Appendix F: Inhomogeneous solution

Even though calculations are tedious, they are also transparent and straightforward. General expression for the inhomogeneous equation of 2n ordinary linear differential equations is found by a standard trick of variable constants (method developed by Lagrange), i.e. assuming that $R = \mathbf{M}(s)A(s)$:

$$\begin{aligned}
&\frac{dR}{ds} = R' = \mathbf{D}\cdot R + \mathbf{C}; \quad \mathbf{M}' = \mathbf{D}\mathbf{M}; \\
&R = \mathbf{M}(s)A(s) \Rightarrow \mathbf{M}'A + \mathbf{M}A' = \mathbf{D}\mathbf{M}A + C \\
&R(0) = 0 \Rightarrow A_o = 0 \\
&A' = \mathbf{M}^{-1}(s)C \Rightarrow A = \int_0^s \mathbf{M}^{-1}(z)C dz = \left(\int_0^s e^{-\mathbf{D}z} dz\right)\cdot C
\end{aligned} \tag{F-1}$$

with well known result of:

$$R = e^{\mathbf{D}s}\left(\int_0^s e^{-\mathbf{D}z} dz\right)\cdot C\,. \tag{F-2}$$

or

$$R = M_{4x4}(s)\left(\int_0^s M^{-1}{}_{4x4}(z) dz\right)\cdot C\,. \tag{F-3}$$

If you use computer, eq. (F-3) is one to use. For analytical folks, you should go though a tedious job is combining all terms together into final form:

$$R(s) = \sum_{k=1}^{m}\left\{\prod_{i\neq k}\left[\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k-\lambda_i}\right]\sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j\right\}\sum_{n=0}^{n_k-1}(\mathbf{D}-\lambda_k\mathbf{I})^n\frac{s^n}{n!}\cdot\sum_{p=0}^{n_k-1}(-1)^{p+1}(\mathbf{D}-\lambda_k\mathbf{I})^p\cdot\mathbf{C}\cdot\left[\sum_{q=0}^{p1}\frac{s^{p-q}}{(p-q)!\,\lambda_k^{\;q+1}}-\frac{e^{\lambda_k}}{\lambda_k^{\;p+1}}\right] \tag{F-4}$$