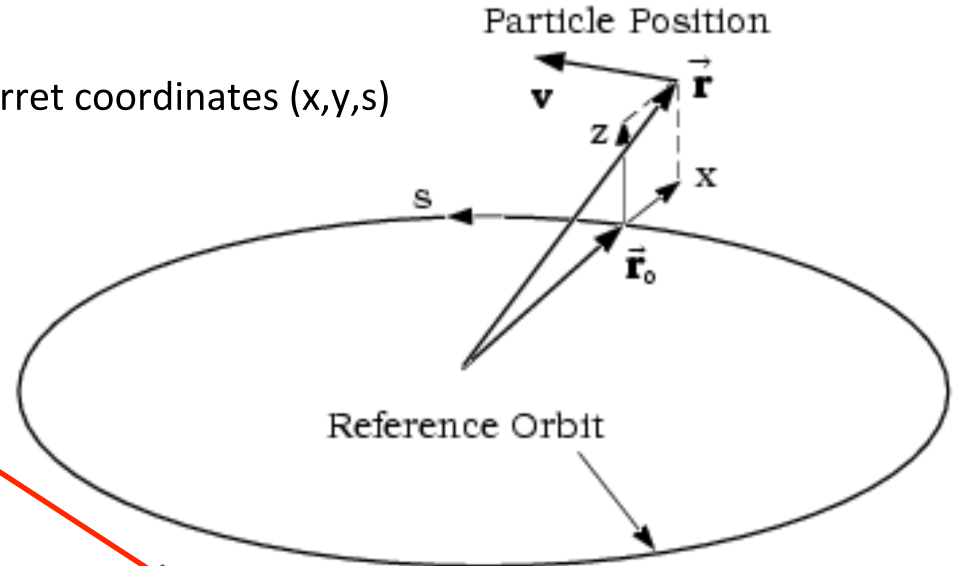
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

What we learned:

Frenet-Serret coordinates (x,y,s)



Hill's equations (derivatives w.r.t. s)

$$x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

Natural focusing from dipoles (curvature)

Focusing from quadrupoles

Higher order magnet, usually field errors

$$\theta = \frac{s}{R} = \frac{\beta ct}{R}$$

Solution of Hill's equations X(s), X'(s) form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

X can be x or y

$$|M(s,s_0)| = 1 \qquad |Trace(M(s,s_0))| \leq 2$$

Stable solution conditions

Courant-Snyder parameterization

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

Where α,β,γ,ϕ are functions of s and describes position dependent beam properties.

Focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Defocusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

Dipole: $K=1/\rho^2$

$$M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Drift space: $K=0$

$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Floquet Theorem

We consider the linear Hill's equation of motion **X'' + K(s)X = 0**, where X(s) and X'(s) are conjugate coordinates, K(s) is the focusing function, and the prime is the derivative with respect to the independent variable s. In many accelerator applications, K(s) is a periodic function of s with period L, i.e. **K(s+L)=K(s)**. Floquet theorem states we can express the solution in amplitude and phase functions which satisfy a periodic boundary condition similar to that of the potential function K(s), i.e.

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

where the phase advance μ in one period is independent of s. Using the Floquet transformation on Hill's equation, we get the differential equation

$$2w'\psi' + w\psi'' = 0, \quad w'' + K(s)w - w\psi'^2 = 0$$

$$\psi' = \frac{1}{w^2}, \quad \psi = \int_{s_0}^{s} \frac{ds}{w^2}, \quad w'' + K(s)w - \frac{1}{w^3} = 0$$

Floquet transformation: $$X'' + K(s)X = 0$$

$$X(s) = aw(s)e^{j\psi(s)} \qquad w'' + K(s)w - \frac{1}{w^3} = 0 \qquad \psi' = \frac{1}{w^2}$$

What is the transfer matrix $M(s_2,s_1)$?

$\psi_2=\psi_1+\mu$, where μ is the phase advance.

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2,s_1)\begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2,s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\mu - w_2 w_1' \sin\mu & w_1 w_2 \sin\mu \\ -\frac{1+w_1w_1'w_2w_2'}{w_1w_2}\sin\mu - (\frac{w_1'}{w_2} - \frac{w_2'}{w_1})\cos\mu & \frac{w_1}{w_2}\cos\mu + w_1 w_2' \sin\mu \end{pmatrix}$$

$$w_1 = w_2, \quad w_1' = w_2', \quad \psi_2 - \psi_1 = \mu$$

$$M(s) = \begin{pmatrix} \cos\mu - ww'\sin\mu & w^2\sin\mu \\ -(1+w'^2)\sin\mu & \cos\mu + ww'\sin\mu \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s}\frac{1}{\beta}ds$$

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}[1+(\frac{\beta'}{2})^2] = 0, \quad \text{or} \quad \alpha' = K\beta - \frac{1}{\beta}[1+\alpha^2]$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\psi & \frac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

$$w_1 = w(s_1), \quad w_2 = w(s_2), \quad w_1' = w'(s_1), \quad w_2' = w'(s_2)$$

The transfer matrix from s_1 to s_2 in any beam transport line becomes

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1 \beta_2}\sin\mu \\ -\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}}\sin\mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1 \sin\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

Floquet theorem: Many accelerator components obey the periodic condition: $K(s+L)=K(s)$. The solution of Hill's equation is periodic. In matrix representation, we obtain

$M_1 M_2 \ldots\ldots\ldots\ldots M_n \quad M_1 M_2 \ldots\ldots\ldots\ldots M_n$

$$M(s_1+L|s_1)=M_n M_{n-1} M_{n-2} \ldots M_2 M_1 = \mathbf{M}(s_1)$$
$$M(s_2+L|s_2)=M_1 M_n M_{n-1} M_{n-2} \ldots M_2 = \mathbf{M}(s_2) = M_1 \mathbf{M}(s_1) M_1^{-1}$$

Each matrix is a product of identical number of matrices. They are related by **similarity** transformation. The eigen-values of the periodic matrix $\mathbf{M}(s_i)$ are identical.

$$M(s_2) = M(s_2 \mid s_1) M(s_1) [M(s_2 \mid s_1)]^{-1}$$

With the similarity transformation of the transfer matrix,

$$M(s_2) = M(s_2 | s_1) M(s_1) [M(s_2 | s_1)]^{-1}$$

the values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\mathrm{M}(s_2) = I\cos\Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin\Phi = I\cos\Phi + J_2 \sin\Phi$$

$$\mathrm{M}(s_1) = I\cos\Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin\Phi = I\cos\Phi + J_1 \sin\Phi$$

$$J_2 = M(s_2 | s_1) J_1 [M(s_2 | s_1)]^{-1}$$

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

M_{ij} is the *ij*-th component of the matrix $M(s_2,s_1)$

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix}\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_1$$

1. The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s-s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Note that γ is constant in a drift space, and $\mathbf{s^*=\alpha_1/\gamma_1}$ is the location for an extremum of the betatron amplitude function with $\alpha(s^*) = 0$.

2. Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

where *f* is the focal length of the quadrupole. Thus a thin-lens quadrupole gives rise to an angular kick to the betatron amplitude function without changing its magnitude.

$$X'' + K(s)X = 0$$

Since $$X(s) = a\sqrt{\beta(s)}\cos(\psi(s) + \psi_0) \quad with \quad \psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Thus $$X' = -\frac{X}{\beta}(\tan\psi - \frac{\beta'}{2})$$

$$\frac{1}{2\beta}[X^2 + (\beta X' + \alpha X)^2] = \frac{X^2}{2\beta}\sec^2\psi = \frac{a^2}{2} \equiv J$$

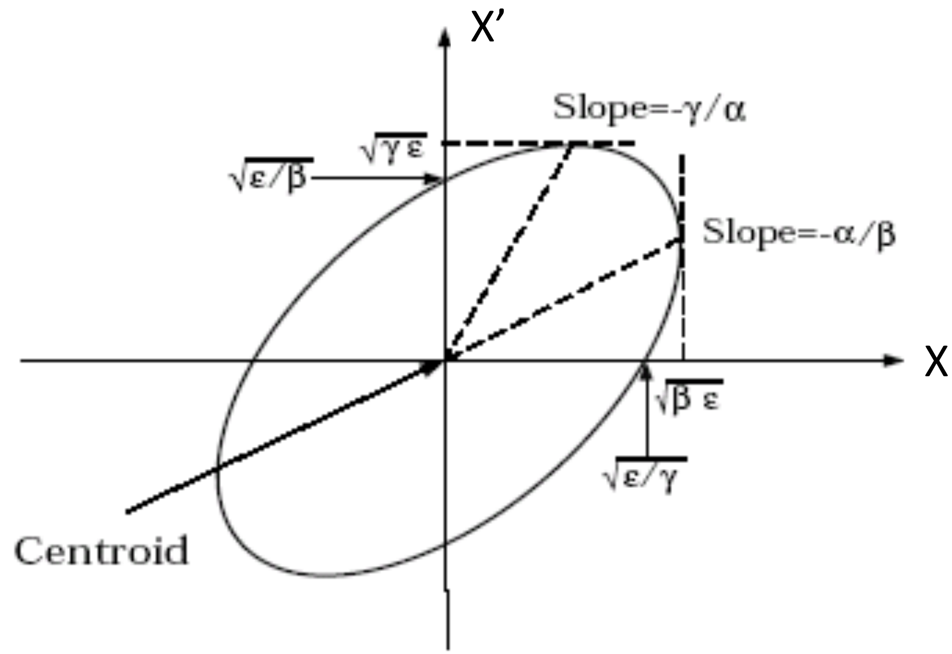
$$X = \sqrt{2\beta J}\cos\psi, \quad X' = -\sqrt{\frac{2J}{\beta}}(\sin\psi + \alpha\cos\psi)$$

Define: $$P_X = \beta X' + \alpha X = -\sqrt{2\beta J}\sin\psi$$

(X,P_X) form a normalized phase space coordinates with $X^2+P_X^2=2\beta J$, here J is called **action**.

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Questions:

1) When we have two particles with different action J_1 and $J_2 = 2J_1$, what will their ellipses look like?

2) Will the ellipses intersect with each other?

$$\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}_{n+1} = M_X \begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}_n \qquad \mathrm{M}_X = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}$$

The horizontal and vertical betatron ellipses for a particle with actions $J_x=J_y=0.5\pi$ mm-mrad at the end of the first dipole (left plots) and the end of the fourth dipole of the AGS lattice. The scale for the ordinate x or y is in mm, and that for the coordinate x' or y' is in mrad. Left plots: β_x=17.0 m, α_x=2.02, β_y=14.7 m, and α_y=−1.84. Right plots: β_x=21.7 m, α_x=−0.33, β_y=10.9 m, and α_y=0.29.