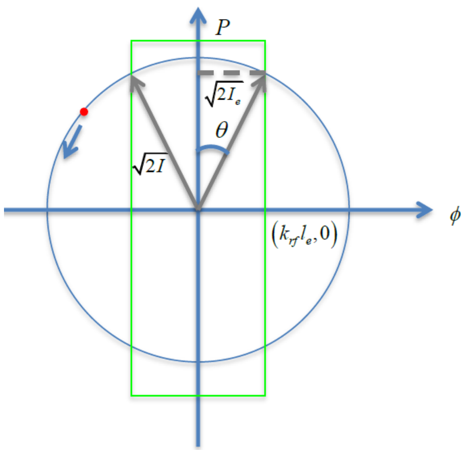
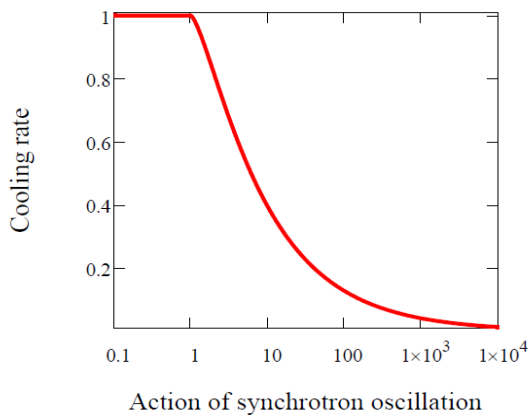
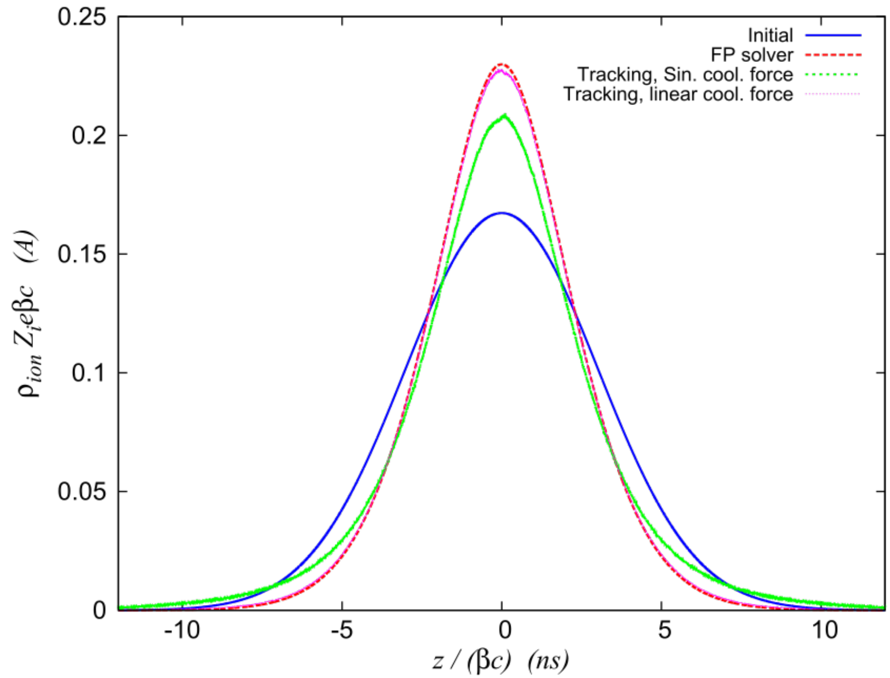
Comparison with Macro-particle Tracking







$$\bar{\xi}(r) = \begin{cases} \dfrac{2}{\pi}\arcsin\left(\dfrac{1}{r}\right) + \dfrac{2}{\pi r^2}\sqrt{r^2-1}; & \text{for } r \ge 1 \\ 1 \quad ; & \text{for } r < 1 \end{cases}$$

Some limitations of the Fokker-Planck solver:

1. The Fokker-Planck solver assumes constant cooling rate (linear cooling force) inside electron bunch while macro-particle tracking assume sinusoidal cooling force;
2. The Fokker-Planck solver does not account for particle leakage from rf bucket or other rf-related effects.
3. The diffusion coefficients applied in Fokker-Planck equation does not change with time.

Future Work and Challenges

- Since PCA-based CeC is planned to be tested in the next two years, it is our main focus to complete and improve the existing model:
  - Develop PCA model for electron bunch with initial energy chirp;
  - Develop PCA model for electron bunch with acceleration;
  - Investigate how transverse-longitudinal coupling affects PCA;
  - Exploring how cooling performance scales with various beam/accelerator parameters and searching for optimal settings;
  - Develop 3-D model for PCA (Plasma oscillation with finite transverse size, transverse mode…).

- Further improvement of MBEC may also include study the 3-D effects to cooling rate (plasma oscillation with finite transverse size) as well as how space charge (longitudinal and transverse) and emittance limit the gain.

- For predicting the evolution of ion beam with cooling, further improvements include solving Fokker-Planck equation for non-linear cooling force and including transverse cooling into ion tracking code.

- Further improvement of theoretical model for FEL-based CeC involves incorporating 3-D model for FEL at diffraction dominated regime, which has relative low priority at the moment.

Backup slides

Debye shielding in an uniform electron plasma with anisotropic velocity distribution

The system can be described by linearized Vlasov-Maxwell equations

$$\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t) + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t) - \frac{e\vec{E}}{m_e}\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_o}\left\{Ze\delta(\vec{x}) - e\tilde{n}_1(\vec{x},t)\right\}$$

$$\tilde{n}_1(\vec{x},t) = \int f_1(\vec{x},\vec{v},t)\, d^3v.$$

$$f_0(\vec{v}) = \frac{1}{\pi^2\beta_x\beta_y\beta_z}\left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2}\right)^{-2}$$

In 3-D Fourier domain, the equations reduces to a non-homogeneous 2nd ODE

$$\frac{d^2}{dt^2}\tilde{H}_1(\vec{k},t) + \omega_p^2\tilde{H}_1(\vec{k},t) = \omega_p^2 Z_i e^{-\lambda(\vec{k})t}$$

$$\tilde{H}_1(\vec{k},t) = \tilde{n}_1(\vec{k},t)e^{-\lambda(\vec{k})t}$$

The solution for zero initial density and velocity modulation in Fourier domain can be found

$$\tilde{\tilde{n}}_1(\vec{k},t) = Z_i\omega_p \sin(\omega_p t)e^{\lambda(\vec{k})t}$$

$$\lambda(\vec{k}) = i\vec{k}\cdot\vec{v}_0 - \sqrt{(k_x\beta_x)^2 + (k_y\beta_y)^2 + (k_z\beta_z)^2}$$

By inverse Fourier transformation, we obtain the density modulation in space domain

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 a_x a_y a_z}\int_0^{\omega_p t}\frac{\tau\sin\tau\cdot d\tau}{\left[\tau^2 + \left(\frac{x}{a_x} + \frac{v_{0,x}}{\beta_x}\tau\right)^2 + \left(\frac{y}{a_y} + \frac{v_{0,y}}{\beta_y}\tau\right)^2 + \left(\frac{z}{a_z} + \frac{v_{0,z}}{\beta_z}\tau\right)^2\right]^2}$$

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1-D integral for energy modulation

$$I_d(z,t) = -\frac{Z_i e \omega_p^2}{\pi} \int_0^t d\tau \left(z + v_{0,z}\tau\right) \left\{ \frac{a_z \sin(\omega_p \tau)}{\left[\bar{\beta}^2 \tau^2 + (z + v_{0,z}\tau)^2\right]\left[1 + \bar{\beta}^2 \tau^2 + (z + v_{0,z}\tau)^2 / a^2\right]} - \cos(\omega_p \tau) \left[\frac{\arctan\left(\left|z + v_{0,z}\tau\right| / \left(\bar{\beta}\tau\right)\right)}{\left|z + v_{0,z}\tau\right|} - \frac{\arctan\left(\sqrt{(z + v_{0,z}\tau)^2 + a^2} / \left(\bar{\beta}\tau\right)\right)}{\sqrt{(z + v_{0,z}\tau)^2 + a^2}} \right] \right\}$$

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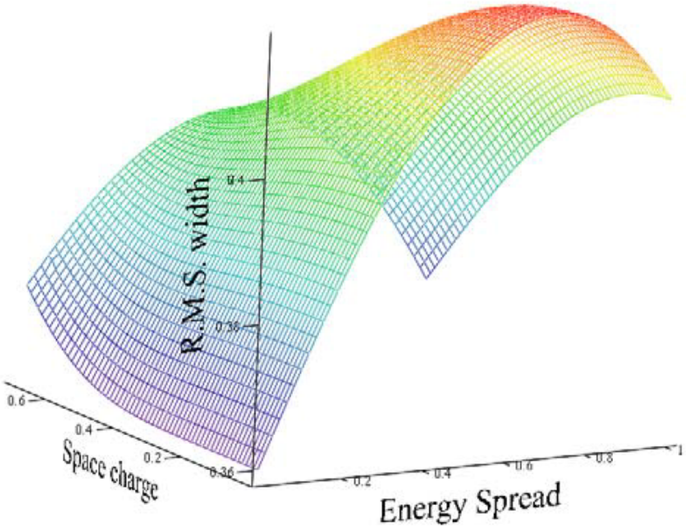
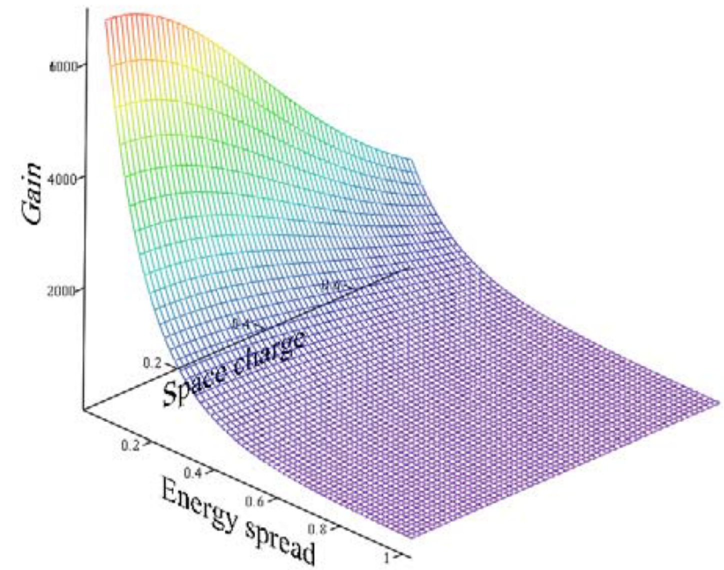
Analytical tools for FEL amplifier II

- The 1-D FEL model with uniform beam provides us with some insights as well as scaling laws of how the wave-packets depends on various beam parameters





- We also used the 1-D FEL model to study beam conditioning for CeC.

Start-to-end simulation for the single pass I

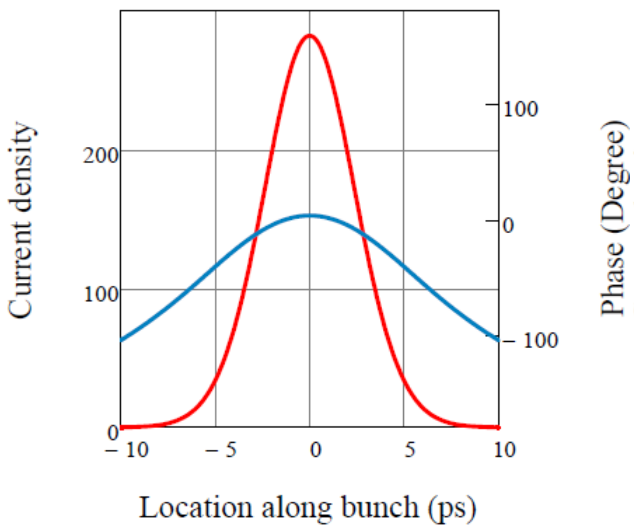
Steps for single pass start-to-end simulation:

1. At the entrance of the FEL, create macro-particles for the whole electron beam with proper shot noise. The 6-D distribution of the particles is determined by the beam dynamic simulation.
2. At the entrance of modulator, create one slice of macro-particles (with duration of one optical wavelength) with proper shot noise and 6-D distribution.
3. Run modulator simulation with the slice created in step 2. Due to periodic condition, the shot noise of the slice will stay correct.
4. Replace the corresponding slice created from step 1 with that output from step 3.
5. Run Genesis simulation. (Need to add macro-particles with negative energy to make it work as Genesis require each slice has the same number of macro-particles).
6. Take a proper portion of macro-particles output from GENESIS and import them into SPACE for kicker simulation.
7. Repeat step 1-6 but without the ion. The difference of step 6 and step 7 provides the single-pass coherent kick solely due to the ion.

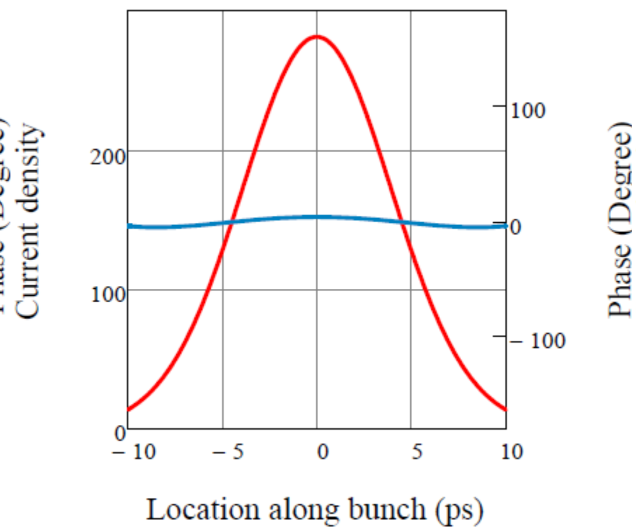
Analytical tools for FEL amplifier III

- As the amplitude and phase of the wave-packets depends both on the local beam current and the local beam energy, it is possible to optimize beam parameters such that the cooling efficiency over the whole ion beam can be improved.

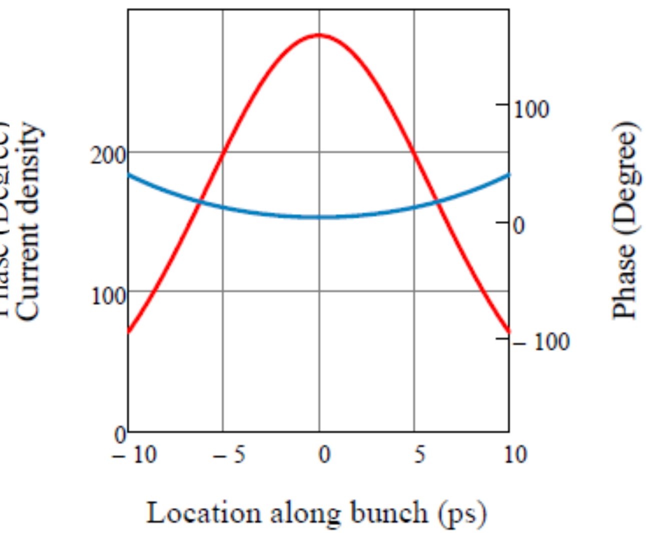
5 ps rms bunch length



8 ps rms bunch length



12 ps rms bunch length



- With the same peak current density, there is an optimum bunch length to minimize phase variation of the wave packet with respect to ions located at different portion of the electron bunch.

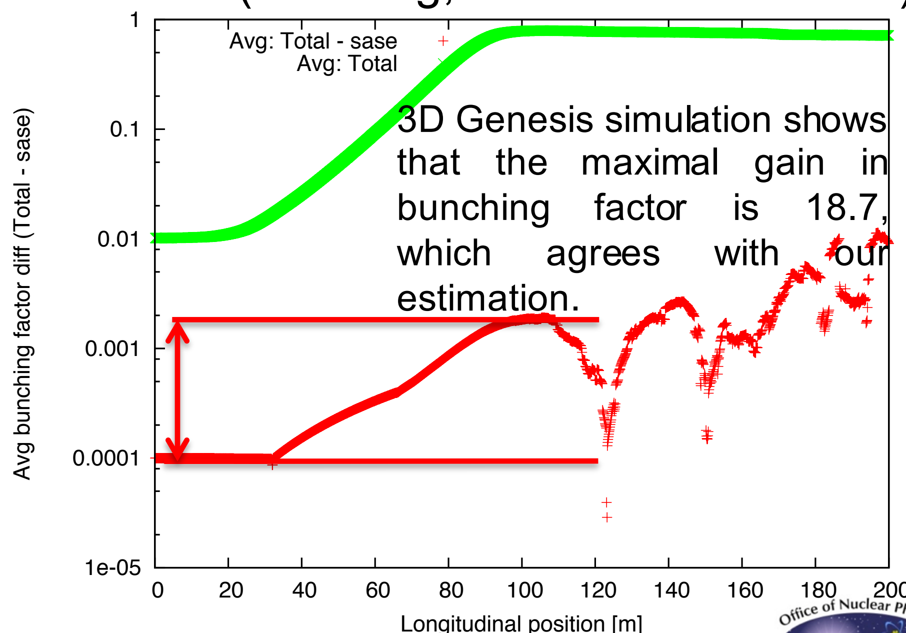
Analytical tools for FEL amplifier IV

- By requiring the relative density variation is smaller than one, we derived the upper limit of the FEL gain for the amplifier to work in the linear regime

$$\left|\delta\hat{n}/n_0\right|_{\max} < 1 \Rightarrow \quad |g|_{\max} < \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}} \Rightarrow \quad g_{\max} \sim 72\cdot\sqrt{\frac{I_e[A]\cdot\lambda_o[\mu m]}{M_c}} = 14.1$$

- γ=7460.52
- Peak current: 30 A
- Norm emittance 1 mm mrad
- RMS energy spread 2.5e-5
- λw=10 cm
- a_w = 10
- λo=90.73 nm
- Mc = 70.6

(© Y. Jing, with code GENESIS)



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Field Reduction due to Finite Transverse Modulation Size

$$\rho(\vec{r}) = \rho_o(r)\cdot\cos(kz);$$

$$\Delta\varphi = 4\pi\rho \Rightarrow \varphi(\vec{r}) = \varphi_o(r)\cdot\cos(kz);$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_o}{dr}\right) - k^2\varphi_o = 4\pi\rho_o(r)$$

$$\varphi(\vec{r}) = -4\pi\cos(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

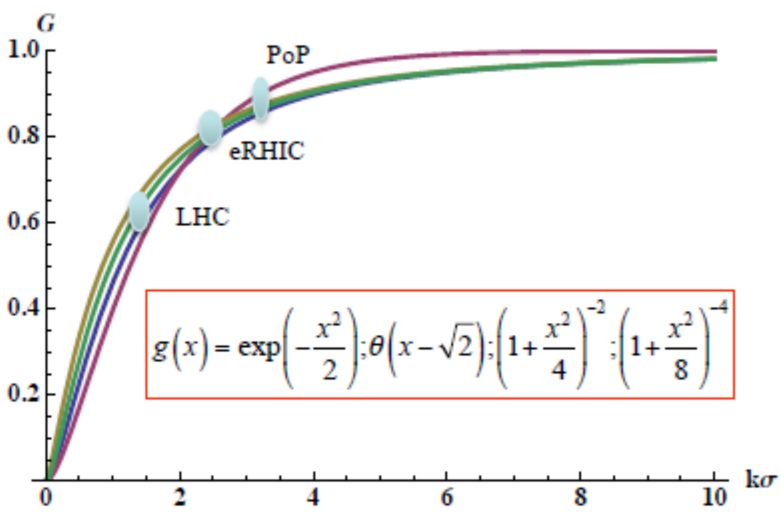
$$E_z = -\frac{\partial\varphi}{\partial z} = -4\pi k\sin(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$E_r = -\frac{\partial\varphi}{\partial r} = 4\pi k\cos(kz)\left\{I_1(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi - K_1(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

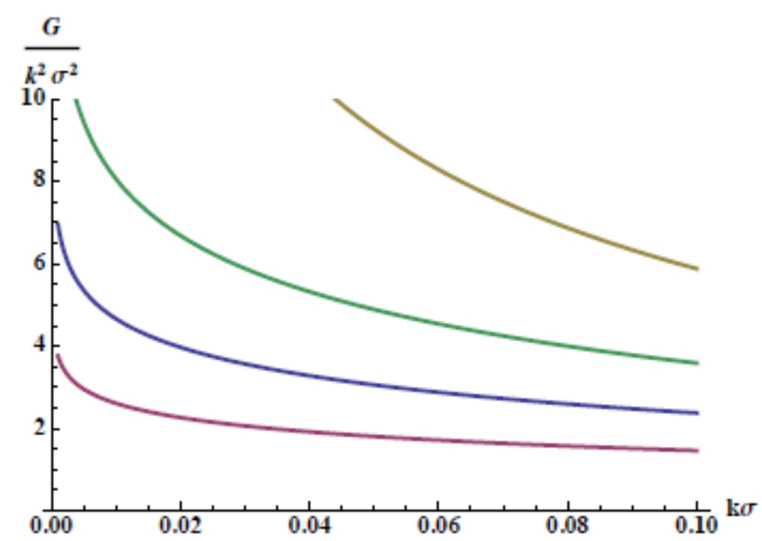
$$\rho(r) = \rho(0)\cdot g(r/\sigma)$$

$$E_{zo}(r=0) \propto -\frac{4\pi\tilde{q}}{\sigma^2}G(k_{cm}\sigma)$$

$$k_{cm}\sigma_\perp = \frac{k_o}{\gamma_o}\sqrt{\frac{\beta_\perp\varepsilon_{n\perp}}{\gamma_o}} = \sqrt{\gamma_o}\sqrt{\beta_\perp\varepsilon_{n\perp}}\,\frac{k_w}{2(1+a_w^2)}$$



$$g(x) = \exp\left(-\frac{x^2}{2}\right); \theta\left(x-\sqrt{2}\right); \left(1+\frac{x^2}{4}\right)^{-2}; \left(1+\frac{x^2}{8}\right)^{-4}$$



Analytical Prediction for FEL Amplifier I

- The 1D FEL amplifier model with collinear radiation field and κ-1 (Lorentzian) energy distribution has been applied to study how the wave-packet forms inside the undulator.

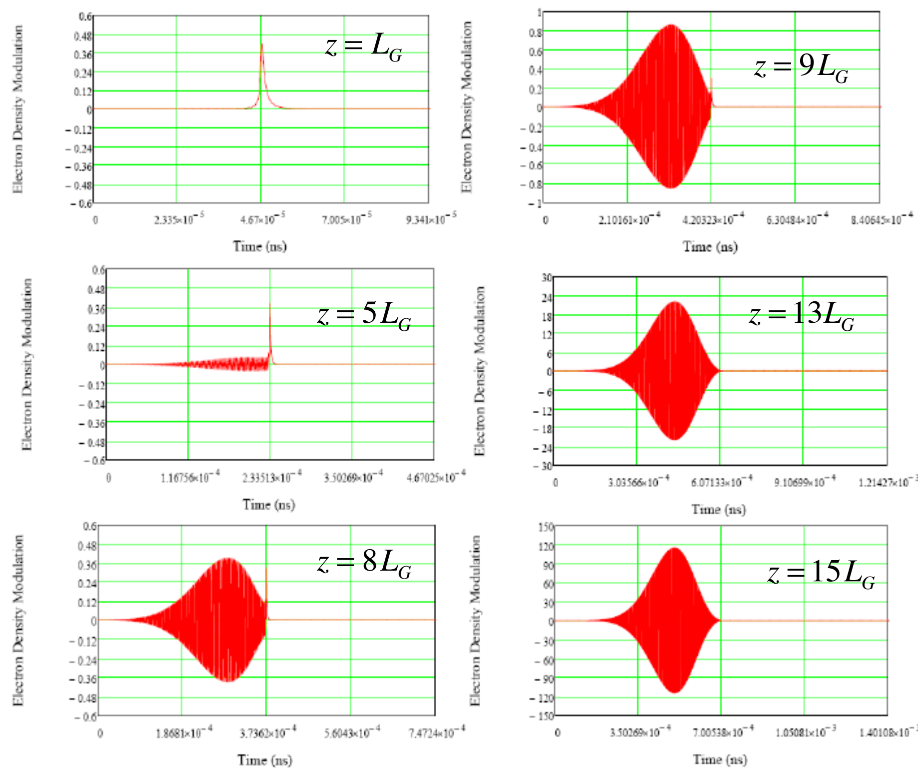
$$\tilde{j}_1(z) = -\left(\frac{\theta_s}{2\varepsilon_0 c}\right)^{-1}\left[A_1\lambda_1 e^{\lambda_1 \hat{z}} + A_2\lambda_2 e^{\lambda_2 \hat{z}} + A_3\lambda_3 e^{\lambda_3 \hat{z}}\right]$$

- In high gain regime, the eigenvalues can be expanded into quadratic order in the detuning parameter $\hat{C}$, i.e.

$$\lambda\left(\hat{C}\right) = c_0 + c_1\left(\hat{C} - \hat{C}_0\right) + c_2\left(\hat{C} - \hat{C}_0\right)^2$$

which make it possible to obtain analytical form of the wave-packet

Evolution of wave-packet along undulator as calculated from 1-D theory, showing that it take a few gain length for the wave-packet to overtake the initial modulation.



$$j_1(z,t) = \frac{Z_i eck_0}{S\sqrt{\pi}}\exp\left[\frac{-\left(t - t_p(z)\right)^2}{2\sigma_t^2}\right]\mathrm{Re}\left\{B_1\left(\hat{C}_0\right)\frac{c_0}{\sqrt{-c_2}}\sqrt{\frac{\rho}{2k_w z}}\exp\left(2c_0\rho k_w z\right)\exp\left[ik_w z + ik_0\left(z - ct\right)\right]\right\}$$

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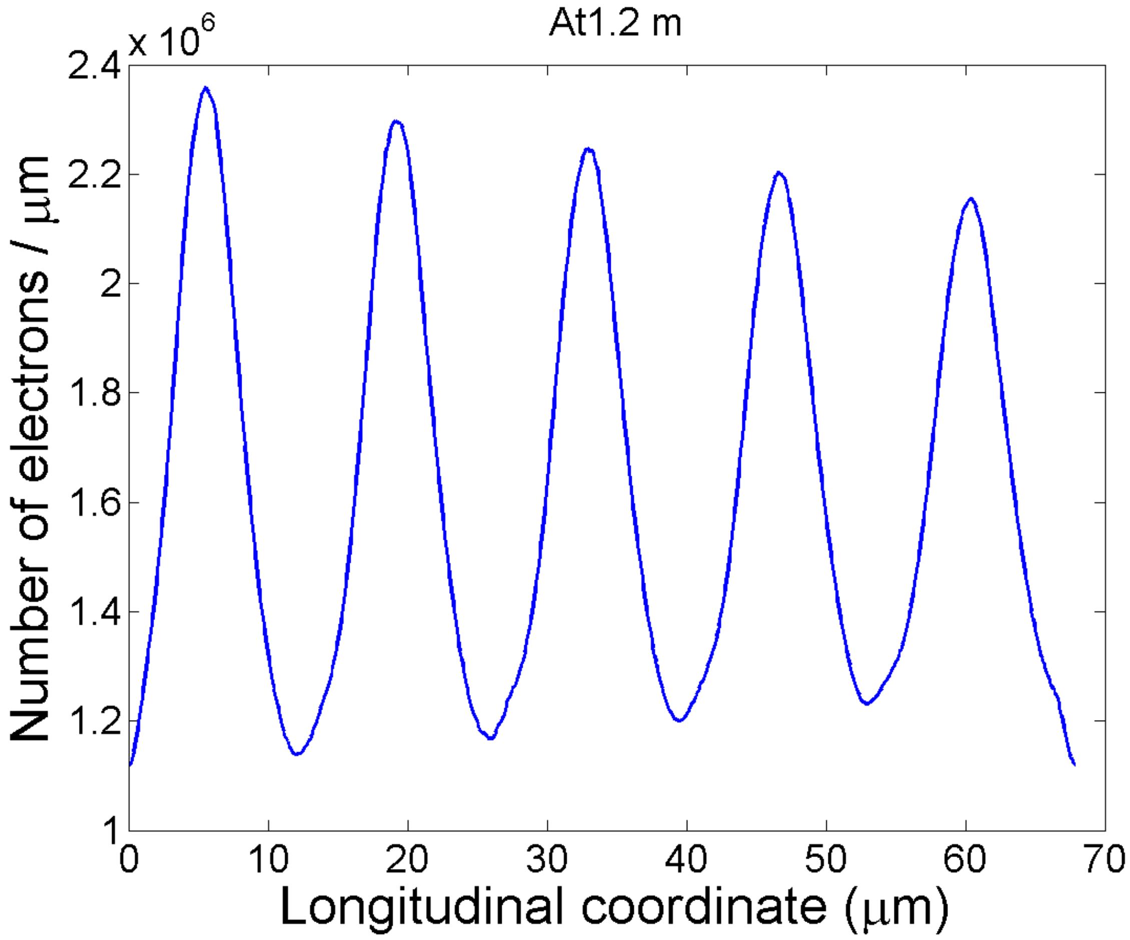
Background electron line density at the entrance of the kicker



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Analytical Tools for Kicker I

- Dynamic equation in Kicker is very similar to that in the modulator except the initial modulation in 6D phase space dominates the process. For κ-2 velocity distribution, the electron density perturbation is determined by:

$$\frac{d^2}{dt^2}\tilde{R}_1(\vec{k},t)+\omega_p^2\tilde{R}_1(\vec{k},t)=Z_i\omega_p^2 e^{-\lambda(\vec{k},\vec{v}_0)t}-\omega_p^2\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k},\vec{v},0)e^{-\lambda(\vec{k},\vec{v})t}d^3v$$

with $\tilde{R}_1(\vec{k},t)\equiv\tilde{n}_1(\vec{k},t)e^{-\lambda(\vec{k})t}-\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k},\vec{v},0)e^{-\lambda(\vec{k},\vec{v})t}d^3v$ and

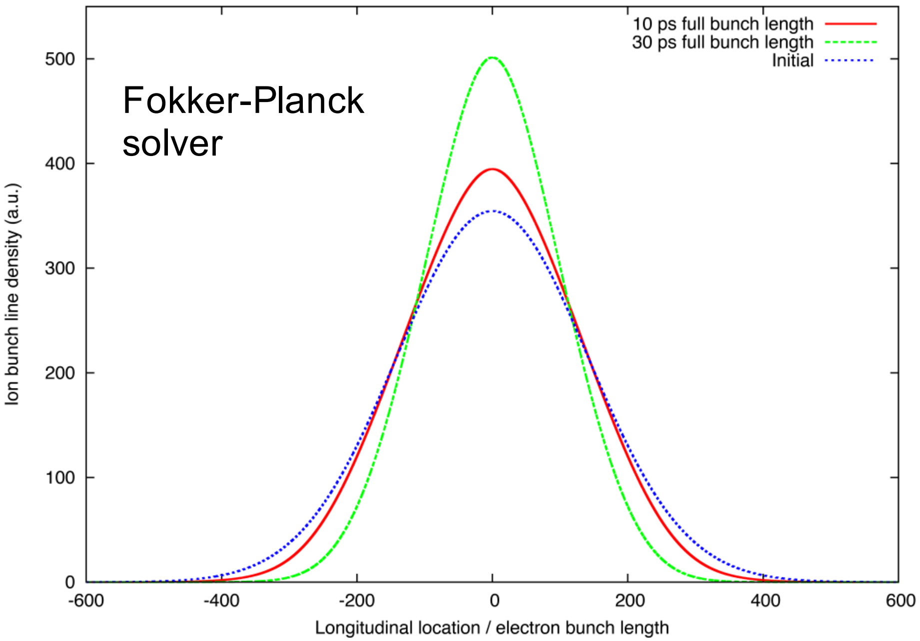
$$\lambda(\vec{k},\vec{v})\equiv i\vec{k}\cdot\vec{v}-\sqrt{(k_x\sigma_x)^2+(k_y\sigma_y)^2+(k_z\sigma_z)^2}$$

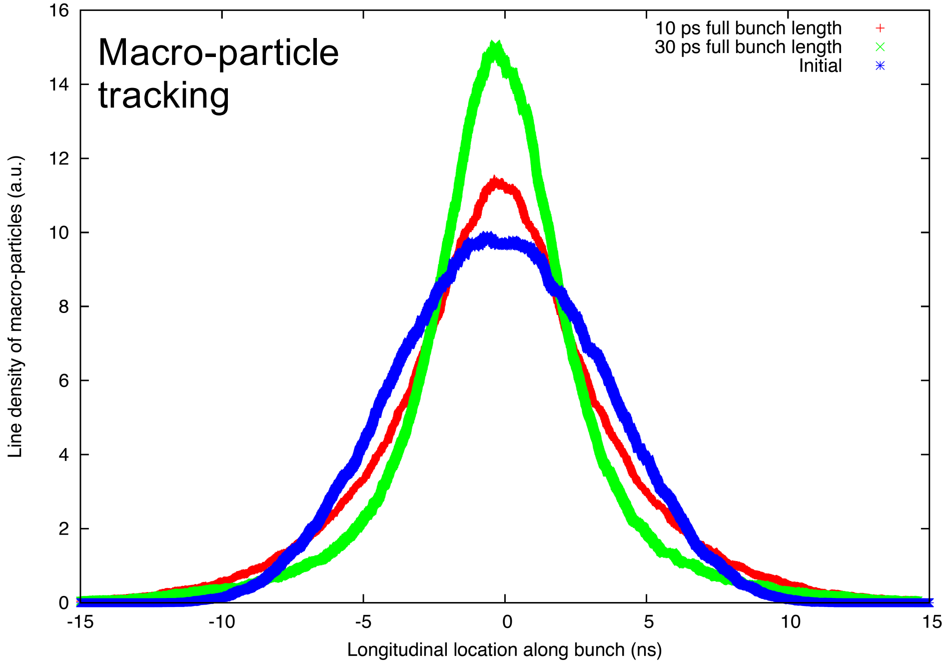
The solution of this inhomogeneous 2nd order differential equation reads

$$\tilde{R}_1(\vec{k},t)=c_1\cos(\omega_p t)+c_2\sin(\omega_p t)$$
$$+\frac{1}{\omega_p}\int_{-\infty}^{\infty}\frac{\omega_p e^{-\lambda t}+\lambda\sin(\omega_p t)-\omega_p\cos(\omega_p t)}{\lambda^2+\omega_p^2}\tilde{f}_1(\vec{k},\vec{v},0)d^3v$$

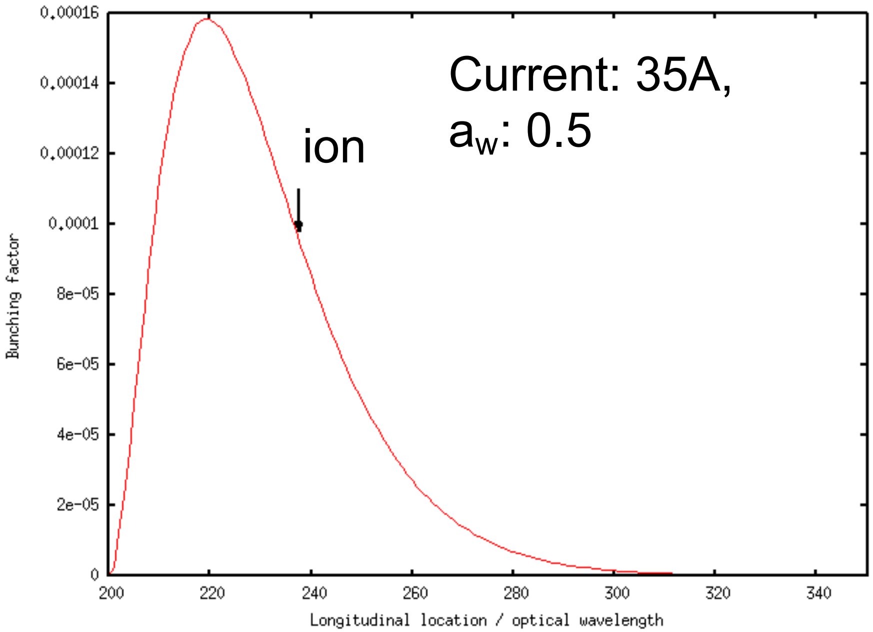
Simulation results for cooling 40 GeV Au beam

The plots show how the longitudinal profile of ion bunch evolves after 40 minutes of cooling with 10 ps (red) and 30 ps (green) electron bunch. The left plot is generated by solving Fokker-Planck equation as described in the previous slides and the right plot is created by the macro-particle tracking code. The parameters applied are for cooling 40 GeV RHIC gold ion beam.

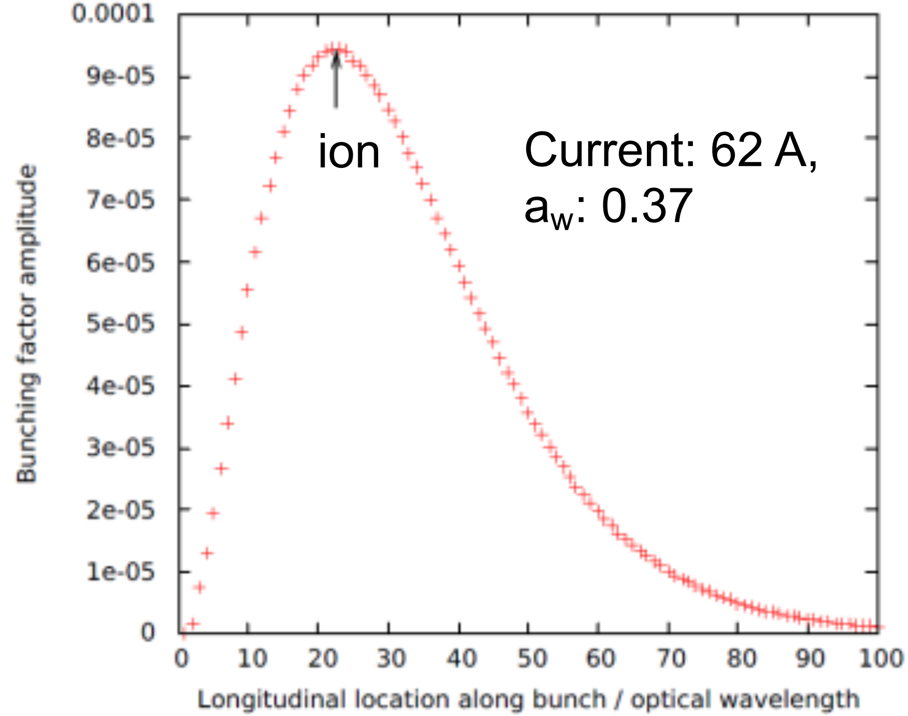




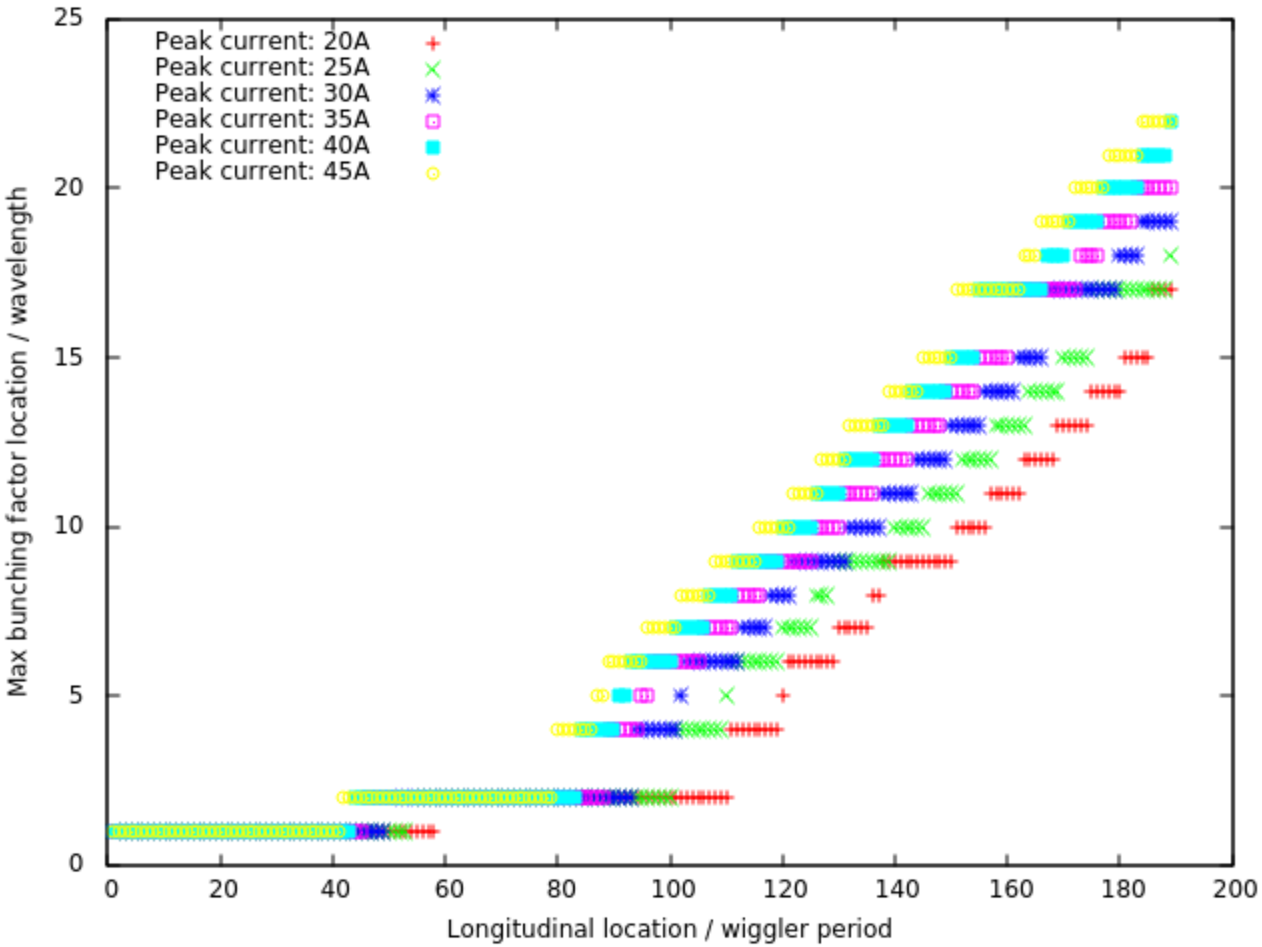
Reduce a_w to align ion with wave-packet



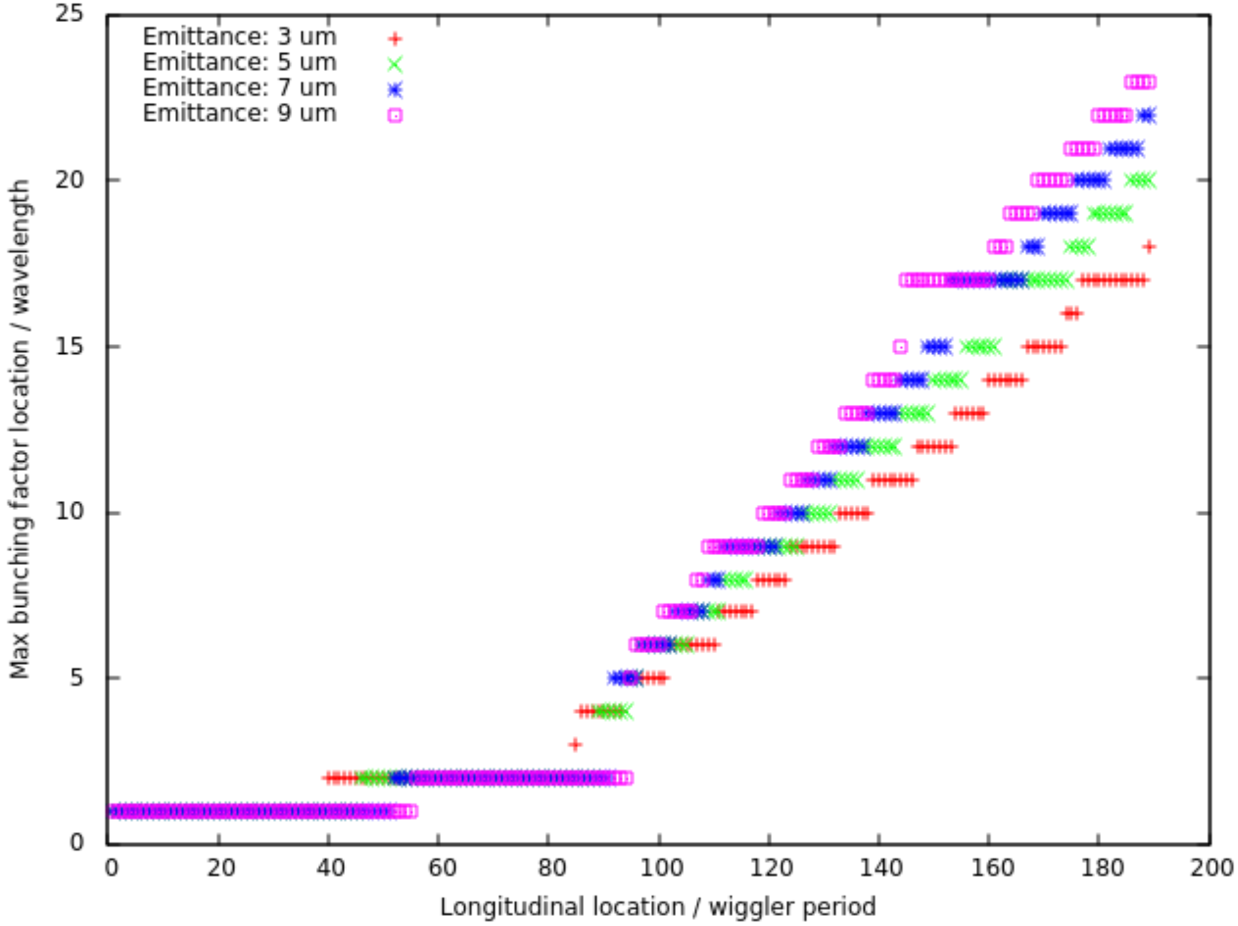
Gamma: 28.66,
Norm. emittance: 5e-6 m



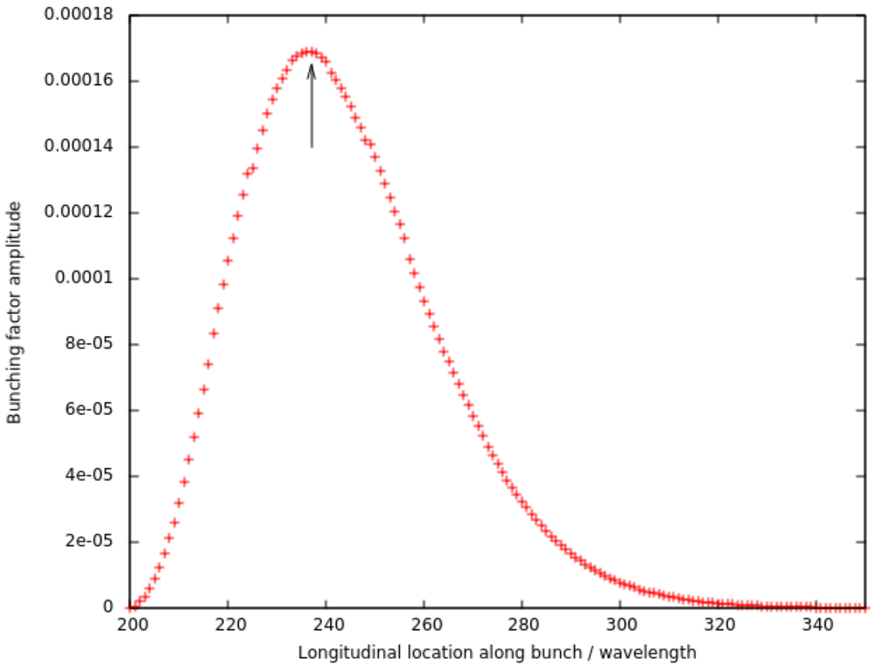
How wave-packet peak location depends on peak current II



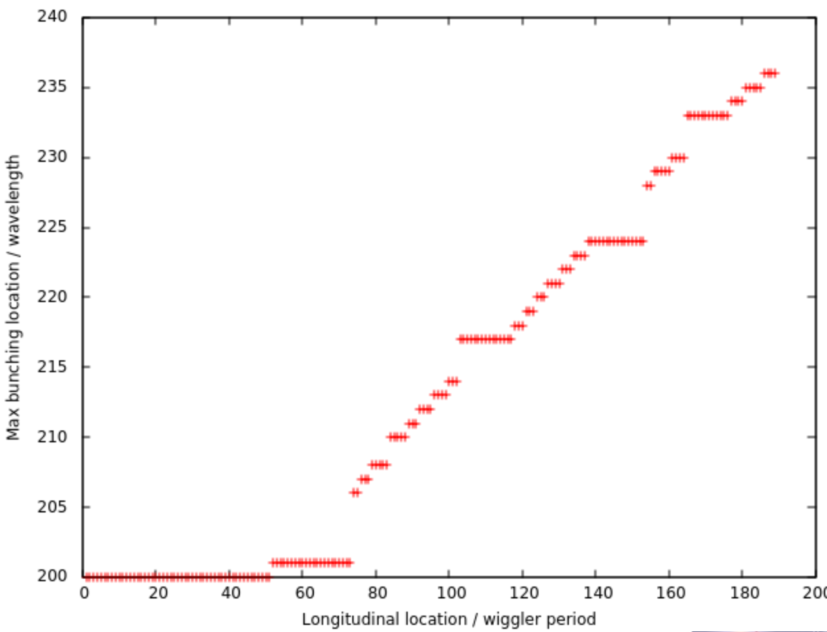
How wave-packet peak location depends on emittance



Combining the above two effects: 22 um emittance and 85 A current



The ion does arrive at slice #237



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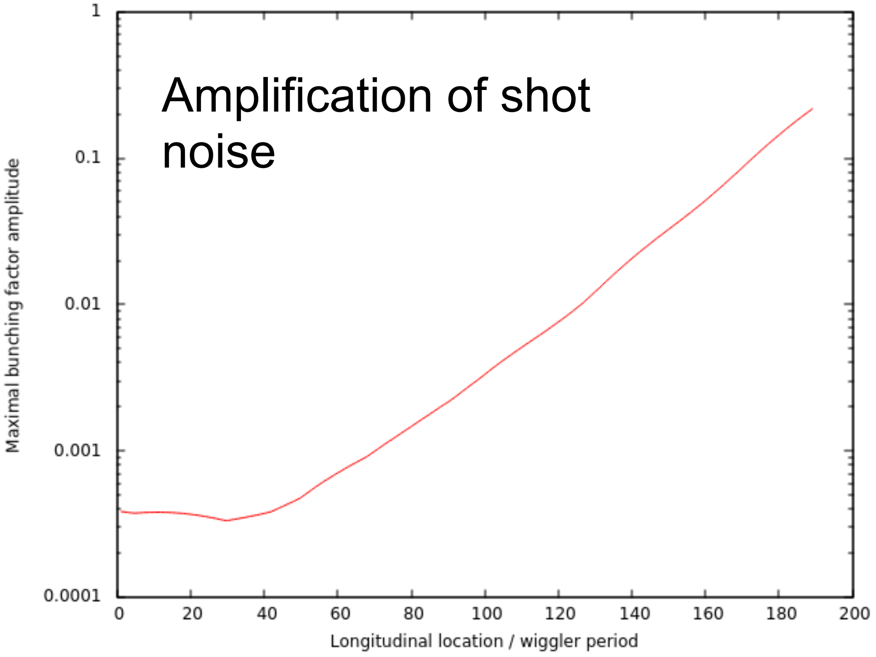
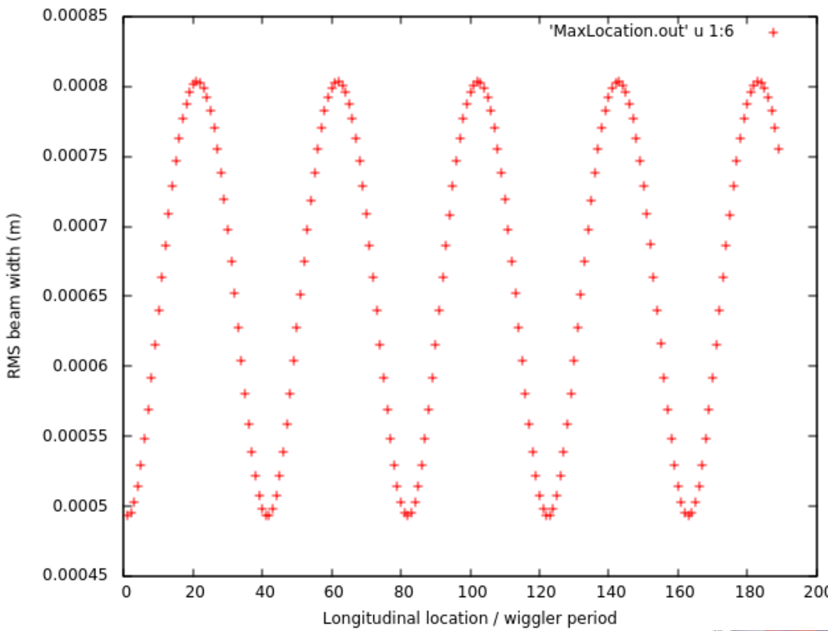
The FEL looks not saturated

Amplification of shot noise



Transverse beam size



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Simulation Tools for the Modulation Process II

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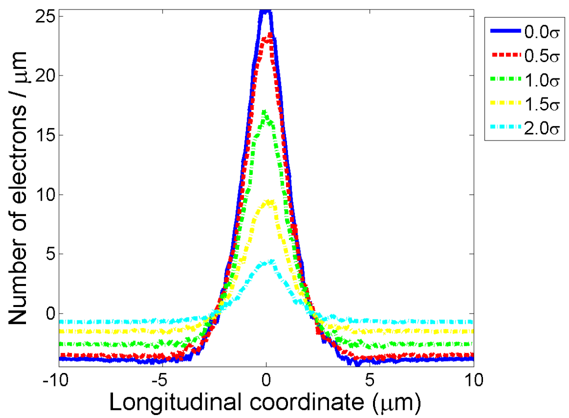
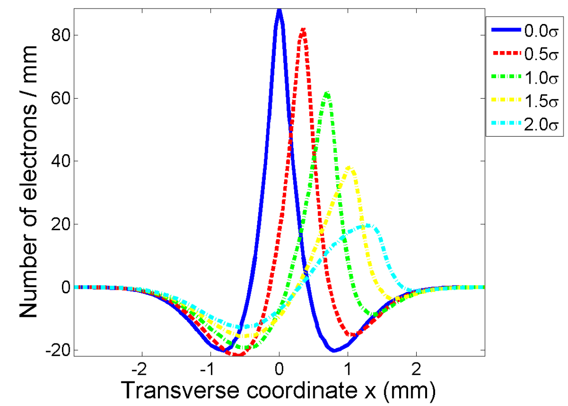
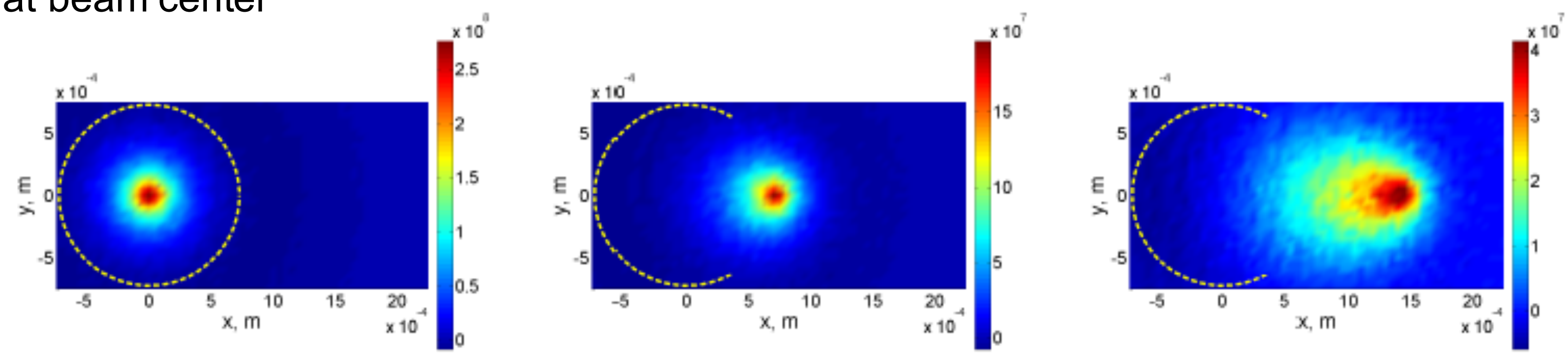
– Simulation results for a continuous focusing channel (Beam is matched and transverse beam size does not vary.)

- Modulation is less effective for an off-centered ion. For an ion sitting at 1σ away from transverse electron beam center, the longitudinal density modulation reduces by ~40%.
- The transverse density modulation profile induced by an off-centered ion is significantly different from that induced by an ion at beam center





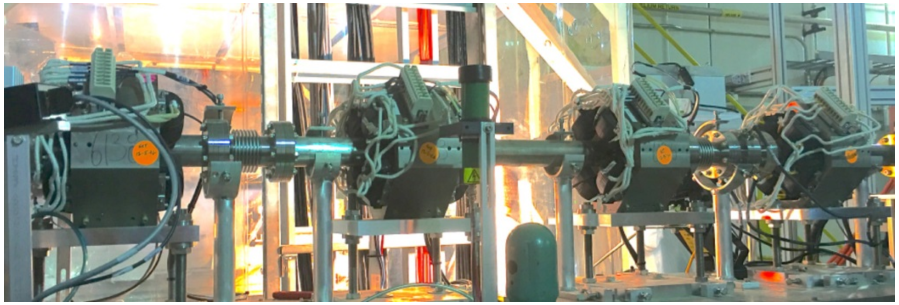


(a) Ion at center

(b) Ion at $x = 1\sigma$

(c) Ion at $x = 2\sigma$

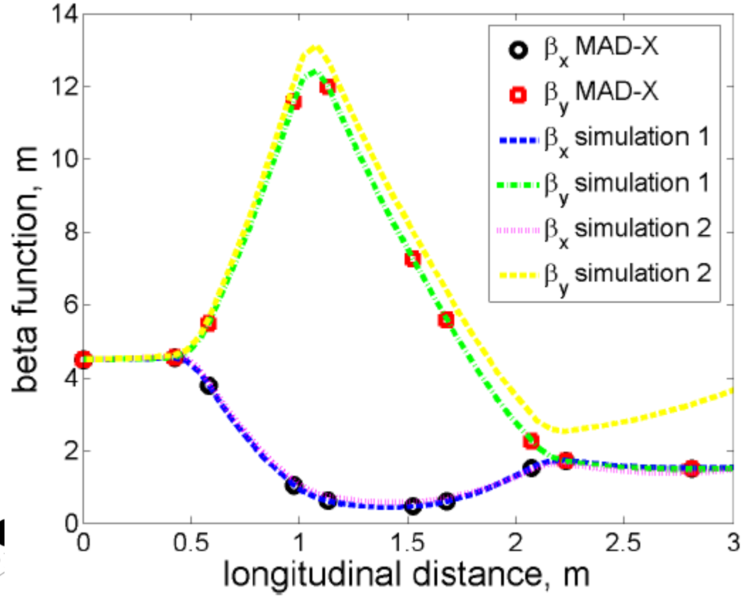
Simulation Tools for the Modulation Process III



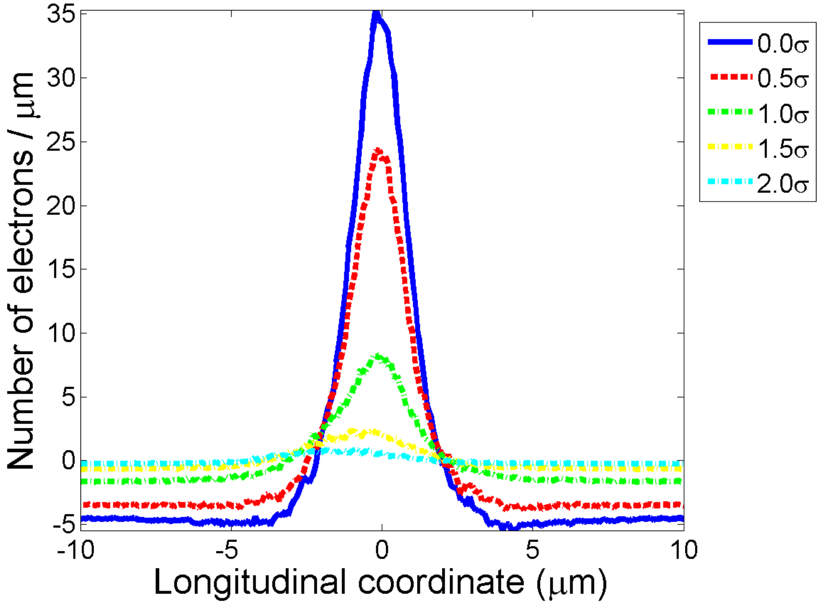
(© J. Ma, with code SPACE)

- Simulation results for a quadrupole focusing channel (Transverse beam size varies along the modulator)
  - Beta function extracted from SPACE agrees with that from MAD-X calculation when space charge is turned off in SPACE simulation.
  - When space charge is turned on, the vertical lattice function at the end of modulator deviates significantly from that calculated by MAD-X.
  - The efficiency of longitudinal density modulation depends strongly on the quadrupole settings, which has to be taken into account in optimizing the system.

Beta function along modulator (not optimized)



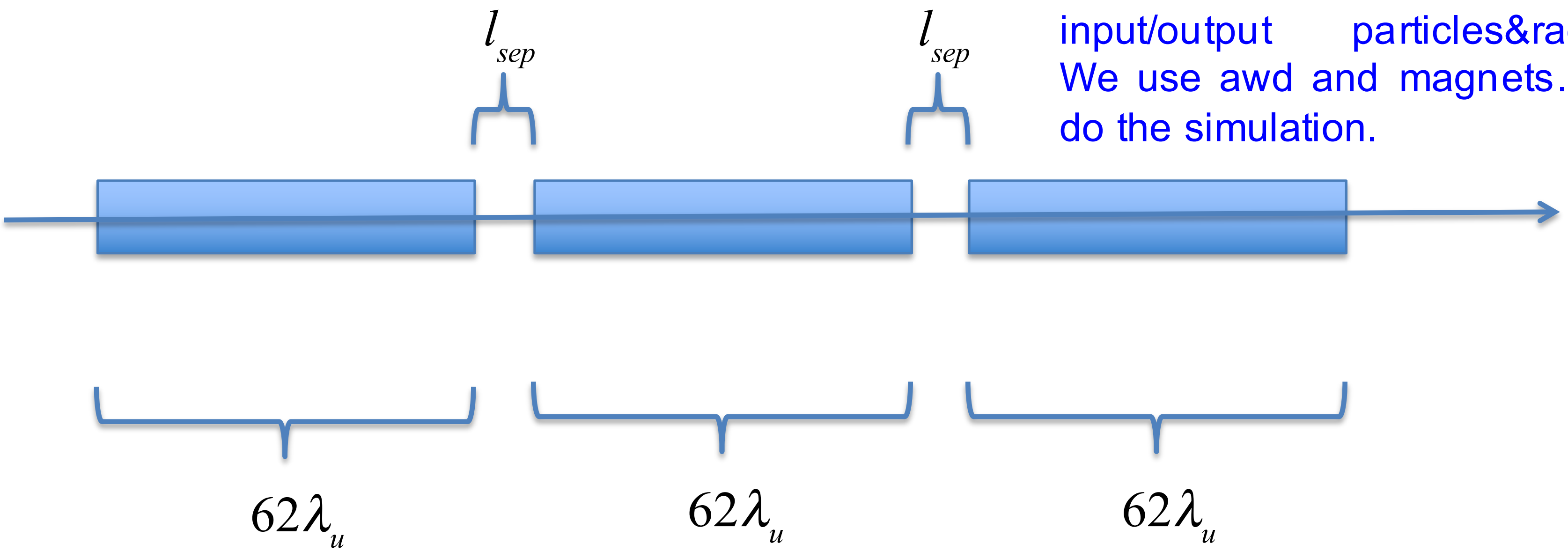
Longitudinal density modulation



Genesis Simulation with Undulator Consisting of 3 Subsections

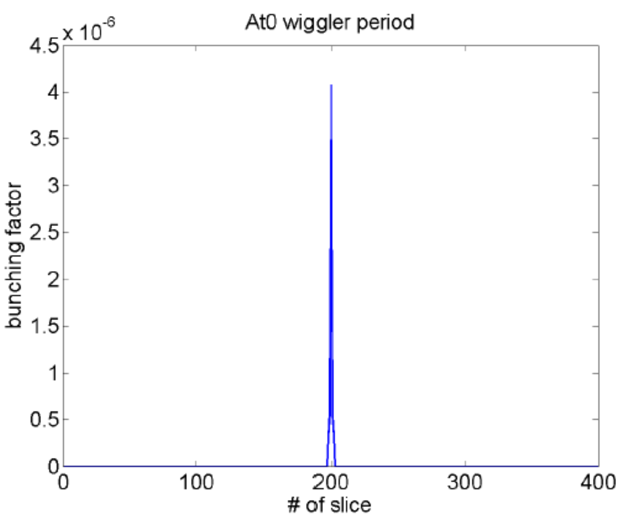
G. Wang and J. Ma

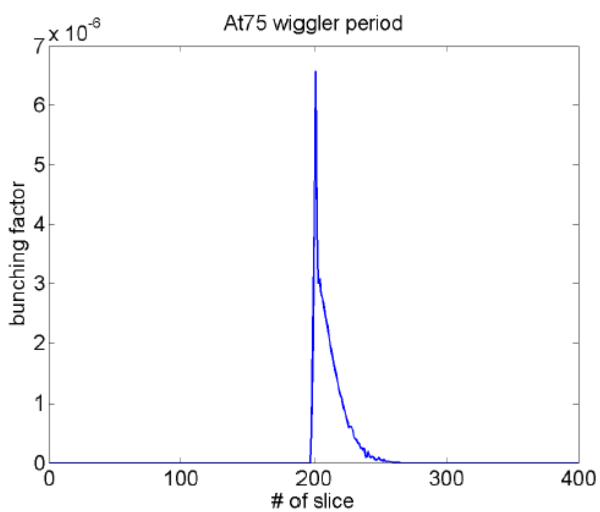
It seems possible to configure Genesis so that it simulates a multi-section undulator without input/output particles&radiations. We use awd and magnets.in file to do the simulation.

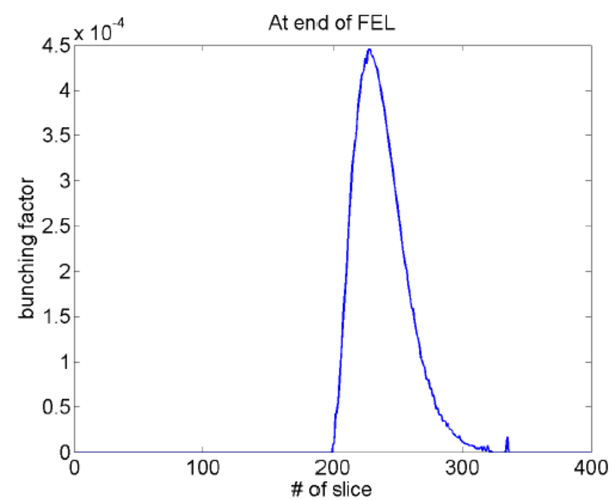


Simulation tools for FEL amplifier

- We use GENESIS 1.3 to simulate the amplification process in the FEL amplifier.
- Following the approach of perturbative trajectories, we run two sets of FEL simulation: one with shot noise plus modulation induced by the ion and the other one with shot noise only. The wave-packet due to the ion is extracted from the difference of the two sets of simulation. The plots are results for 20 MeV electrons.







Simplified Genesis Simulation for 14.6 MeV Electron Beam

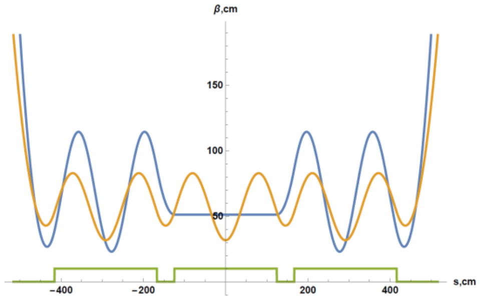
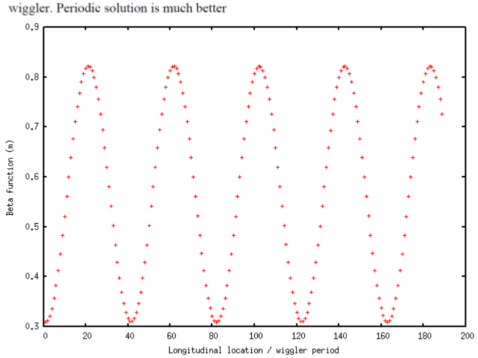
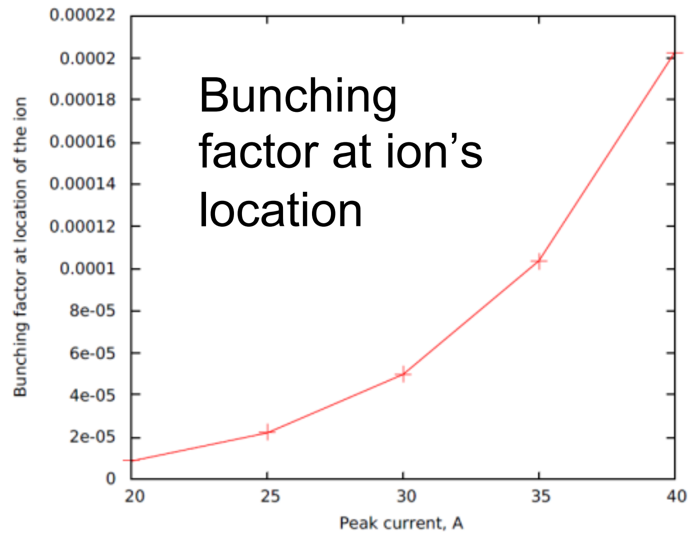
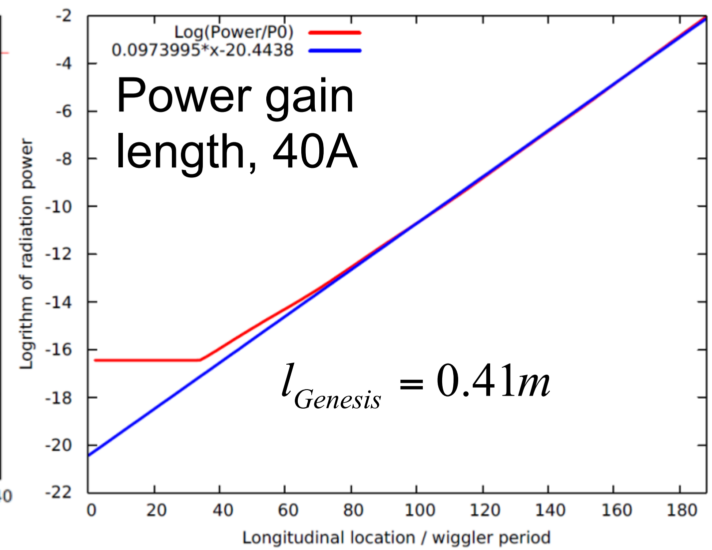


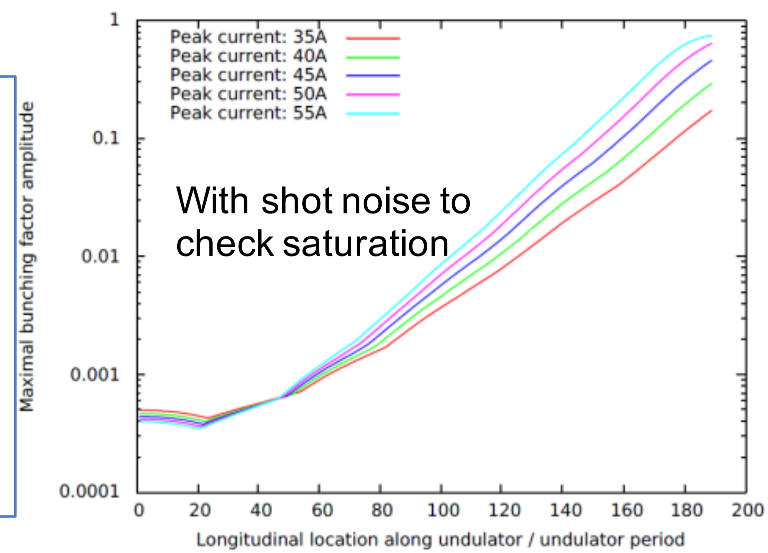
Fig. 1 Periodic solution for CeC FEL system (yellow) and "matched" beta in the middle wiggler. Periodic solution is much better







- We assume one long undulator and mismatch is intentionally introduced to reflect the beam size variation for a three subsection undulator.
- The power gain length is about 10% longer than that derived from Ming Xie's formula.
- With a peak current of 35A~40A, we should get a gain of 100~200 for the bunching factor.
- Below 40A, the FEL works in the linear regime.



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Simulation tools for kicker

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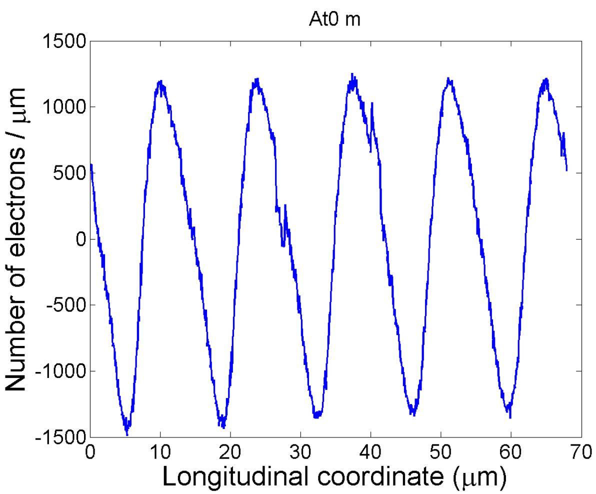
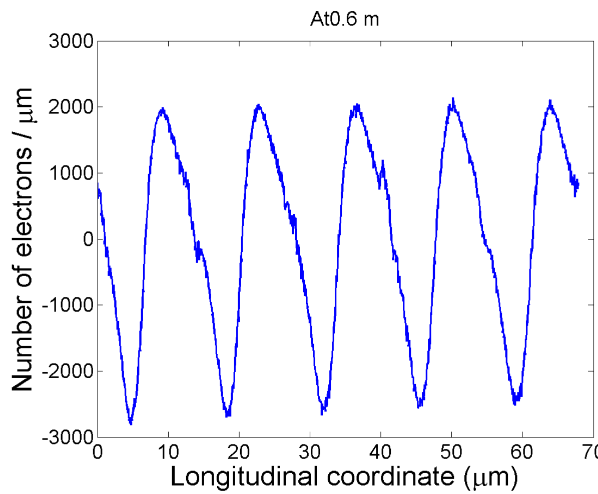
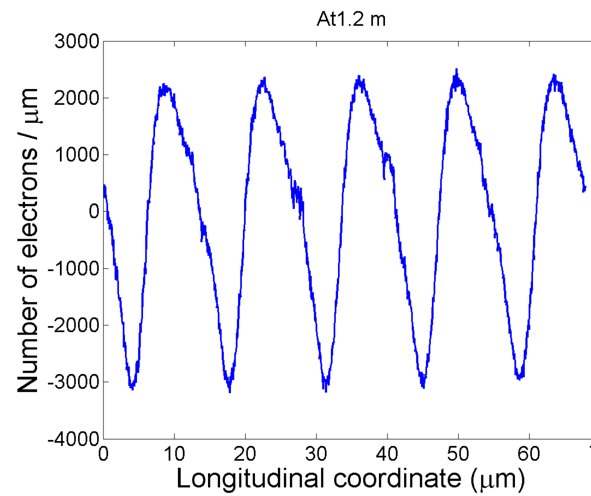
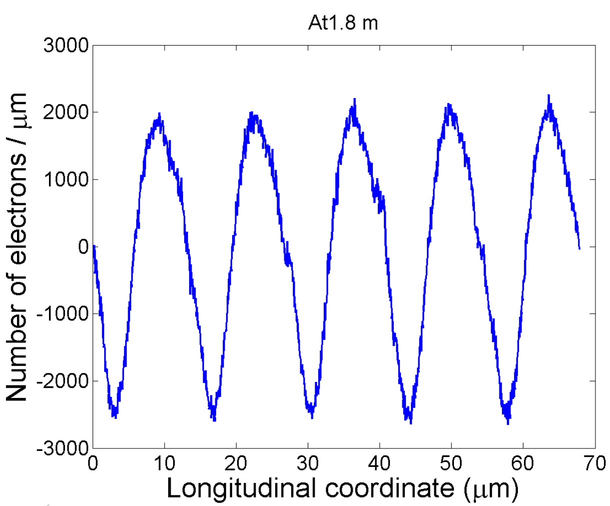
- The macro-particles from GENESIS simulation are imported into SPACE for the kicker simulation.

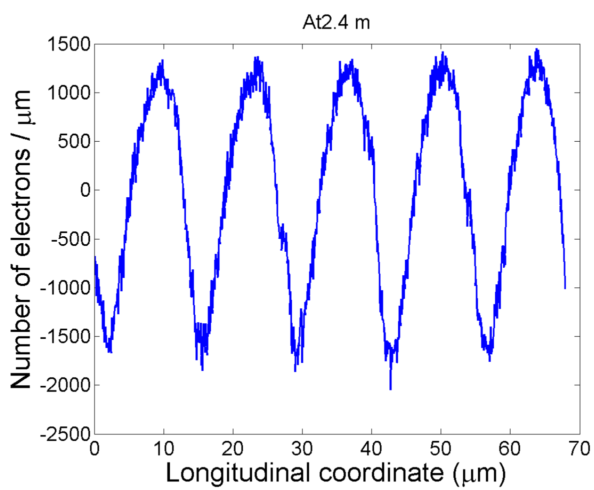
Background line density is 2e6/μm.











The evolution of the wave-packet shows similar behavior as that obtained from analytical model, i.e. the wave-packet amplitude increases initially and then starts to decrease.

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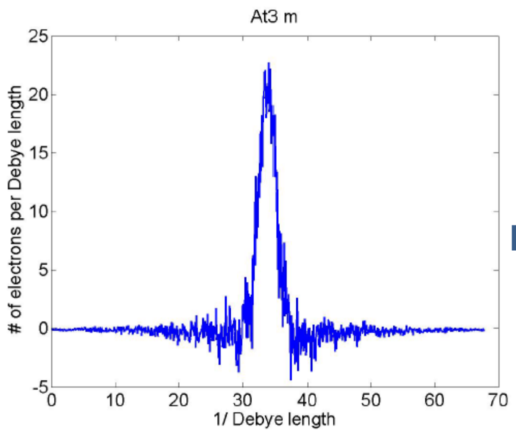
Start-to-end simulation for the single pass

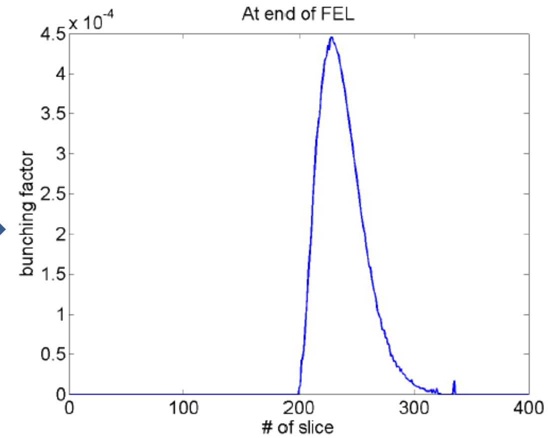
(© J. Ma, with code SPACE and GENESIS)

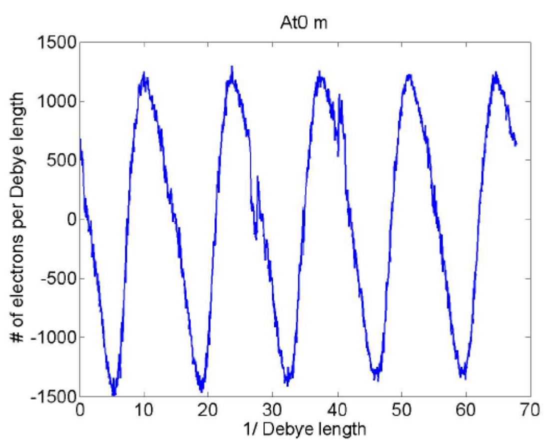
- One example of start-to-end simulation (to cool 40 GeV Au)

Modulator

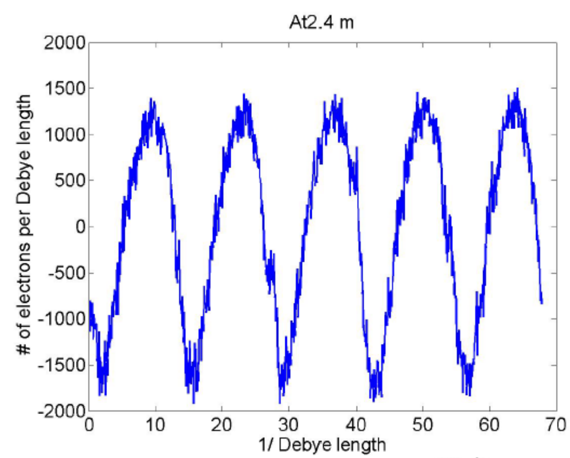


Amplifier





Kicker



Simulation tools for predicting the influences of CeC to a circulating ion beam II

- Assuming the ion density does not vary significantly over the width of the wave-packet

$$\left\langle \Delta E_{inc,j}^{2} \right\rangle = \frac{\left(Z_i e E_p l_1\right)^2}{2} \int_{-\infty}^{\infty} \rho_{ion}\left(z_i\right) e^{-\frac{\left(z_i - z_i\right)^2}{\sigma_{z,rms}^2}} dz_i \approx \frac{\left(Z_i e E_p l_1\right)^2}{2} \sqrt{\pi} \rho_{ion}\left(z_j\right) \sigma_{z,rms}$$

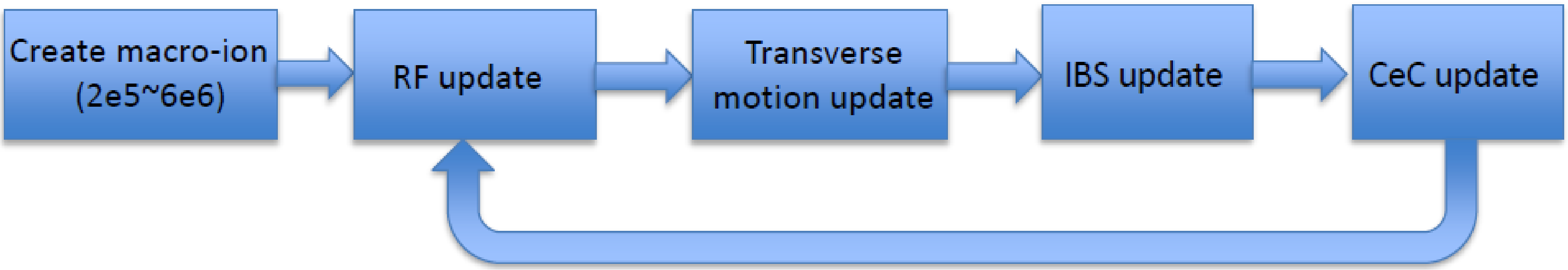
- The one-turn energy kick due to CeC is

$$\Delta E_{j,N} \approx -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right) + Z_i e E_p l \sqrt{\frac{3}{2}\sqrt{\pi}\rho_{ion}\left(z_j\right)\sigma_{z,rms}} \cdot X_{j,N} + \Delta E^{e}{}_{j,N}$$

Diffusive kick induced by neighbor electrons, i.e. electrons' shot noise

$$\Delta E_{j,N}^{e} \approx e E_p l \sqrt{\frac{3}{2}\sqrt{\pi}\rho_e\left(z_j\right)\sigma_{z,rms}} \cdot Y_{j,N}$$

Simulation tools for predicting the influences of CeC to a circulating ion beam I



Energy kicks from CeC is $\Delta E_j = \Delta E_{coh,j} + \Delta E_{inc,j}$

Coherent kick induced by the ion itself $\Delta E_{coh,j} \equiv -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right)$

Incoherent kick induced by the neighbor ions (using the Gaussian profile as obtained by quadratic expansion of FEL eigenvalues)

$$\Delta E_{inc,j} \equiv -Z_i e E_p l \sum_{i \neq j} \exp\left[-\frac{\left(z_j - z_i\right)^2}{2\sigma_{z,rms}^2}\right] \sin\left(k_0\left(D\delta_j + z_j - z_i\right) - k_2^2\left(z_j - z_i\right)^2\right)$$

z_i: longitudinal location of the i^{th} ion; $\sigma_{z,rms}$: RMS width of the wave-packet; $D : R_{56}$.

Since there is no correlation between any successive incoherent kicks, one can use a random kick to represent the incoherent kicks

$$\Delta E_{j,N} \approx -Z_i e E_p l_1 \sin\left(k_0 D \cdot \delta_j\right) + \sqrt{\frac{\left\langle \Delta E_{inc,j}^2 \right\rangle}{\left\langle X^2 \right\rangle}} \cdot X_{j,N}$$

For a random number uniformly distributed between -1 and 1

$$\left\langle X^2 \right\rangle = \frac{1}{2}\int_{-1}^{1} X^2 dX = \frac{1}{3}$$

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