Analytical Solution of Fokker-Planck Equation

A tool for benchmarking numerical methods in predicting ion beam evolution under CeC

Intention

- An macro-particle tracking code is currently under development to predict ion beam evolution in the presence of CeC.
- To benchmark/validate the code, we need results from an independant approach, even if it has unrealistic assumptions.
- Solving Fokker-Planck equation provides such an approach.

1D Fokker-Planck Equation

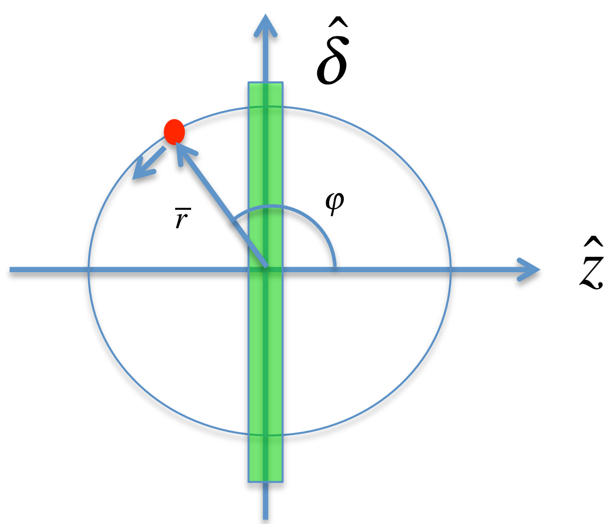
- After averaging over the synchrotron phase, the evolution of the ion bunch over the time scale of CeC local cooling time (3 seconds in PoP case) can be described by the following 1-D Fokker-Planck equaton:

$$\frac{\partial}{\partial t}F(I,t)-\frac{\partial}{\partial I}\left(\zeta(I)\cdot I\cdot F(I,t)\right)-\frac{\partial}{\partial I}\left(I\cdot D(I)\cdot\frac{\partial F(I,t)}{\partial I}\right)=0$$

$\zeta(I)$: cooling rate for an ion with action I;

$D(I)$: diffusion coefficient for an ion with action I.

$$\bar{r}=\sqrt{I}=\sqrt{\hat{z}^2+\hat{\delta}^2}$$



In the absence of diffusion

- In the case of $D(I)=0$

$$\frac{\partial}{\partial t}\tilde{F}(I,t)-\alpha(I)\frac{\partial}{\partial I}\tilde{F}(I,t)=0$$

$$\tilde{F}(I,t)\equiv\zeta(I)\cdot I\cdot F(I,t)$$

$$\alpha(I)\equiv I\cdot\zeta(I)$$

- Above equation is almost like a wave-equation , which has the following solutions

$$\tilde{F}(t,I)=G(C)$$

$$C=(t-t_0)+\int\frac{1}{\alpha(I)}dI$$

- The initial condition is imposed by

$$G\left(\int\frac{1}{\alpha(I)}dI\right)=\tilde{F}(t_0,I)$$

Solutions

- General solution:

$$\alpha(I) \equiv I \cdot \zeta(I)$$

$$\tilde{F}(t,I) = G(C) = \tilde{F}\left(t_0, h^{-1}(C)\right)$$

$$h(I) \equiv \int \frac{1}{\alpha(I)} dI$$

- For the following cooling rate profile:

$$\zeta(I) = \zeta_0 \frac{1}{1+\frac{I}{I_e}}$$

the solution reads

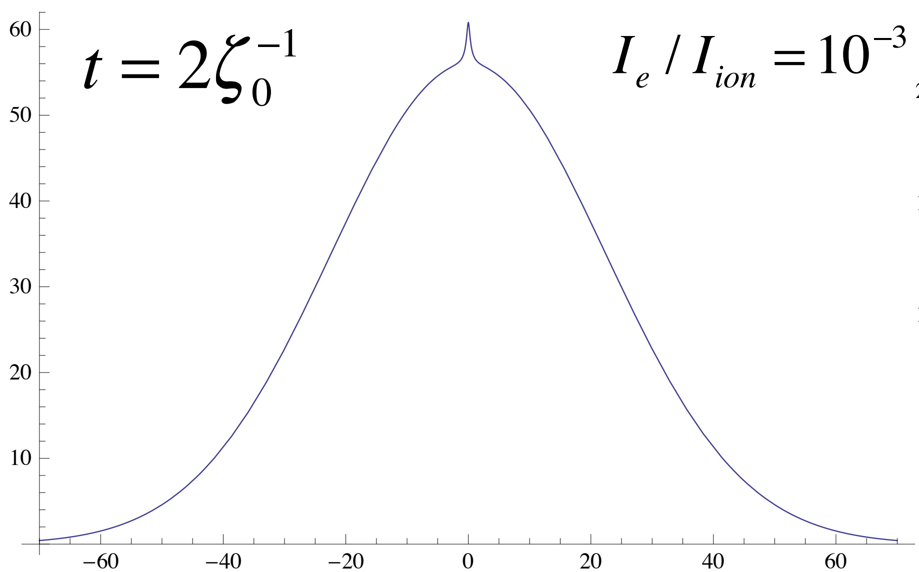
$$F(t,I) = \frac{I_e}{I} \cdot \frac{P_{\log}\left[\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right]\left(1+\frac{I}{I_e}\right)}{1+P_{\log}\left[\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right]} F_0\left(I_e \cdot P_{\log}\left[\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right]\right)$$

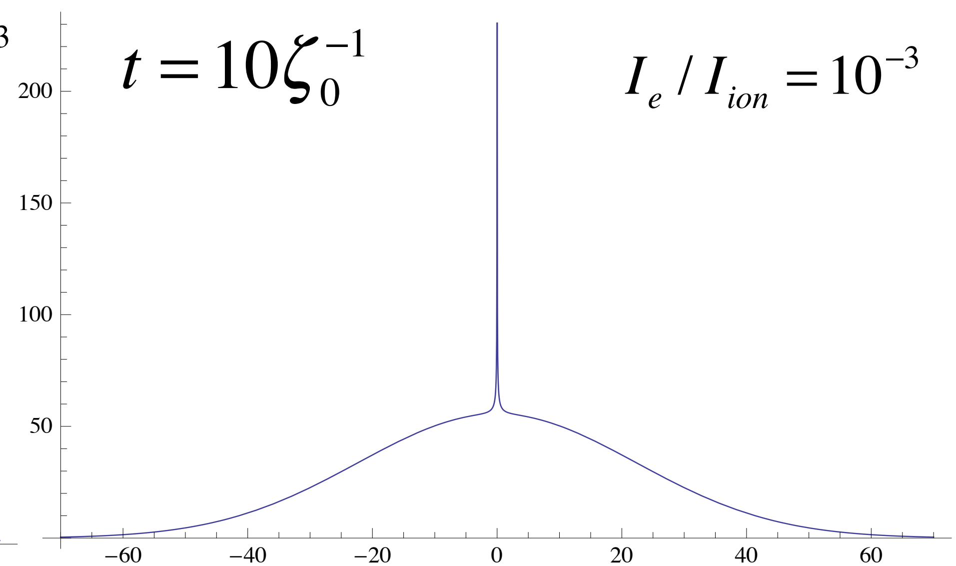
Product logarithm function: $P_{\log}(x) = w^{-1}(x) \qquad w(x) = xe^x$

Numerical Evaluation for Bunch Profile

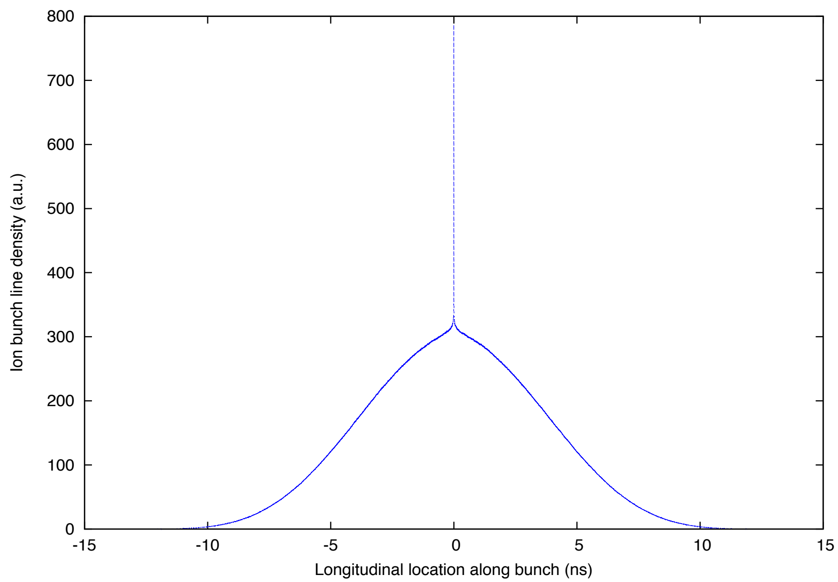
- Ion bunch line density is calculated from:

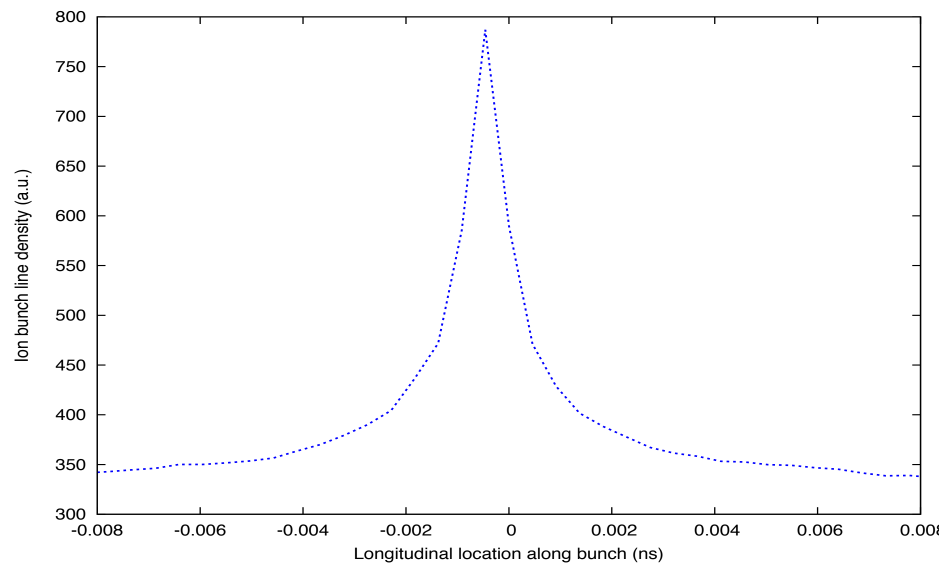
$$\rho_{ion}\left(t,\hat{z}\right)=\int_{-\infty}^{\infty}F\left(t,\hat{z}^2+\hat{\delta}^2\right)d\hat{\delta}$$





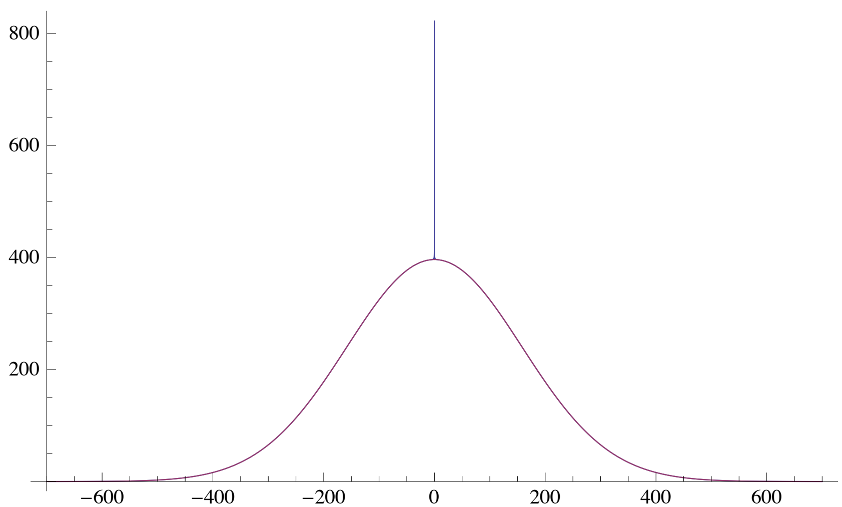
Comparison with Macro-particle Tracking (not a fair comparison yet)

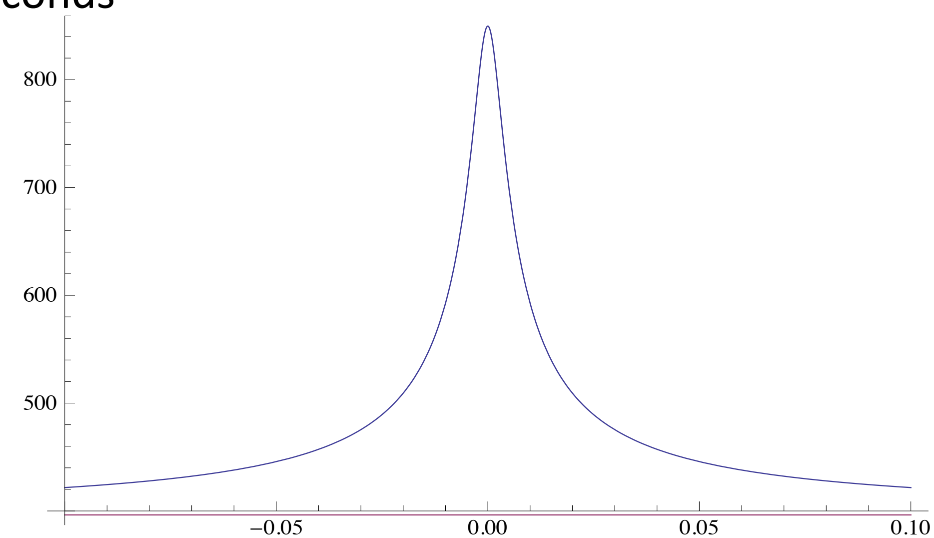




$$I_e / I_i = (15\,ps / 3.55\,ns)^2 \approx 2 \times 10^{-5}$$

After 30.7 seconds

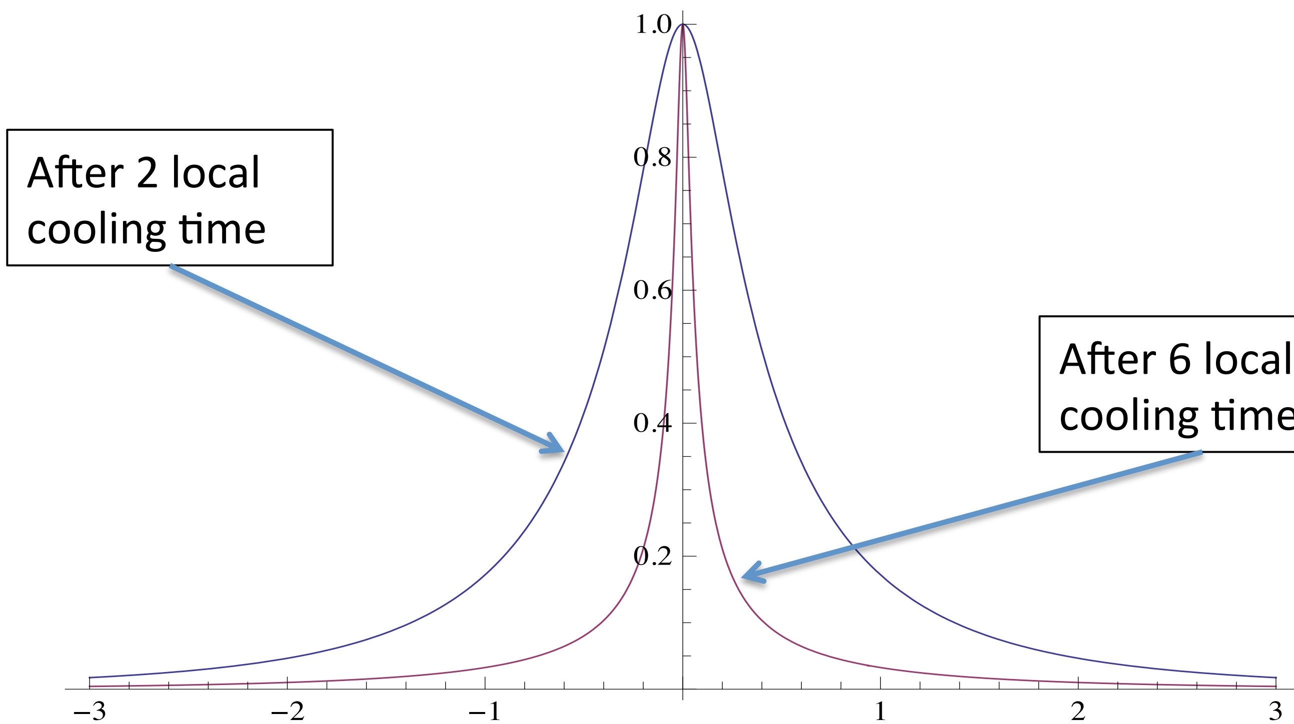




The Central Peak Gets Narrower with Time

$$\left[\rho_{ion}(t,z)-\rho_{ion}(0,z)\right]/\left[\rho_{ion}(t,0)-\rho_{ion}(0,0)\right]$$

Ion line density change normalized to central values



Longitudinal location / electron bunch length

Summary

- An analytical solution for 1D Fokker-Planck equation is found, which in principal can be solved for any CeC cooling rate profile. However, the solution involves getting inverse function, which can be practically difficult.
- For a specific cooling rate profile, an explicit expression is obtained, which can be easily evaluated by Mathematica.
- The specific solution shows qualitatively similar properties as observed from macro-particle tracking, i.e. the appearance of a delta-like peak after the ions being cooled for a few local cooling time.