Hadron Beam Cooling in Particle Accelerators
Day 1 - Tools

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Storage ring

There is large variety of accelerators, but in this course we will focus on storage rings which are usually used for cooling of the hadron beam. The reason for this is that all known cooling processes are very slow and required from thousands to millions of turns in storage ring for hadron beam to be cooled. Hence we will consider hadrons circulating in a storage ring – a nearly perfect periodic structure with practically zero energy losses: even for LHC operating at top energy of 7 TeV synchrotron radiation is so feeble that its effect can be seen only in abut 12 hours.

Hence, let's consider a periodic Hamiltonian for such storage ring with circumference C:

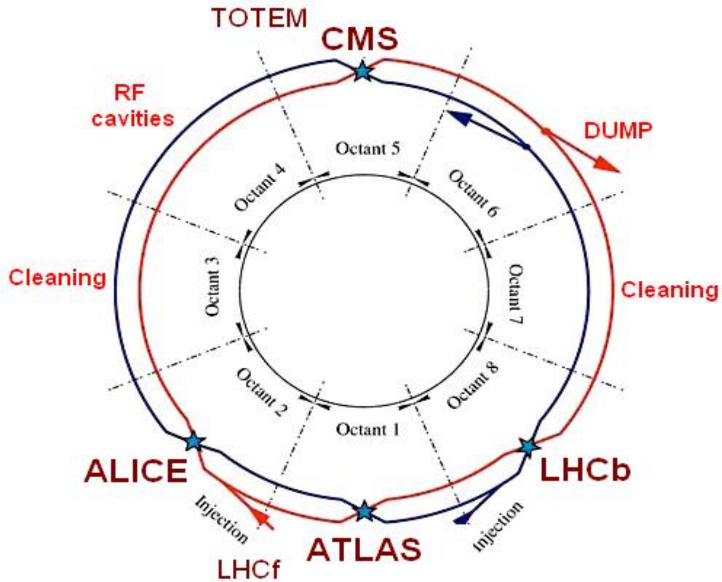
$$X^T = \begin{bmatrix} q^1 & P_1 & ..... & .... & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & ..... & .... & x_{2n-1} & x_{2n} \end{bmatrix};$$

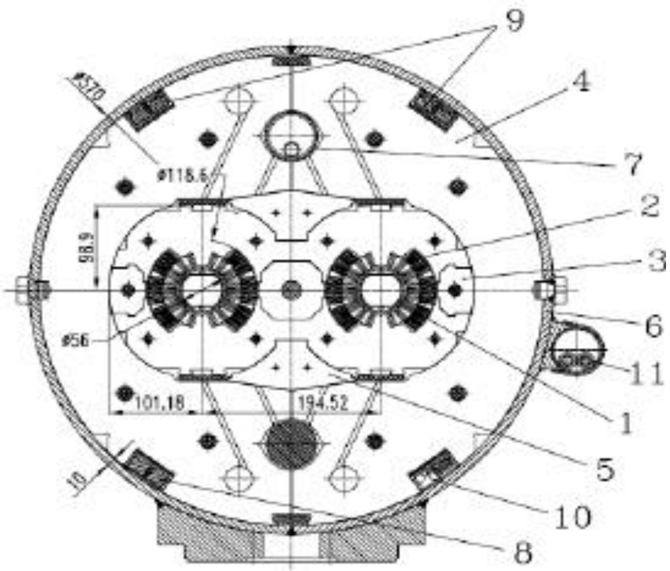
$$H(X,s) = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s) x_i x_j \equiv \frac{1}{2} X^T \cdot \mathbf{H}(s) \cdot X; \;\; \mathbf{H}(s+C) \equiv \mathbf{H}(s);$$

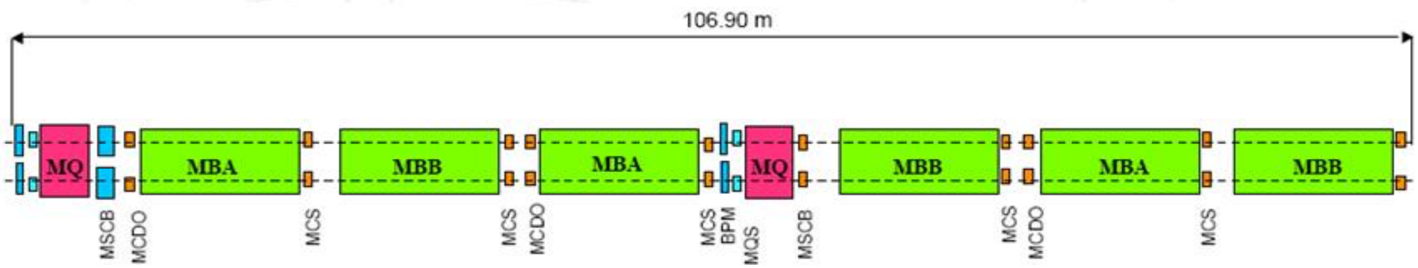
$$\mathbf{D}(s) = \mathbf{S}\cdot\mathbf{H}(s)\mathbf{S} = \begin{bmatrix} \sigma & 0 & ... & 0 \\ 0 & \sigma & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sigma \end{bmatrix}_{2n x 2n}; \;\; \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \rightarrow \mathbf{D}(s+C) = \mathbf{D}(s); \qquad \text{(M2.1)}$$

$$X(s) = \mathbf{M}(s_o|s)X(s_o); \;\; \mathbf{M}'(s_o|s) \equiv \frac{d\mathbf{M}(s_o|s)}{ds} = \mathbf{D}(s)\cdot\mathbf{M}(s_o|s);$$

which defines periodicity of the system. Naturally: $\mathbf{M}(s_o + C|s + C) \equiv \mathbf{M}(s_o|s)$.







The LHC is not a perfect circle. It is made of eight arcs and eight ‘insertions’. LHC consists of eight 2.45-km-long arcs, and eight 545-m-long straight sections. The arcs contain the dipole ‘bending’ magnets, with 154 in each arc. An insertion consists of a long straight section plus two (one at each end) transition regions — the so-called ‘dispersion suppressors’. The exact layout of the straight section depends on the specific use of the insertion: physics (beam collisions within an experiment), injection, beam dumping or beam cleaning. Each arc, with a regular lattice structure, contains 23 arc cells, and each arc cell has a FODO structure (main dipole magnets + quadrupole magnets + other multipoles magnets), 106.9 m long

https://www.lhc-closer.es/taking_a_closer_look_at_lhc/0.lhc_layout

Transport matrices

We will skip details of how matrices of individual elements are calculated – if you are interest in this, look though lectures 6-9 in our Advanced Accelerator Physics course: **[2]** http://case.physics.stonybrook.edu/index.php/PHY564_fall_2022

Just let us pay attention on how transport matrices are manipulated, multiplied, inverted:

$$
\begin{aligned}
&1.\ X(s)=\mathbf{M}\left(s_o|s\right)X(s_o)\rightarrow \mathbf{M}\left(s_o|s_o\right)=\mathbf{I};\\
&2.\ X(s_1)=\mathbf{M}\left(s_o|s_1\right)X(s_o);\ X(s_2)=\mathbf{M}\left(s_1|s_2\right)X(s_1)=\mathbf{M}\left(s_1|s_2\right)\mathbf{M}\left(s_o|s_1\right)X(s_o);\\
&\quad X(s_2)=\mathbf{M}\left(s_0|s_2\right)X(s_o)\rightarrow \mathbf{M}\left(s_o|s_2\right)\equiv\mathbf{M}\left(s_1|s_2\right)\mathbf{M}\left(s_o|s_1\right);\\
&3.\ \mathbf{M}\left(s_o|s_o\right)\equiv\mathbf{M}\left(s_o|s\right)\mathbf{M}\left(s|s_o\right)=\mathbf{I}\rightarrow\mathbf{M}\left(s|s_o\right)=\mathbf{M}^{-1}\left(s_o|s\right).
\end{aligned}
\quad ;\ (\text{M2.2})
$$

Specifically, transport matrices a multiplied from right to left in order of the appearances

$$\mathbf{M}_{tl}=\prod_{ordered\ n=1}^{N}\mathbf{M}_n\equiv\mathbf{M}_N\mathbf{M}_{N-1}....\mathbf{M}_2\mathbf{M}_1.$$

Matrices can describe motion forward (s_1->s_2: s_2> s_1) or backwards (s_1->s_2: s_2<s_1), which is - naturally – is an inverse matrix of that for motion forward.

Storage ring: stability

Since hadrons are circulating in storage rings for millions and billions of turns, it is very important that their motion is stable at each turn. Properties of one turn transport matrix are the key for investigating stability of the particle's motion:

$$\mathbf{T}(s) = \mathbf{M}\left(s\middle|s+C\right);\ \ \mathbf{T}(s+n\cdot C) \equiv \mathbf{T}(s),\ \ n = \text{int} \qquad \text{(M2.3)}$$

with it eigen values λ_i

$$\det[\mathbf{T} - \lambda_i \cdot \mathbf{I}] = 0 \qquad \text{(M2.4)}$$

determining stability of the system. System is unstable of some of $|\lambda_i| > 1$. Before making specific statements about the stability, we look at the properties of the eigen values.

First, eigen values are a function of periodic system and do not depend on the azimuth, s. It is easy to show that a one-turn matrix is transformed by the transport matrix as

$$\mathbf{T}(s_1) = \mathbf{M}\left(s\middle|s_1\right)\cdot\mathbf{T}(s)\cdot\mathbf{M}^{-1}\left(s\middle|s_1\right) \qquad \text{(M2.5)}$$

$$\mathbf{T}(s_1) = \mathbf{M}\left(s_1\middle|s_1+C\right) = \mathbf{M}\left(s+C\middle|s_1+C\right)\mathbf{M}\left(s_1\middle|s+C\right) = \mathbf{M}\left(s+C\middle|s_1+C\right)\mathbf{M}\left(s\middle|s+C\right)\mathbf{M}\left(s_1\middle|s\right)$$

$$\mathbf{M}\left(s+C\middle|s_1+C\right) \equiv \mathbf{M}\left(s\middle|s_1\right),\ \ \mathbf{M}\left(s_1\middle|s\right) \equiv \mathbf{M}^{-1}\left(s\middle|s_1\right) \Rightarrow \mathbf{T}(s_1) = \mathbf{M}\left(s\middle|s_1\right)\mathbf{T}(s)\mathbf{M}^{-1}\left(s\middle|s_1\right)\#$$

It means that $\mathbf{T}(s_1)$ has the same eigen values as $\mathbf{T}(s)$, i.e. eigne values are unique feature of the storage ring and do not depend upon choose of s for calculating one-turn matrix:

$$\begin{gathered}\det\left[\mathbf{MTM}^{-1} - \lambda_i\cdot\mathbf{I}\right] = \det\left[\mathbf{M}\left(\mathbf{T} - \lambda_i\cdot\mathbf{I}\right)\mathbf{M}^{-1}\right] = \det\left[\mathbf{T} - \lambda_i\cdot\mathbf{I}\right] \\ \Rightarrow \det\left[\mathbf{T}\left(s_1\right) - \lambda_i\cdot\mathbf{I}\right] = \det\left[\mathbf{T}\left(s\right) - \lambda_i\cdot\mathbf{I}\right] = 0\end{gathered} \qquad \text{(M2.6)}$$

The matrix $\mathbf{T}$ is a real, complex conjugate of eigen value λ_i^* which is also eigen value of $\mathbf{T}$

$$[\mathbf{T}-\lambda_i\cdot\mathbf{I}]^* = [\mathbf{T}-\lambda^*{}_i\cdot\mathbf{I}] = 0$$

Furthermore, the symplecticity of $\mathbf{T}$ requires that λ_i^{-1} also is eigen value of $\mathbf{T}$. Proving that the inverse matrix $\mathbf{T}^{-1}$ has λ_i^{-1} as a eigen value is easy.

$$\mathbf{T}Y_i = \lambda_i Y_i;\ \ \mathbf{T}^{-1}\mathbf{T} = \mathbf{I} \to \left(\mathbf{T}^{-1}\mathbf{T}\right)Y_i = \mathbf{I}Y_i = Y_i$$

$$\mathbf{T}^{-1}\mathbf{T}Y_i = \lambda_i\mathbf{T}^{-1}Y_i = Y_i \to \mathbf{T}^{-1}Y_i = \lambda_i^{-1}Y_i$$

At the same time

$$0 = \det\left[\mathbf{T}^{-1} - \lambda_i^{-1}\mathbf{I}\right] = \det\left(\mathbf{S}\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right]\mathbf{S}^{-1}\right) = \det\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right] = \det\left[\mathbf{T} - \lambda_i^{-1}\mathbf{I}\right]$$

and here, symplectic conditions help us again. Thus, the real symplectic matrix has n pairs of eigen values as follows: a) inverse $\{\lambda_i, \lambda_i^{-1}\}$, and b) complex conjugate $\{\lambda_i, \lambda_i^*\}$.

Let's assume that matrix $\mathbf{T}$ can be diagonalized. Therefore, repeating the matrix T again and again undoubtedly will cause an exponentially growing solution if $|\lambda_i| > 1$. The set of eigen vectors Y_i of matrix $\mathbf{T}$

$$\mathbf{T}\cdot Y_i = \lambda_i \cdot Y_i; \qquad i = 1,2....2n \tag{M2.7}$$

is complete and an arbitrary vector X can be expanded about this basis:

$$X = \sum_{i=1}^{2n} a_i Y_i \equiv \mathbf{U}\cdot A, \quad \mathbf{U} = \left[Y_1.......Y_{2n}\right], \qquad A^T = \left[a_1.......a_{2n}\right]. \tag{M2.8}$$

where we introduce matrix $\mathbf{U}$ built from eigen vector of the matrix $\mathbf{T}$:

$$\mathbf{T}\cdot\mathbf{U} = \mathbf{U}\cdot\Lambda, \ \ \Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & ... & .... \\ 0 & .... & \lambda_{2n} \end{bmatrix} \tag{M2.9}$$

Diagonalization of the matrix **T** gives:

$$\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U}=\Lambda,\ or\ \ \mathbf{T}=\mathbf{U}\cdot\Lambda\cdot\mathbf{U}^{-1} \tag{M2.10}$$

Multiple application of matrix **T** (i.e., passes around the ring)

$$\mathbf{T}^n\cdot X=\sum_{i=1}^{2n}\lambda_i^{\ n}a_iY_i$$

exhibit exponentially growing terms if the module of even one eigen value is larger than 1, $\lambda_k=|\lambda|e^{i\mu},\ |\lambda|>1$; we easily observe that a solution with the initial condition $X_o=\operatorname{Re}a_kY_k$ grows exponentially:

$$\mathbf{T}^nX_o=|\lambda|^n\operatorname{Re}a_kY_ke^{in\mu}.$$

Immediately this suggests that the only possible stable system is when all eigen values are uni-modular

$$|\lambda_i|=1. \tag{M2.11}$$

otherwise assuming $|\lambda_i|<1$ means that there is eigen value $\lambda_k=\lambda_i^{-1};\ |\lambda_k|=1/|\lambda_i|>1$.

In general case of multiplicity of eigen vectors, the matrix cannot be diagonalized but can be brought to Jordan normal form $\{Y_{k,1},...,Y_{k,h}\}$ that belong to a eigen value λ_k with multiplicity h:

$$\mathbf{T}\cdot Y_{k,h}=\lambda_kY_{k,h};\ \mathbf{T}\cdot Y_{k,m}=\lambda_kY_{k,m}+Y_{k,m+1};\ m=1...h-1.$$

The result is even stronger than in the diagonal case: motion is unstable even when $|\lambda_k|=1$:

$$\mathbf{T}\cdot Y_{k,h-1}=\lambda_kY_{k,h-1}+Y_{k,h}\Rightarrow\mathbf{T}^n\cdot Y_{k,h-1}=Y_{k,h-1}+n\cdot Y_{k,h}$$

There is no good reason to study exotic case of unstable periodic system, unless you are interested in blowing up the beam size and loose particles. Hence, let's focus on case of stable motion with 2N linearly independent eigen vectors. In other words, there are n pairs of eigen vectors, which determine modes of oscillations:

$$\lambda_k \equiv 1/\lambda_{k+1} \equiv \lambda^*{}_{k+1} \equiv e^{i\mu_k}; \;\; \mu_k \equiv 2\pi\nu_k, \; \{k=1,...n\}. \tag{M2.12}$$

where the complex conjugate pairs are identical to the inverse pairs.

Eq. (M2.10) can be rewritten as

$$\mathbf{T}(s) = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s); \; \Lambda = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_1^* & & 0 \\ & & ... & 0 \\ 0 & 0 & 0 & \lambda_n^* \end{pmatrix}; \;\; \mathbf{T}(s)\cdot\mathbf{U}(s) = \mathbf{U}(s)\cdot\Lambda \tag{M2.13}$$

and matrix **U** built from complex conjugate eigen vectors of **T**:

$$\mathbf{U}(s) = \left[Y_1, Y_1^* ... Y_n, Y_n^*\right]; \qquad \mathbf{T}(s)Y_k(s) = \lambda_k Y_k(s) \quad \Leftrightarrow \; \mathbf{T}(s)Y_k^*(s) = \lambda_k^* Y_k^*(s) \tag{M2.14}$$

Thus, eigen vectors can be transported from one azimuth to another by the transport matrix:

$$\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\tilde{Y}_k(s) \;\Leftrightarrow\; \frac{d}{ds}\tilde{Y}_k = \mathbf{D}(s)\cdot\tilde{Y}_k \tag{M2.15}$$

It is eigen vector of $\mathbf{T}(s_1)$ - just add (M2.5) to ((M2.15):

$$\mathbf{T}(s_1)\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1)\mathbf{M}(s|s_1)\tilde{Y}_k(s) = \mathbf{M}(s|s_1)\mathbf{T}(s)\tilde{Y}(s) = \lambda_k\mathbf{M}(s|s_1)\tilde{Y}(s) = \lambda_k\tilde{Y}_k(s_1)\#$$

Similarly,

$$\tilde{\mathbf{U}}(s_1) = \mathbf{M}(s|s_1)\tilde{\mathbf{U}}(s) \Leftrightarrow \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}(s)\cdot\tilde{\mathbf{U}} \tag{M2.16}$$

with the obvious follow-up by

$$\tilde{\mathbf{U}}(s+C) = \tilde{\mathbf{U}}(s)\cdot\Lambda,\ \tilde{Y}_k(s+C) = \lambda_k\tilde{Y}_k(s) = e^{i\mu_k}\tilde{Y}_k(s) \tag{M2.17}$$

The k^{th} eigen vectors are multiplied by $e^{i\mu_k}$ after each pass through the period. Hence, we can write

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)};\quad Y_k(s+C) = Y_k(s);\qquad \psi_k(s+C) = \psi_k(s) + \mu_k \tag{M2.18}$$

$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s)\cdot\Psi(s),\ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \tag{M2.19}$$

It is remarkable that the symplectic products (M2.12) of the eigen vectors are non-zero only for complex conjugate pairs: in other words, the structure of the Hamiltonian metrics is preserved here. $Y_k^T\cdot\mathbf{S}\cdot Y_k \equiv 0$ is obvious. Using only the symplecticity of $\mathbf{T}$ gives us desirable yields

$$Y_k^T\cdot\mathbf{S}\cdot Y_j = Y_k^T\cdot\mathbf{T}^T\mathbf{S}\mathbf{T}\cdot Y_j = \lambda_k\lambda_j\left(Y_k^T\cdot\mathbf{S}\cdot Y_j\right) \Rightarrow (1-\lambda_k\lambda_j)\left(Y_k^T\cdot\mathbf{S}\cdot Y_j\right) = 0$$

for $\lambda_k\lambda_j \neq 1$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_{j\neq k} = 0;\quad Y_k^T\cdot\mathbf{S}\cdot Y_j = 0;\ . \tag{M2.20}$$

and only the nonzero products for $\lambda_k = 1/\lambda_j = \lambda^*{}_j$ are clearly pure imaginary[1]:

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_k = 2i, \tag{M2.21}$$

where we chose the calibration of purely imaginary values as *2i* for the following expansion to be symplectic.

[1] $\left(A^{*T}\cdot\mathbf{S}\cdot A\right)^* = \left(A^T\cdot\mathbf{S}\cdot A^*\right) = -\left(A^{*T}\cdot\mathbf{S}\cdot A\right)^T = -\left(A^{*T}\cdot\mathbf{S}\cdot A\right)$

Eqs. (M2.20-M.2.21) in compact matrix form is

$$\mathbf{U}^T \cdot \mathbf{S} \cdot \mathbf{U} \equiv \tilde{\mathbf{U}}^T \cdot \mathbf{S} \cdot \tilde{\mathbf{U}} = -2i\mathbf{S},\ \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S} \cdot \mathbf{U}^T \cdot \mathbf{S}. \tag{M2.22}$$

The expressions for the transport matrices through β, α-functions, and phase advances often derived as a "miraculous" result, and hence called matrix gymnastics, is just a trivial consequence of our parametrization::

$$\mathbf{M}(s|s_1) = \tilde{\mathbf{U}}(s_1)\tilde{\mathbf{U}}^{-1}(s) = \frac{1}{2i}\tilde{\mathbf{U}}(s_1) \cdot \mathbf{S} \cdot \tilde{\mathbf{U}}^T(s) \cdot \mathbf{S} = \frac{1}{2i}\mathbf{U}(s_1) \cdot \mathbf{\Psi}(s_1) \cdot \mathbf{S} \cdot \mathbf{\Psi}^{-1}(s) \cdot \mathbf{U}^T(s_1)$$

with a specific case of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S}$$

S-orthogonality (M2.20) provides an excellent tool of finding complex coefficients in the expansion of an arbitrary solution X(s)

$$X_o = \sum_{i=1}^{2n} a_i Y_i \Rightarrow X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a_k{}^*\tilde{Y}_k{}^*\right) \equiv \mathrm{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv \frac{1}{2}\tilde{\mathbf{U}} \cdot A = \frac{1}{2}\mathbf{U} \cdot \mathbf{\Psi} \cdot A = \frac{1}{2}\mathbf{U} \cdot \tilde{A}. \tag{M2.23}$$

where $2n$ complex coefficients, which are constants of motion* can be found by a simple multiplication (instead of solving a system of 2n linear equations(M2.8))

$$a_i = \frac{1}{2i}\tilde{Y}_i^{*T}SX;\ \tilde{a}_i \equiv a_i e^{i\psi_i} = \frac{1}{2i}Y_i^{*T}SX;$$
$$A = 2\tilde{\mathbf{U}}^{-1} \cdot X = -i\mathbf{\Psi}^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X;\ \tilde{A} = \mathbf{\Psi}A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X. \tag{M2.24}$$

Equation (M2.23) is nothing else but a general parameterization of motion in the linear Hamiltonian system. It is very powerful tool and we will use this many times in this course.

* In matrix form using (M2.16) we have

$$X = \frac{1}{2}\tilde{\mathbf{U}}A,\ X' = \frac{1}{2}\left(\tilde{\mathbf{U}}'A + \tilde{\mathbf{U}}A'\right) = \mathbf{D}X = \frac{1}{2}\mathbf{D}\tilde{\mathbf{U}} \cdot A = \frac{1}{2}\tilde{\mathbf{U}}' \cdot A \Rightarrow A' = 0$$

1D system with a linear periodical Hamiltonian:

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}. \tag{M2.25}$$

The equations of motion are simple

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix}\ (i.e.\ x' \equiv p). \tag{M2.26}$$

A one-turn matrix within its determinant (*ad-bc=1*)

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix}; \tag{M2.27}$$

$$Y = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix}; \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi}; \mathbf{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ u + i/\mathrm{w} & u - i/\mathrm{w} \end{bmatrix}; \tilde{\mathbf{U}} = \mathbf{U} \cdot \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \tag{M2.28}$$

where w(s)* and u(s) are real functions and calibration was used for (21). **T** has a trace

$$Trace(\mathbf{T}) = Trace(\Lambda) = 2\cos\mu \tag{M2.29}$$

(because $Trace(ABA^{-1}) = Trace(B)$). Thus, the stability of motion - μ is real - is easy to check:

$$-2 < Trace(\mathbf{T}) < 2 \tag{M2.30}$$

where some well-known resonances are excluded: The integer $\mu = 2\pi m$, and the half-integer $\mu = 2(m+1)\pi$ as being unstable (troublesome!).

Y* by $e^{i\phi}$ to make ***a* real number. In other words we define the choice of our phase as* $\tilde{Y}(s) = \begin{pmatrix} \tilde{y}_1(s) \\ \tilde{y}_2(s) \end{pmatrix}$; $\mathrm{w}(s) = \left|\tilde{y}_1(s)\right|$; $\psi(s) = \arg\left(\tilde{y}_1(s)\right)$.

Combining (M2.28) into the equations of motion (M2.26)

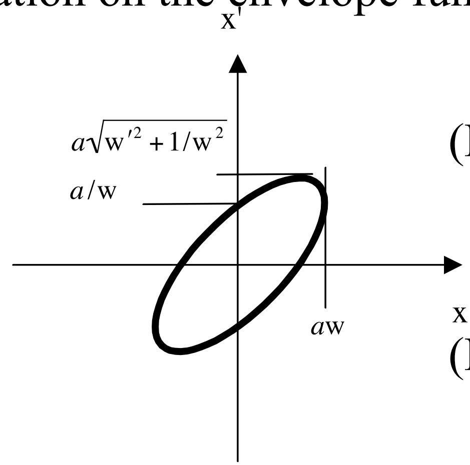
$$\frac{d}{ds}\begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi} \Rightarrow \begin{matrix} \mathrm{w}' + i\mathrm{w}\psi' = u + i/\mathrm{w} \\ u' - i\mathrm{w}'/\mathrm{w}^2 + i\psi'(u + i/\mathrm{w}) = -K_1\mathrm{w} \end{matrix} . \quad \text{(M2.31)}$$

Then, separating the real and imaginary parts, we have from the first equation:

$$u = \mathrm{w}'; \quad \psi' = \frac{1}{\mathrm{w}^2} . \quad \text{(M2.32)}$$

Plugging these into the second equation yields one nontrivial equation on the envelope function, w(s):

$$\mathrm{w}'' + K_1(\mathrm{s})\mathrm{w} = \frac{1}{\mathrm{w}^3} . \quad \text{(M2.33)}$$



Thus, the final form of the eigen vector can be rewritten as

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \ \psi' = \frac{1}{\mathrm{w}^2}; \ \tilde{Y} = Ye^{i\psi} \quad \text{(M2.34)}$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(ae^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$
$$x = a \cdot \mathrm{w}(\mathrm{s}) \cdot \cos(\psi(s) + \varphi)$$
$$x' = a \cdot \left(\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - \sin(\psi(s) + \varphi)/\mathrm{w}(\mathrm{s}) \right) \quad \text{(M2.35)}$$

where ***a*** and ***φ*** are the constants of motion.

Tradition in accelerator physics calls for using the so-called β-function, which simply a square of the envelope function:

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta. \tag{M.2.36}$$

and a wavelength of oscillations divided by 2π. Subservient functions are defined as

$$\alpha \equiv -\beta'/2 \equiv -\mathrm{w}\,\mathrm{w}',\ \gamma \equiv \frac{1+\alpha^2}{\beta}. \tag{M.2.37}$$

While α,β,γ are frequently used in accelerator physics, unless they are equipped with indices $\alpha_{x,y},\beta_{x,y},\gamma_{x,y}$, they can be easily mistaken with relativistic factors β and γ. Beware of this possibility and see in what contexts α,β,γ are used.

Manipulations with them is much less transparent, and oscillation (35) looks like

$$\begin{aligned} x &= a\cdot\sqrt{\beta(\mathrm{s})}\cdot\cos\big(\psi(s)+\varphi\big) \\ x' &= -\frac{a}{\sqrt{\beta(\mathrm{s})}}\cdot\Big(\alpha(\mathrm{s})\cdot\cos\big(\psi(s)+\varphi\big)+\sin\big(\psi(s)+\varphi\big)\Big) \end{aligned} \tag{M.2.38}$$

Finally, it gives us a well-known feature in AP parameterization of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;\quad \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \mathbf{J}^2 = -\mathbf{I} \tag{M.2.39}$$

Summary of 1D treatment

1D - ACCELERATOR

$$Y=\begin{bmatrix} w \\ w'+i/w \end{bmatrix};\ \tilde{Y}=\begin{bmatrix} w \\ w'+i/w \end{bmatrix}e^{i\psi};\ \mathbf{U}=\begin{bmatrix} w & w \\ w'+i/w & w'-i/w \end{bmatrix};\tilde{\mathbf{U}}=\mathbf{U}\cdot\begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

$$w''+K_1(s)w=\frac{1}{w^3},\quad \psi'=1/w^2;\ \begin{bmatrix} x \\ x' \end{bmatrix}=\mathrm{Re}\left(ae^{i\varphi}\begin{bmatrix} w \\ w'+i/w \end{bmatrix}e^{i\psi}\right);$$

$$x=a\cdot w(s)\cdot\cos(\psi(s)+\varphi)$$

$$x'=a\cdot(w'(s)\cdot\cos(\psi(s)+\varphi)-sin(\psi(s)+\varphi)/w(s))$$

$$\beta\equiv w^2\ \Rightarrow\ \psi'=1/\beta;\alpha\equiv-\beta'\equiv-w\,w',\ \gamma\equiv\frac{1+\alpha^2}{\beta}\ \text{- definitions}$$

$$x=a\cdot\sqrt{\beta(s)}\cdot\cos(\psi(s)+\varphi)$$

$$x'=-\frac{a}{\sqrt{\beta(s)}}\cdot(\alpha(s)\cdot\cos(\psi(s)+\varphi)+sin(\psi(s)+\varphi))$$

Complex amplitude and real amplitude and phase are easy to calculate. Expression for a^2 is called Currant-Snyder invariant.

$$X=\mathrm{Re}\,\tilde{a}Y;$$

$$ae^{i\varphi}=-iY^{*T}SX=\begin{bmatrix} w \\ w'-i/w \end{bmatrix}^T\cdot\begin{bmatrix} x' \\ -x \end{bmatrix}=x/w+i(w'x-wx')$$

$$a^2=\frac{x^2}{w^2}+(w'x-wx')^2\equiv\frac{x^2+(\alpha x+\beta x')^2}{\beta}\equiv\gamma x^2+2\alpha xx'+\beta x'^2$$

$$\varphi=\arg(x/w+i(w'x-wx'))=\tan^{-1}\frac{ww'x-w^2x'}{x}=-\tan^{-1}\frac{\alpha x+\beta x'}{x};$$

$$\varphi=\sin^{-1}\frac{w'x-wx'}{\sqrt{x^2+w^2(w'x-wx')^2}}=-\sin^{-1}\frac{\alpha x+\beta x'}{\sqrt{\gamma x^2+2\alpha xx'+\beta x'^2}}$$

Summary of 1D treatment

1D - ACCELERATOR

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix};\ \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix};\ \frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}\cdot\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix}\ (i.e.\ x' \equiv p).$$

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix};$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};\quad \mathbf{J}^2 = -\mathbf{I}$$

HENCE:

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};\quad \mathbf{J}^2 = -\mathbf{I};\quad \gamma = \left(1+\alpha^2\right)/\beta$$

$$\cos\mu = Trace(\mathbf{T})/2 = \frac{T_{11}+T_{22}}{2}$$

$$Stability\ if: -1 < Trace(\mathbf{T})/2 < 1$$

$$w^2 \equiv \beta = \frac{T_{12}}{\sin\mu} = \frac{|T_{12}|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}};\ w = \sqrt{\frac{|T_{12}|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}}};$$

$$ww' \equiv -\alpha = \frac{T_{22}-T_{11}}{2\cos\mu} = -\frac{T_{11}-T_{22}}{T_{11}+T_{22}}$$

$$M(s_1|s_2)=\frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}_2'+i/\mathrm{w}_2 & \mathrm{w}_2'-i/\mathrm{w}_2 \end{bmatrix}\cdot\begin{pmatrix} e^{i\Delta\psi} & 0 \\ 0 & e^{-i\Delta\psi} \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{bmatrix} \mathrm{w}_1 & \mathrm{w}_1'+i/\mathrm{w}_1 \\ \mathrm{w}_1 & \mathrm{w}_1'-i/\mathrm{w}_1 \end{bmatrix}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta\psi=\psi(s_2)-\psi(s_1);$$

$$M(s_1|s_2)=\frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}_2'+i/\mathrm{w}_2 & \mathrm{w}_2'-i/\mathrm{w}_2 \end{bmatrix}\cdot\begin{bmatrix} -(\mathrm{w}_1'-i/\mathrm{w}_1)e^{i\Delta\psi} & \mathrm{w}_1 e^{i\Delta\psi} \\ (\mathrm{w}_1'+i/\mathrm{w}_1)e^{-i\Delta\psi} & \text{-}\mathrm{w}_1 e^{-i\Delta\psi} \end{bmatrix}=$$

$$\begin{bmatrix} \mathrm{w}_2/\mathrm{w}_1\cos\Delta\psi-\mathrm{w}_2\mathrm{w}_1'\sin\Delta\psi & \mathrm{w}_1\mathrm{w}_2\sin\Delta\psi \\ (\mathrm{w}_2'/\mathrm{w}_1-\mathrm{w}_1'/\mathrm{w}_2)\cos\Delta\psi & \mathrm{w}_1/\mathrm{w}_2\cos\Delta\psi+\mathrm{w}_1\mathrm{w}_2'\sin\Delta\psi \\ -\left(\mathrm{w}_1'\mathrm{w}_2'+1/(\mathrm{w}_2\mathrm{w}_1)\right)\sin\Delta\psi & \end{bmatrix}$$

use $\mathrm{w}=\sqrt{\beta};\ \ \mathrm{w}'=\text{-}\alpha/\sqrt{\beta}$ to get a standard

$$M(s_1|s_2)=\begin{bmatrix} \dfrac{\cos\Delta\psi+\alpha_1\sin\Delta\psi}{\sqrt{\beta_1/\beta_2}} & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{(\alpha_2-\alpha_1)\cos\Delta\psi+(1+\alpha_1\alpha_2)\sin\Delta\psi}{\sqrt{\beta_1\beta_2}} & \dfrac{\cos\Delta\psi-\alpha_2\sin\Delta\psi}{\sqrt{\beta_2/\beta_1}} \end{bmatrix}$$

A little bit more complex is fully coupled 2D case. First, the equation for eigen values of symplectic matrix has symmetric coefficients:

$$\det\left[\mathbf{T}-\lambda\mathbf{I}\right]=c_4\lambda^4+c_3\lambda^3+c_2\lambda^2+c_1\lambda+c_0=0\Rightarrow$$
$$\lambda^4\det\left[\mathbf{T}-\lambda^{-1}\mathbf{I}\right]=\lambda^4\left(c_4\lambda^{-4}+c_3\lambda^{-3}+c_2\lambda^{-2}+c_1\lambda^{-1}+c_0\right)=0 \tag{M2.40}$$
$$\Rightarrow c_4+c_3\lambda+c_2\lambda^2+c_1\lambda^3+c_0\lambda^4=0$$
$$\Rightarrow c_0=c_4; c_3=c_1; c_0=1; c_3=-Trace[\mathbf{T}];$$

with only one coefficient c_2 remaining unknown.

$$T(s)=\begin{bmatrix} A & B \\ C & D \end{bmatrix};\ \det\left[T-\lambda I\right]=\lambda^4-\lambda^3 Tr\left[T\right]+\lambda^2 c_2-\lambda Tr\left[T\right]+1=0;$$

$$c_2=\underbrace{a_{11}\,a_{22}-a_{12}\,a_{21}}_{\det A}+\underbrace{d_{11}\,d_{22}-d_{12}\,d_{21}}_{\det D}-(\underbrace{b_{11}\,c_{11}\ b_{12}\,c_{21}+b_{21}\,c_{12}+b_{22}\,c_{22}}_{Tr[B\cdot C]}) \tag{M2.41}$$
$$+\underbrace{a_{11}\,d_{11}+a_{11}\,d_{22}+a_{22}\,d_{11}+a_{22}\,d_{22}}_{Tr[A]\cdot Tr[B]}$$

$$\mathbf{T}(s)=\begin{bmatrix} A & B \\ C & D \end{bmatrix};\ \det\left[\mathbf{T}-\lambda\mathbf{I}\right]=\lambda^4-\lambda^3 Tr\left[\mathbf{T}\right]+a\lambda^2-\lambda Tr\left[\mathbf{T}\right]+1=0$$
$$a=2\det A+Tr\left[A\right]\cdot Tr\left[D\right]-Tr\left[BC\right] \tag{M2.42}$$
$$(\mathit{note}\ \det A=\det D; \det C=\det B=1-\det A)$$

Finding roots:

$$d(\lambda)=\lambda^4-\lambda^3 Tr[\mathbf{T}]+a\lambda^2-\lambda Tr[\mathbf{T}]+1=(\lambda-\lambda_1)(\lambda-\lambda_1^{-1})(\lambda-\lambda_2)(\lambda-\lambda_2^{-1}); z_k=\lambda_k+\lambda_k^{-1}$$

$$d(\lambda)=(\lambda^2-\lambda z_1+1)(\lambda^2-\lambda z_2+1)=\lambda^4-\lambda^3(z_1+z_2)+\lambda^2(z_1z_2+2)-\lambda(z_1+z_2)+1$$

$$z_1+z_2=Tr[\mathbf{T}]=Tr[A+B]; z_1z_2=a-2\rightarrow(z-z_1)(z-z_2)=z^2-z(z_1+z_2)+z_1z_2=0$$

$$z^2-z\cdot Tr[A+B]+a-2=0\Rightarrow z_k=\frac{Tr[A+B]}{2}\pm\left\{\frac{Tr[A+B]^2}{4}+2-a\right\}^{1/2}; \qquad \text{(M.2.43)}$$

$$\frac{Tr[A+B]^2}{4}-Tr[A]\cdot Tr[D]+2(1-\det A)+Tr[BC]=\frac{Tr[A-B]^2}{4}+2\det C+Tr[BC];$$

$$z_k=2\cos\mu_k=\frac{Tr[A+B]}{2}\pm\sqrt{\frac{Tr[A-B]^2}{4}+2\det C+Tr[BC]}.$$

Stability conditions for coupled 2D motions is:

$$|\cos\mu_k|\le 1; k=1,2 \qquad \text{(M.2.44)}$$

And finally, the parameterization

$$X=\begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix}=\operatorname{Re}\tilde{a}_1Y_1+\operatorname{Re}\tilde{a}_1Y_2=\operatorname{Re}a_1\tilde{Y}_1+\operatorname{Re}\tilde{a}_1\tilde{Y}_2$$

$$Y_k=R_k+iQ_k;\quad \tilde{Y}_k=\begin{bmatrix} w_{kx}e^{i\psi_{kx}} \\ (u_{kx}+iv_{kx})e^{i\psi_{kx}} \\ w_{ky}e^{i\psi_{ky}} \\ (u_{ky}+iv_{ky})e^{i\psi_{ky}} \end{bmatrix};\quad \psi_{kx}(s+C)=\psi_{kx}(s)+\mu_k; \psi_{ky}(s+C)=\psi_{ky}(s)+\mu_k; \qquad \text{(M.2.45)}$$

$$w_{kx}v_{kx}+w_{ky}v_{ky}=1;$$

Conditions: there are

$$
\begin{aligned}
&Y_k^{*T} S Y_k = 2i; \quad Y_1^{*T} S Y_2 = 0; \; Y_1^T S Y_2 = 0; \quad \theta_k = \psi_{kx} - \psi_{ky} \\
&a) \; w_{1x} v_{1x} = w_{2y} v_{2y} = 1-q \quad \Rightarrow v_{1x} = \frac{1-q}{w_{1x}}; \;\; v_{2y} = \frac{1-q}{w_{2y}} \\
&b) \; w_{1y} v_{1y} = w_{2x} v_{2x} = q \quad \Rightarrow v_{2x} = \frac{q}{w_{2x}}; \;\; w_{1y} = \frac{q}{w_{1y}} \\
&c) \;\; c = w_{1x} w_{1y} \sin\theta_1 = -w_{2x} w_{2y} \sin\theta_2 \\
&d) \;\; d = w_{1x}\left(u_{1y}\sin\theta_1 - v_{1y}\cos\theta_1\right) = -w_{2x}\left(u_{2y}\sin\theta_2 - v_{2y}\cos\theta_2\right) \\
&e) \;\; e = w_{1y}\left(u_{1x}\sin\theta_1 + v_{1x}\cos\theta_1\right) = -w_{2y}\left(u_{2x}\sin\theta_2 + v_{2x}\cos\theta_2\right)
\end{aligned}
\tag{M.2.46}
$$

Conditions are result of symplecticity. Conditions a) and b) are equivalent to Poincaré's invariants conserving sum of projections on (x-px) and (y-py) planes.

$$
Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i\dfrac{q}{w_{1x}}\right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i\dfrac{1-q}{w_{1y}}\right) e^{i\varphi_{1y}} \end{bmatrix}; \quad
Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i\dfrac{1-q}{w_{2x}}\right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i\dfrac{q}{w_{2y}}\right) e^{i\varphi_{2y}} \end{bmatrix}
\tag{M.2.47}
$$

One should note for completeness that there is another way of parameterization of coupled motion proposed by Edward and Teng (*D.A.Edwards, L.C.Teng, IEEE Trans. Nucl. Sci.* ***NS-20*** *(1973) 885*), which differs from what we are discussing here.

Full 3D treatment.

Just for completeness, full 3D linearized motion requires 6x6 transport matrix of the storage ring (again, look at Lecture 11 in our Advanced Accelerator Physics course: **[2]** http://case.physics.stonybrook.edu/index.php/PHY564_fall_2022). We than can solve cubic equation on $\left(\lambda+\lambda^{-1}\right)$ of its eigen values

$$\det\left[T(s)-\lambda I\right]=0\rightarrow\left(\lambda+\lambda^{-1}\right)^{3}+b_{2}\left(\lambda+\lambda^{-1}\right)^{2}+b_{1}\left(\lambda+\lambda^{-1}\right)+b_{o}=0;$$

$$\mathbf{T}=\left[\begin{pmatrix} A & B & C\\ D & E & F\\ G & H & J\end{pmatrix}\right] \qquad \text{(M2-48)}$$

$$p_6(\lambda)=\prod_{k=1}^{3}\left(\lambda^{-1}{}_k-\lambda\right)\left(\lambda_k-\lambda\right)=0;\ \left(\lambda^{-1}{}_k-\lambda\right)\left(\frac{\lambda_k}{\lambda}\right)=-\left(\lambda_k-\lambda^{-1}\right);$$

$$\left(\lambda_k-\lambda^{-1}\right)\left(\lambda_k-\lambda\right)=\left(\lambda^{2}{}_k-\lambda_k\left(\lambda+\lambda^{-1}\right)+1\right)=-\lambda_k\left(\left(\lambda+\lambda^{-1}\right)-\left(\lambda_k+\lambda_k{}^{-1}\right)\right);$$

$$\frac{1}{\lambda^3}p_6(\lambda)=\prod_{k=1}^{3}\left(\left(\lambda+\lambda^{-1}\right)-\left(\lambda_k+\lambda_k{}^{-1}\right)\right)=\tilde{p}_3\left(\lambda+\lambda^{-1}\right)\rightarrow\tilde{p}_3\left(\lambda+\lambda^{-1}\right)=0.$$

$$\lambda^{6}+\alpha_5\lambda^{5}+\alpha_4\lambda^{4}+\alpha_3\lambda^{3}+\alpha_2\lambda^{2}+\alpha_1\lambda+1=0$$

$$\lambda^{-6}+\alpha_5\lambda^{-5}+\alpha_4\lambda^{-4}+\alpha_3\lambda^{-3}+\alpha_2\lambda^{-2}+\alpha_1\lambda^{-1}+1=0$$

$$1+\alpha_5\lambda+\alpha_4\lambda^{2}+\alpha_3\lambda^{3}+\alpha_2\lambda^{4}+\alpha_1\lambda^{5}+\lambda^{6}=0\Rightarrow$$

$$\alpha_1=\alpha_5=-Trace[\mathbf{T}];$$

$$\alpha_2=\alpha_4=\det A+\det E+\det J+Tr(A)\cdot Tr(E)+Tr(A)\cdot Tr(J)+Tr(J)\cdot Tr(E)$$
$$-Tr(BD)-Tr(CG)-Tr(FH)$$

....

and check that the 3D motion is stable

$$|\lambda_k| = 1;\ \lambda_k = e^{i\mu_k};\ \mu_k = 2\pi Q_k;\ k = 1,2,3 \qquad \text{(M2-49}$$

and define three eigen vectors and their complex conjugates:

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}}\right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right) e^{i\chi_{k\tau}} \end{bmatrix}; Y_k(s+C) = Y_k(s); T(s) Y_k(s) = e^{i\mu_k} Y_k(s); k = 1,2,3 \qquad \text{(M2-50)}$$

with the symplectic orthogonality relations that we already discussed:

$$Y_k^{\ T} S Y_j = 0;\quad Y^{*}{}_j^{\ T} S Y_k = 2i\delta_{kj}; \qquad \text{(M2-51)}$$

which will apply multiple (15 to be exact!) relations on the component of the eigen vectors, with the simples being:

$$q_{kx} + q_{ky} + q_{k\tau} = 1;\ k = 1,2,3 \qquad \text{(M2-52)}$$

Frequently there are a lot simpler cases, some of which we going to consider. It is also much better known in accelerator literature and papers. Hence, step back to 2D.

Accelerator with constant energy – closed orbit.

One of the most used approximations (and simplification) is coming from the fact that in the most of the accelerators (especially in storage rings) longitudinal (or so called synchrotron) oscillations are very slow, when compared with transverse (or so called betatron) oscillations. Specifically, in most of typical storage rings it takes from few hundreds to few thousands of turns to complete one synchrotron oscillation. Furthermore, in hadron storage rings, where losses on synchrotron are practically absent, one can operate beam in so-called coasting mode – e.g. without any AC (RF) fields. Thus, let's consider such an accelerator and study how particles motion depends on their energy (momentum p_o) and explicitly no time dependence.

$$\mathcal{H}_L = \frac{mc}{p_o}\cdot\frac{\pi_x^{\ 2}+\pi_y^{\ 2}}{2} + \left(\frac{mc}{p_o}\right)^3\cdot\frac{\pi_\tau^{\ 2}}{2} +$$
$$L\left(x\pi_y - y\pi_x\right) + g_x x\pi_\tau + g_y y\pi_\tau + \frac{F}{mc}\frac{x^2}{2} + \frac{N}{mc}xy + \frac{G}{mc}\frac{y^2}{2}; \qquad \text{(M2-52)}$$
$$\pi_x = \frac{P_1}{mc};\ \ \pi_y = \frac{P_3}{mc};\ \ \pi_\tau = \frac{\delta}{mc};$$

Since the energy of the particle is constant but time is slipping:

$$\frac{d}{ds}\pi_\tau = -\frac{\partial H}{\partial \tau} = 0 \rightarrow \pi_\tau = const;$$
$$\frac{d}{ds}\tau_\tau = \frac{\partial H}{\partial \pi_\tau} = g_x x + g_y y + \left(\frac{mc}{p_o}\right)^3\cdot\pi_\tau \qquad \text{(M2-53)}$$

we can simplify the equations of motion for 2D case plus energy dependence and time slippage:

$$\mathcal{H}_L = \mathcal{H}_\beta + \mathcal{H}_\delta;\ Z^T = \left(x, \pi_x, y, \pi_y\right);$$

$$\mathcal{H}_\beta = \frac{mc}{p_o}\cdot\frac{\pi_x{}^2 + \pi_y{}^2}{2} + \frac{F}{mc}\frac{x^2}{2} + \frac{N}{mc}xy + \frac{G}{mc}\frac{y^2}{2} + L\left(x\pi_y - y\pi_x\right);$$

$$\mathcal{H}_\delta = \left(\frac{mc}{p_o}\right)^3\cdot\frac{\pi_\tau{}^2}{2} + g_x x\pi_\tau + g_y y\pi_\tau; \tag{M2-54}$$

$$\frac{d}{ds}Z = D_\beta\cdot Z + \pi_\tau\cdot F_\delta;\ \ \pi_\tau\cdot C = S\frac{\partial}{\partial Z}H_\delta;\ F_\delta{}^T = C\cdot\begin{bmatrix} 0 & -g_x & 0 & -g_y \end{bmatrix};$$

or in explicit matrix form:

$$\frac{dZ}{ds} = D\cdot Z + \pi_\tau\cdot C;\quad D = \begin{bmatrix} 0 & 1 & -L & 0 \\ -f & 0 & -n & -L \\ L & 0 & 0 & 1 \\ -n & L & -g & 0 \end{bmatrix}; C = \begin{bmatrix} 0 \\ -g_x \\ 0 \\ -g_y \end{bmatrix}. \tag{M2-55}$$

$$\frac{d\tau}{ds} = g_x x + g_y y + \left(\frac{mc}{p_o}\right)^3\pi_\tau;\ \ g_x = \left(\frac{mc}{p_o}\right)^2\frac{eE_x}{p_o c} - \frac{mc^2}{p_o \mathrm{v}_o}K;\ g_y = \left(\frac{mc}{p_o}\right)^2\frac{eE_y}{p_o c}$$

We shall note that for ultra-relativistic particles (or in the absence of the electric fields!) only the curvature K of the trajectory remains as the driving term g_x for transverse motion. Solution for of the in-homogeneous equation for Z can be trivially expressed using 4x4 transport matrix:

$$Z = Z_\beta + \pi_\tau \cdot R;\ \frac{dZ_\beta}{ds} = D \cdot Z_\beta; \frac{dR}{ds} = D \cdot R + C;$$

$$\mathbf{M} \equiv \mathbf{M}_{4x4}; \frac{d\mathbf{M}(s)}{ds} = D \cdot \mathbf{M}(s);\quad Z_\beta(s) = \mathbf{M}(s) Z_{\beta o};$$

$$R(s) = \mathbf{M}(s) A(s);\quad \mathbf{M}(s)\frac{dA}{ds} = \pi_\tau \cdot C \Rightarrow \frac{dA}{ds} = \mathbf{M}^{-1}(s) C(s); \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S}; \tag{M2-56}$$

$$\Rightarrow A(s) = \int_o^s \mathbf{M}^{-1}(\xi) C(\xi) d\xi;\ R(s) = \mathbf{M}(s)\left(A_o + \int_o^s \mathbf{M}^{-1}(\xi) C(\xi) d\xi \right).$$

$$R(s) = \int_o^s \mathbf{M}(\xi|s) C(\xi) d\xi; \eta$$

For periodic system we can find "periodic transverse orbit" for an off-momentum particle:

$$\eta(s+C) = \int_o^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi = \mathbf{T}(s)\eta(s) + \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi$$

$$\mathbf{M}(\xi|s+C) = \mathbf{T}(s)\mathbf{M}(\xi|s); \mathbf{T}(s) \equiv \mathbf{M}(s|s+C); \tag{M2-57}$$

$$\eta(s+C) = \eta(s) \Rightarrow (\mathbf{I} - \mathbf{T})\eta(s) = \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi;$$

$$\eta(s) = (\mathbf{I} - \mathbf{T}(s))^{-1} \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi;\quad Z = Z_\beta + \pi_\tau \cdot \eta(s).$$

We will find expression for such closed periodical orbit expressed via eigen vectors – naturally the results would be identical. The $\eta = \begin{bmatrix} \eta_x & \eta_{px} & \eta_y & \eta_{py} \end{bmatrix}$ - function is called transverse dispersion (picking analogy from optics). Unfortunately in accelerator physics terminology there is a number of confusions… and frequently the dispersion is represented by $D = \begin{bmatrix} D_x & D_{px} & D_y & D_{py} \end{bmatrix}$. Read the context to be sure…

Next natural step is to look onto the slippage of the particle in time for a particle without betatron oscillations $Z_\beta = 0$ (we will add them later):

$$Z = \pi_\tau \cdot \eta(s); \frac{d\tau}{ds} = \left(g_x \eta_x + g_y \eta_y + \left(\frac{mc}{p_o} \right)^3 \right) \pi_\tau;$$

$$\tau(s) = f_\tau(s)\pi_\tau; \quad f_\tau(s) = f_\tau(0) + \left(\frac{mc}{p_o} \right)^3 \cdot s + \int_0^s \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi) \right) d\xi. \tag{M2-58}$$

First (red) term corresponds to the velocity dependence on the particles energy – it is weak for ultra-relativistic particles moving very-very close to the speed of the light, but it is important for hadron accelerators. Hence, we will keep it. Again, for a periodic system we

$$f_\tau(s+C) = f_\tau(s) + \left(\frac{mc}{p_o} \right)^3 \cdot C + \int_s^{s+C} \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi) \right) d\xi = \eta_\tau \cdot C;$$

$$\eta_\tau = \frac{1}{C} \int_0^C \left(g_x \eta_x + g_y \eta_y \right) ds + \left(\frac{mc}{p_o} \right)^3; \tag{M2-59}$$

It worth expressing it for a simple case when electric field is zero

$$\eta_\delta = \frac{p_o}{mc}\eta_\tau = \left(\frac{mc}{p_o}\right)^2 - \frac{1}{C}\frac{c}{\mathrm{v}_o}\int_0^C K(s)\eta_x(s)ds = \frac{1}{\beta_o{}^2\gamma_o{}^2} - \frac{1}{\beta_o}\langle K\eta_x\rangle \tag{M2-60}$$

e.g. the dependence of the travel time around the storage ring on particles momentum as two components: one corresponds to increase in velocity (kinematic) and the other (geometrical) to - typically - elongation of the trajectory in bending magnets – particles with higher energy travel at larger radius. In general, η_τ can be either negative or positive. When two terms cancel each other, travel time around storage ring is energy independent – this energy is called critical. If the geometrical term $\langle g_x\eta_x + g_y\eta_y\rangle$ in the accelerator is positive, the accelerator does not have critical energy. Such conditions do not come naturally and require a special, so call negative compaction factor lattice:

$$\alpha_c = \langle g_x\eta_x + g_y\eta_y\rangle > 0 .$$

Now, let's find the full set of eigen vectors for 3D motion using $\mathbf{T_{6x6}}$ one turn transport matrix.

Let's start from obvious eigen vector:

$$Y_\tau = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; T_{6x6} Y_\tau = Y_\tau; \lambda_\tau = 1. \qquad \text{(M2-61)}$$

nothing depends on the time shift! A particle following the reference particle with some time delay follow the same trajectory but with the given time delay. Next eigen vector is not a simple vector but a root vector:

$$Y_\delta = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix}; \quad T_{6x6} Y_\delta = Y_\delta + \eta_\tau Y_\tau; \quad \lambda_\delta = 1. \qquad \text{(M2-62)}$$

Note, this is clearly degenerated case when matrix $\mathbf{T_{6x6}}$ can not be diagonalized and we have to use root vectors, but the symplectic product

$$Y_\tau{}^T S Y_\delta = 1 \qquad \text{(M2-63)}$$

is well behaving.

What it left is to define the structure of 6-component betatron eigen vectors. Again, since energy is constant, it does not depend on the transverse motion, e.g. the corresponding element is simply zero:

$$Y_k = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} \qquad \text{(M2-64)}$$

which is generally not true for the time component. While it can be calculated directly and after long manipulations brought to the form we derive easily using symplectic orthogonality of eigen vector of symplectic matrix T:

$$\begin{gathered} Y_i^T \left(T^T S T \right) Y_k = Y_i^T S Y_k \to \lambda_i \lambda_k \cdot Y_i^T S Y_k = Y_i^T S Y_k; \\ \left(\lambda_i \lambda_k - 1 \right) Y_i^T S Y_k = 0. \end{gathered} . \qquad \text{(M2-65)}$$

With root vectors is just a bit different, but still trivial. Note that betatron eigen vectors is a regular are symplectic-orthogonal to Y_τ is a regular eigen vector with eigen value of 1 and, naturally,

$$Y_\tau^T S Y_k = 0; \; k = 1,2. \qquad \text{(M2-66)}$$

Note, that this is also requirement is equivalent to requirement that 6th component of betatron eigen vectors Y_k is equal zero. It takes one extra step to prove that for root eigen vector Y_δ:

$$\begin{gathered} Y_k^{\ T}\left(T^T S T\right) Y_\delta = Y_k^{\ T} S Y_\delta;\ \ T Y_\delta = Y_\delta + \eta_\tau Y_\tau \\ \lambda_k \cdot \left(Y_k^{\ T} S Y_\delta + \eta_\tau Y_k^{\ T} S Y_\tau\right) = Y_k^{\ T} S Y_\delta;\ \ Y_k^{\ T} S Y_\tau = 0;. \\ \left(\lambda_k - 1\right) Y_k^{\ T} S Y_\delta = 0 \rightarrow Y_k^{\ T} S Y_\delta = 0. \end{gathered} \tag{M2-67}$$

This gives us automatically explicit expression for 5th component of the betatron eigen vectors:

$$Y_k^{\ T} S Y_\delta = 0 \rightarrow \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix}^T \begin{bmatrix} S_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = Y_{k\beta}^{\ T} S \eta + \mathrm{y}_{k\tau} = 0;$$

$$\mathrm{y}_{k\tau} = -Y_{k\beta}^{\ T} S \eta = \eta^T S Y_{k\beta} = \qquad . \tag{M2-68}$$

$$\left(\eta_x\left(\mathrm{v}_{kx} + \frac{iq_k}{\mathrm{w}_{kx}}\right) - \eta_{px}\mathrm{w}_{kx}\right)e^{i\chi_{kx}} + \left(\eta_y\left(\mathrm{v}_{ky} + \frac{i(1-q_k)}{\mathrm{w}_{ky}}\right) - \eta_{py}\mathrm{w}_{ky}\right)e^{i\chi_{ky}}.$$

This equation makes explicit the dependence of the arrival time on amplitudes and phases of betatron oscillation. In locations where dispersion is zero (called achromatic), this dependence vanishes

$$\eta = 0 \Leftrightarrow \mathrm{y}_{k\tau} = 0. \tag{M2-69}$$

Important note for people who dealing with dependence of particle's arrival time of particles on transverse (betatron) motion – a term important for optical stochastic and coherent electron cooling. Let's consider a transfer line where particles propagate with constant energy. One can try simply to integrate equation for arrival time, but because we are using Hamiltonian system, there is a simpler and much more convenient way of doing it using symplectic conditions, i.e. that symplectic product of two solutions is an invariant of motion"

$$\frac{d}{ds}X_{1,2}(s) = \mathbf{SH}(s)\cdot X_{1,2}(s);\ X_1^T(s)\mathbf{S}X_2(s) = X_1^T(s_o)\mathbf{S}X_2(s_o) = inv,$$

which is easy to prove

$$\begin{gathered}\left(X_1^T\mathbf{S}X_2\right)' = X_1^T\mathbf{S}X_2' + X_1^{T\prime}\mathbf{S}X_2 = X_1^T(\mathbf{SSH})X_2 + X_1^T(\mathbf{SH})^T\mathbf{S}X_2 \\ \mathbf{SS} = -\mathbf{I};\ (\mathbf{SH})^T = -\mathbf{HS} \to \left(X_1^T\mathbf{S}X_2\right)' = -X_1^T\mathbf{H}X_2 + X_1^T\mathbf{HS}X_2 = 0\,\#\end{gathered},$$

Now, let's select two initial conditions:

$$X_1(s_o) = \begin{bmatrix} X_\beta(s_o) \\ 0 \\ 0 \end{bmatrix};X_\beta(s_o) = \begin{bmatrix} x_o \\ \pi_{xo} \\ y_o \\ \pi_{y0} \end{bmatrix};X_2(s_o) = \begin{bmatrix} \eta(s_o) \\ 0 \\ 1 \end{bmatrix};$$

$$X_1^T(s_o)\mathbf{S}_{6x6}X_2(s_o) = X_\beta(s_o)^T\mathbf{S}_{4x4}\eta(s_o);$$

which gives us explicit expression for change in arriving time caused by betatron oscillations:

$$X_1(s)=\begin{bmatrix} X_\beta(s) \\ \Delta\tau(s) \\ 0 \end{bmatrix}; X_2(s)=\begin{bmatrix} \eta(s) \\ 0 \\ 1 \end{bmatrix};$$

$$X_1^T\mathbf{S}_{6x6}X_2 = X_\beta^T\mathbf{S}_{4x4}\eta + \begin{bmatrix} \Delta\tau(s) & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = X_\beta(s)^T\mathbf{S}_{4x4}\eta(s)+\Delta\tau(s);$$

$$\Delta\tau(s)=X_\beta(s_o)^T\mathbf{S}_{4x4}\eta(s_o)-X_\beta(s)^T\mathbf{S}_{4x4}\eta(s).$$

What is most interesting, that in absence of transverse field (i.e. curvature), this $\Delta\tau(s)$ is zero. It is relatively easy to show by remembering that $\eta(s)$ is solution of inhomogeneous equation (M2-55):

$$\frac{d\eta}{ds}=\mathbf{D}_{4x4}\cdot\eta + C;\ ;C^T=\left[0,\ \left(\frac{mc}{p_o}\right)^2\frac{eE_x}{p_oc}-\frac{mc^2}{p_o\mathrm{v}_o}K,\ 0,\ \left(\frac{mc}{p_o}\right)^2\frac{eE_y}{p_oc}\right];$$

while X_β is solution of homogeneous equation

$$\frac{dX_\beta}{ds}=\mathbf{D}_{4x4}\cdot X_\beta;\mathbf{D}_{4x4}=\mathbf{S}_{4x4}\mathbf{H}_{4x4};$$

Hence

$$\left(X_\beta^T\mathbf{S}_{4x4}\eta\right)' = X_\beta^{T\prime}\mathbf{S}_{4x4}\eta + X_\beta^T\mathbf{S}_{4x4}\eta' = X_\beta^T\mathbf{S}_{4x4}C$$

is zero in the abcense of transverse fields when *C=0*. In other words, one need to bend beam's trajectory to couple travel time with transverse positions and angles.

Separating betatron oscillations. Note that in accelerator jargon we call fast transverse motions (e.g. not related to energy of particles) “betatron” oscillations. Naming is historical and related to oscillations studied in one of the accelerator types – betatron. Let’s formally separate energy dependent motion from transverse “betatron” oscillations using a Canonical transformation:

$$\tilde{H}\left(X_\beta\right) = H\left(\tilde{X} + X_\delta\right) - \frac{\partial F}{\partial s} = H\left(\tilde{X} + X_\delta\right) + \\ +\eta_x'\tilde{\pi}_\tau\left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) - \eta_{px}'\tilde{\pi}_\tau\tilde{x} + \eta_y'\tilde{\pi}_\tau\left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) - \eta_{py}'\tilde{\pi}_\tau\tilde{y} \qquad \text{(M2-70)}$$

while we can prove that matrix of such transformation is symplectic, it is also very easy to do using a generation function noticing that $\tilde{\pi}_\tau = \pi_\tau$ is not changing during the transformation

$$F\left(q,\tilde{P}\right) = \left(x - \eta_x\tilde{\pi}_\tau\right)\left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) + \left(y - \eta_y\tilde{\pi}_\tau\right)\left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) \\ +\tau\tilde{\pi}_\tau - \left(\eta_x\eta_{px} + \eta_y\eta_{py}\right)\frac{\tilde{\pi}_\tau^{\ 2}}{2}$$

$$\pi_\tau = \frac{\partial F}{\partial \tau} = \tilde{\pi}_\tau; \;\; \tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x\tilde{\pi}_x + \eta_{px}\tilde{x} - \eta_y\tilde{\pi}_y + \eta_{py}\tilde{y}; \qquad \text{(M2-71)}$$

$$x_\beta = \tilde{x} = \frac{\partial F}{\partial \tilde{\pi}_x} = x - \eta_x\tilde{\pi}_\tau; y_\beta = \tilde{y} = \frac{\partial F}{\partial \tilde{\pi}_y} = y - \eta_y\tilde{\pi}_\tau;$$

$$\pi_x = \frac{\partial F}{\partial x} = \tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau; \;\; \pi_y = \frac{\partial F}{\partial y} = \tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau.$$

We should note that transformation (M2-71) is linear:

$$\tilde{X} == \mathbf{L}\cdot X = \mathbf{I}\cdot X - \pi_\tau \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ -\eta_x\left(\pi_x - \eta_{px}\pi_\tau\right) + \eta_{px}\left(x - \eta_x\pi_x\right) - \eta_y\left(\pi_y - \eta_{py}\pi_\tau\right) + \eta_{py}\left(y - \eta_y\pi_\tau\right) \\ 0 \end{bmatrix} \tag{M2-72}$$

with matrix L naturally being symplectic – you can directly check that it is correct. We can follow a direct way of finding form of Hamiltonian in the new variables (e.g. transforming coefficients in Hamiltonian (M2-70) and adding *s*-derivative of the transfer function), but since the transformation is simply linear we can use a short-cut rewriting (M2-72) as

$$\begin{gathered} \tilde{X} = \begin{bmatrix} \tilde{Z} \\ \tilde{\tau} \\ \tilde{\pi}_\tau \end{bmatrix} = X - \begin{bmatrix} \pi_\tau\eta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} O \\ \eta^T S\tilde{Z} \\ 0 \end{bmatrix}; X = \begin{bmatrix} Z \\ \tau \\ \pi_\tau \end{bmatrix}; \tilde{Z} = Z - \pi_\tau\eta; \\ X = \tilde{X} + \begin{bmatrix} \tilde{\pi}_\tau\eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ \eta^T S\tilde{Z} \\ 0 \end{bmatrix}; \end{gathered} \tag{M2-73}$$

where we used that $\eta^T S\eta = 0$.

Now it is relatively straight-forward to write equations of motion is new variables:

$$\frac{dX}{ds}=\mathbf{D}X;\frac{d\tilde{X}}{ds}=\tilde{\mathbf{D}}\tilde{X};\mathbf{D}=\begin{bmatrix} D_{4\times4} & 0 \quad C \\ .... & .... \end{bmatrix};\frac{dZ}{ds}=D_{4\times4}Z+C\pi_\tau;\eta'=D_{4\times4}\eta+C;$$

$$\frac{d\tilde{X}}{ds}=\frac{dX}{ds}-\pi_\tau\begin{bmatrix}\eta'\\0\\0\end{bmatrix}-\begin{bmatrix}O\\ \frac{d}{ds}\left(\eta^T S\tilde{Z}\right)\\0\end{bmatrix}= \tag{M2-74}$$

$$\begin{bmatrix} Z'-\pi_\tau\eta' \\ \frac{d}{ds}\left(\tau-\eta^T S\tilde{Z}\right) \\ 0 \end{bmatrix}=\begin{bmatrix} D_{4x4}\left(Z-\pi_\tau\eta\right) \\ \frac{d}{ds}\tilde{\tau} \\ 0 \end{bmatrix}=\begin{bmatrix} D_{4x4}\tilde{Z} \\ \tilde{\tau}' \\ 0 \end{bmatrix}.$$

It means that the transverse part of the Hamiltonian remain the same since $D_{4x4}=S_{4x4}\cdot H_{4x4}$

$$\frac{d\tilde{Z}}{ds}=D_{4\times4}\tilde{Z};\quad \tilde{D}_{4\times4}\equiv D_{4\times4}; \tag{M2-75}$$

but the C components, as expected, vanish. It means that in new Hamiltonian mixed components between longitudinal $\left\{\tilde{\tau},\tilde{\pi}_\tau\right\}$ and $\tilde{Z}^T=\left\{x_\beta,\pi_{x\beta},y_\beta,\pi_{y\beta}\right\}$ disappear. A non-zero component of type $\left(a\tilde{\tau}+b\tilde{\pi}_\tau\right)\tilde{z}_i$ in the new Hamiltonian will generate non-zero additional component in (M2-75):

$$\frac{d\tilde{z}_k}{ds}=S_{ki}\frac{\partial\tilde{H}}{\partial\tilde{z}_i}=S_{ki}\left(a\tilde{\tau}+b\tilde{\pi}_\tau\right)$$

which contradict the findings.

Hence, both the Hamiltonian and new D matrix have a block diagonal form:

$$\tilde{\mathbf{H}} = \begin{bmatrix} H_{4x4} & O_{4x2} \\ O_{2x4} & \tilde{H}_{l2x2} \end{bmatrix}; \tilde{\mathbf{D}} = \mathbf{S}\cdot\tilde{\mathbf{H}} = \begin{bmatrix} D_{4x4} & O_{4x2} \\ O_{2x4} & \tilde{D}_{l2x2} \end{bmatrix}; \ \tilde{H}_{4x4} = H_{4x4} \tag{M2-76}$$

It is also possible to prove it explicitly by considering in detail the only remaining equation in (M2-74) for $\tilde{\tau}'$. This also allows us to find explicitly expression for the longitudinal Hamiltonian $\tilde{H}_l$.

$$\frac{d\tilde{\tau}}{ds} = \frac{d}{ds}\left(\tau - \eta^T SZ\right) = \frac{d}{ds}\left(\tau - \eta^T S\tilde{Z}\right)$$

$$\frac{d\tau}{ds} == \left(g_x x + g_y y + \left(\frac{mc}{p_o}\right)^2\right)\pi_\tau = \pi_\tau\left(\frac{mc}{p_o}\right)^2 + C^T S\left(\tilde{Z} + \tilde{\pi}_\tau \eta^T\right); \tag{M2-77}$$

$$\left(\eta^T S\tilde{Z}\right)' = \left(\eta^T D^T{}_{4x4} + C^T\right)S\tilde{Z} + \eta^T SD^T{}_{4x4}\tilde{Z} = C^T S\tilde{Z}; \ \rightarrow \frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau\left(\left(\frac{mc}{p_o}\right)^3 + C^T S\eta^T\right)$$

with $C^T = \begin{bmatrix} 0 & -g_x & 0 & -g_y \end{bmatrix}$ and we used obvious

$$\eta^T D^T{}_{4x4} S\tilde{Z} + \eta^T SD_{4x4}\tilde{Z} = \eta^T H_{4x4}\tilde{Z} - \eta^T H_{4x4}\tilde{Z} = 0\,.$$

We can re-write $\tilde{\tau}'$ explicitly as

$$\frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau \left(\left(\frac{mc}{p_o} \right)^2 + \eta_x g_x + \eta_y g_y \right); \tag{M2-78}$$

and the new Hamiltonian as

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta;$$

$$\mathcal{H}_\beta = \frac{\pi_{x\beta}{}^2 + \pi_{y\beta}{}^2}{2} + \frac{F}{p_o c}\frac{x_\beta{}^2}{2} + \frac{N}{p_o c} x_\beta y_\beta + \frac{G}{p_o c}\frac{y_\beta{}^2}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o} \right)^2 + g_x \eta_x + g_y \eta_y \right) \frac{\pi_\tau{}^2}{2} = c_\tau \frac{\pi_\tau{}^2}{2} \tag{M2-79}$$

$$\frac{dX_\beta}{ds} = S \frac{\partial \mathcal{H}_\beta}{\partial X_\beta} = D_\beta X_\beta; \ \frac{d}{ds}\begin{bmatrix} \tilde{\tau} \\ \pi_\tau \end{bmatrix} = \begin{bmatrix} c_\tau \\ 0 \end{bmatrix} \pi_\tau; \rightarrow \tilde{\tau} = \pi_\tau \int_o^s c_\tau(\xi) d\xi$$

where I dropped tilde for compactness. It is important to remember that in this new variables

$$\tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x \pi_{x\beta} + \eta_{px} x_\beta - \eta_y \tilde{\pi}_y + \eta_{py} y_\beta;$$
$$\tau = c\left(t_o(s) - t\right). \tag{M2-80}$$

It means that arrival time dependence on transverse oscillations is well hidden in

$$t = t_o(s) - \frac{\tau}{c} = t_o(s) - \frac{\tilde{\tau}}{c} - \frac{\eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta}{c} \qquad \text{(M2-81)}$$

Since, without time dependent components in the Hamiltonian, the betatron and longitudinal motions are fully decoupled. It also means that in new variables our eigen vectors become:

$$Y_{k\beta} = \begin{bmatrix} Y_{k\beta} \\ 0 \\ 0 \end{bmatrix}; \; Y_{\tilde{\tau}} = \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix}; Y_{\tilde{\delta}} = \begin{bmatrix} O \\ 0 \\ 1 \end{bmatrix}; \qquad \text{(M2-82)}$$

The reason for disappearance of the transverse component in $Y_{\tilde{\delta}}$: it is caused by measuring the transverse orbit from the closed orbit for deviated energy. Similarly, disappearance time component in betatron eigen vectors is caused by its explicit inclusion into the new time variable.

Synchrotron oscillations: Intuitive approach

$$V_{rf} = V_o \cdot \sin\left(2\pi \cdot f_{rf} \cdot t\right); \quad f_{rf} = h/T_o$$

Synchronous particle: n is just a turn number

$$2\pi \cdot f_{rf} \cdot t_s(n) = N\pi \quad \rightarrow \sin\left(2\pi \cdot f_{rf} \cdot t_s(n)\right) = 0$$

$$t(n) = t_s(n) + \tau(n)$$

$$\tau(n+1) = \tau(n) + \eta_c T_o \frac{\Delta \mathbf{E}(n)}{\mathbf{E}_o};$$

$$\Delta \mathbf{E}(n+1) = \Delta \mathbf{E}(n) \pm qV_o \cdot \sin\left(f_{rf} \cdot \tau(n+1)\right)$$

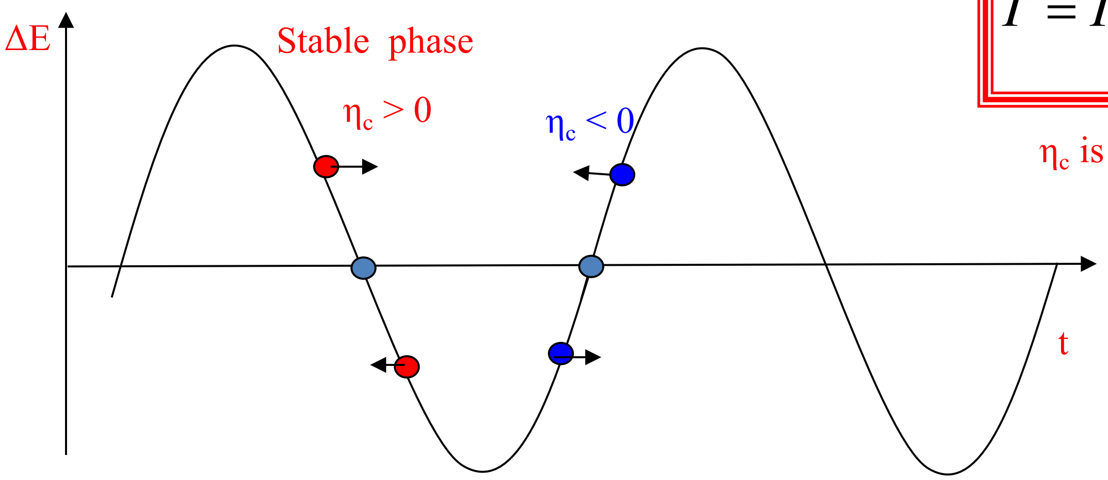
Revolution Time T = Circumference/velocity

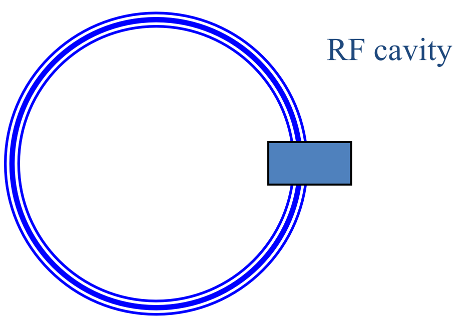
$$C = C(\mathrm{E}); \; \mathrm{v} = c \cdot \sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$$

$$T = \frac{C(\mathrm{E})}{c \cdot \sqrt{1 - \left(\frac{mc^2}{E}\right)^2}}$$

$$T = T_o \cdot \left(1 + \eta_c \frac{\Delta \mathbf{E}}{\mathbf{E}_o} + \ldots\right); \; \Delta \mathbf{E} \equiv \mathbf{E} - \mathbf{E}_o$$

η_c is a function of the accelerator lattice





η_c = 0 is a special case and called a transition energy

Synchrotron oscillations – much more details

With constant longitudinal momentum (energy) of particle, it propagates along a periodic orbit determined by dispersion function and momentum deviation for the reference particle, and also executes transverse betatron oscillation with respect to this orbit:

$$Z = Z_\beta + \pi_\tau \cdot \eta(s); \eta(s+C) = \eta(s); \eta' = D\eta + C;$$

$$Z'_\beta = DZ'_\beta; Z_\beta = \mathrm{Re}\sum_{k=1}^{2} a_k Y_{k\beta}(s) e^{i(\psi_k(s)+\varphi_k)}; \tag{M2-83}$$

where are periodic eigen vectors of the transverse oscillations:

$$T_{4x4} Y_{k\beta} = e^{i\mu_k} Y_{k\beta}.$$

In addition, we found that particles with energy deviation are slipping in time as follows:

$$\tau(s) = \pi_\tau\left(\eta_\tau \cdot s + \chi_\tau(s)\right) + \tau_\beta(s); \chi_\tau(s+C) = \chi_\tau(s)$$

$$\eta_\tau \cdot s + \chi_\tau(s) = \left(\frac{mc}{p_o}\right)^2 \cdot s + \int_0^s \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi)\right) d\xi; \tag{M2-84}$$

$$\eta_\tau = \frac{1}{C}\int_0^C \left(g_x\eta_x + g_y\eta_y\right) ds + \left(\frac{mc}{p_o}\right)^2;$$

with τ_β is the contribution from the betatron motion. To be exact, we just separated two parts of the linear motion using the fact that solution o linear differential equation are additive (linear combination of solution is a solution) and that there no time dependence.

Adding RF fields. Finally we are ready to move to synchrotron oscillations. Let's consider that we adding alternating (AC) longitudinal electric field on the beam axis

$$\frac{d\mathrm{E}}{ds} = -eE_s(s,t) \tag{M2-85}$$

For a moment we do not need to pick any specific form of this field, as far it does supports our assumption that the reference particle's trajectory in time, space and momentum exist. Specifically, it means that we request that

$$\frac{d\mathrm{E}_o}{ds} = -eE_s\left(s,t_o\left(s\right)\right) = 0 \tag{M2-86}$$

e.g. that alternating electric field crosses zero at the time of the passing of the reference particle. For a storage ring (a periodic system) the accelerating field has to be is periodic. The AC field (called RF field in the accelerators) has to satisfy the same condition.

$$\begin{gathered} E_s\left(s+C,t\right) = E_s\left(s+C,t\right);\; E_s\left(s+nC,t_o\left(s+nC\right)\right) = 0; \\ t_o\left(s+nC\right) = t_o\left(s\right) + nT_o;\; T_o = \frac{C}{\mathrm{v}_o}. \end{gathered} \tag{M2-87}$$

where T_o is called revolution period in the storage ring.

In practice, the alternating EM fields are generated in resonant cavities and have a sine-wave time dependence:

$$E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = \sum_n |E_n(s)| \sin(-\omega_n t + \phi_n(s)); \tag{M2-88}$$

where we simply numerated various RF frequencies ω_n, which frequently can be just a single frequency. In combination with (12-29) it yields requirement that all RF frequencies have to be harmonic of the revolution frequency:

$$T_o\omega_n = 2\pi h_n; h_n - inger;\ \ \omega_n = h_n\omega_o$$

$$f_{RFn} = \frac{\omega_n}{2\pi} = h_n f_{rev};\ \ f_{rev} = \frac{\omega_o}{2\pi} = \frac{1}{T_o} = \frac{\mathrm{v}_o}{C}. \tag{M2-89}$$

with the field on axis of:

$$E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = e\sum_n |E_n(s)| \sin(h_n\omega_o(t - t_o(s)) - \phi_n(s));$$

$$\sum_n |E_n(s)| \sin(\phi_n(s)) = 0.$$

For a single harmonic RF it becomes $\phi_n(s) = \pm\pi$ and

$$E_s(s,t) = eE_{rf}(s)\sin(h_{rf}\omega_o(t - t_o(s)))$$

Note that the sign of E_{rf} depends on what node of the sin-wave we choose. We can add the term corresponding to this longitudinal field using our full accelerator Hamiltonian (which doable but not necessary), or by noticing that

$$\frac{d\pi_\tau}{ds} = \frac{1}{p_o c}\frac{d(\mathrm{E}\text{-}\mathrm{E}_o)}{ds} = \frac{e}{p_o c}\sum_n \left|E_n(s)\right| \sin\left(h_n \omega_o t - \phi_n(s)\right) \qquad \text{(M2-90)}$$

corresponds to a term in Hamiltonian of

$$\delta H = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\cos\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o};\; k_o = \frac{\omega_o}{c} = \frac{2\pi}{C}\frac{\mathrm{v}_o}{c};$$

$$\tau_{add} = \eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta; \qquad \text{(M2-91)}$$

$$\frac{d\pi_\tau}{ds} = -\frac{\partial(\delta H)}{\partial\tilde{\tau}} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\sin\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o};$$

Thus, we can write a generic Hamiltonian without expansion in time domain:

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta + \delta\mathcal{H}$$

$$\mathcal{H}_\beta = \frac{mc}{p_o}\cdot\frac{\pi_{x\beta}^2 + \pi_{y\beta}^2}{2} + \frac{F}{p_o}\frac{x_\beta^2}{2} + \frac{N}{mc}x_\beta y_\beta + \frac{G}{p_o}\frac{y_\beta^2}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o}\right)^2 + g_x \eta_x + g_y \eta_y\right)\frac{\pi_\tau^2}{2} = c_\tau \frac{\pi_\tau^2}{2} \qquad \text{(M2-92)}$$

$$\delta\mathcal{H} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\cos\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o}$$

Linearized part of the additional Hamiltonian term is

$$\delta\mathcal{H}_\tau = -\frac{e\left(\tilde{\tau}+\tau_{add}\right)^2}{p_o c}\sum_n h_n k_o \left|E_n\right|\cos\left(\phi_n\right). \tag{M2-93}$$

While looking simpler than original Hamiltonian, adding RF fields made the Hamiltonian fully 3D coupled through τ_{add}. Hence, next step – let's consider case without betatron oscillations $\tilde{\tau}=\tau$:

$$\mathcal{H}_s = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} + \frac{e}{p_o c}\sum_n \frac{\left|E_n(s)\right|\cos\left(h_n k_o \tau + \phi_n\right)}{h_n k_o} \tag{M2-94}$$

or in linear case

$$\mathcal{H}_{sL} = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} - \frac{\tau^2}{2}\frac{e}{p_o c}\sum_n h_n k_o \left|E_n(s)\right|\cos\left(\phi_n\right) \tag{M2-95}$$

Coefficients in both Hamiltonians are s-dependent, and the Hamiltonians are not constants of motion. Naturally the linear system, when stable, can be solved using 1D parameterization

$$\tau = a_s \mathrm{w}_s(s)\cos\left(\psi_s(s)+\varphi_s\right);\ \ \psi_s' = \frac{1}{\mathrm{w}_s^{\ 2}};$$

$$\pi_\tau = \left\{a_s \mathrm{w}_s'(s)\cos\left(\psi_s(s)+\varphi_s\right) - \frac{1}{\mathrm{w}_s(s)}\sin\left(\psi_s(s)+\varphi_s\right)\right\}. \tag{M2-96}$$

This said, in majority of the storage rings, synchrotron oscillations are very slow and it takes from hundreds to tens of thousands turns to complete a single synchrotron oscillations. In this case small variations during one pass around a ring can be averaged. The easiest way is just to average the Hamiltonians (M2-94) and (M2-95): Beware, this is an approximation which brakes if synchrotron tune is relatively larger (let's say ~ 0.1). Still, it is easy way to get something useful – lets' do it:

$$\langle \mathcal{H}_s \rangle = \left(\eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{eU_{RF}(\tilde{\tau})}{p_o c} \right); U'_{RF}(0) = 0;$$

$$U_{RF}(\tilde{\tau}) = \frac{1}{C} \cdot \frac{e}{mc} \sum_n \frac{V_n}{h_n k_o} \cos\left(h_n k_o \tilde{\tau} + \bar{\phi}_n\right);\ \eta_\tau = \left(\frac{mc}{p_o}\right)^3 + \left\langle g_x \eta_x + g_y \eta_y \right\rangle; \qquad \text{(M2-97)}$$

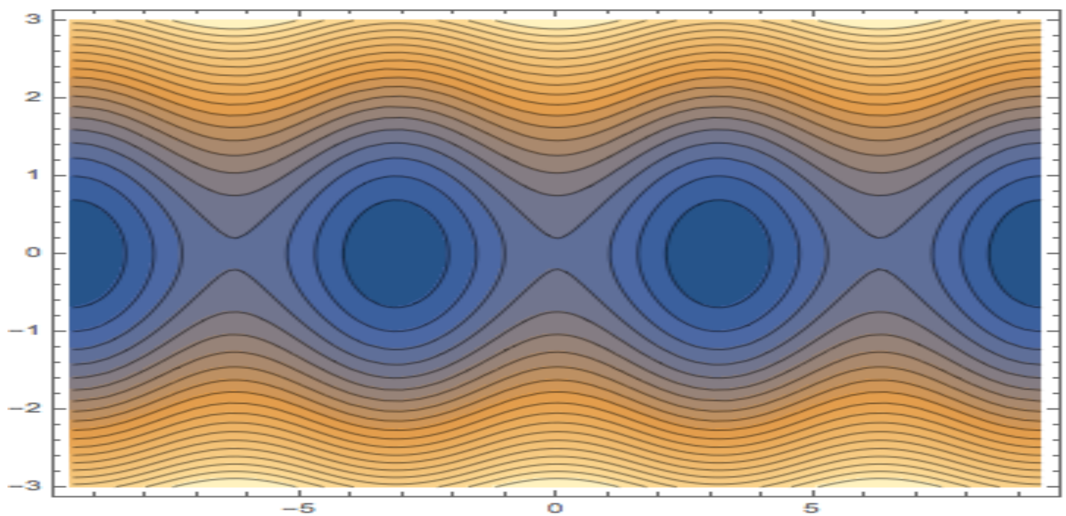
$$V_n \cos\left(\theta + \bar{\phi}_n\right) = \frac{\cos\theta}{h_n k_o} \int_o^C \left| E_n(s) \right| \cos\phi_n(s) - \frac{\sin\theta}{h_n k_o} \int_o^C \left| E_n(s) \right| \sin\phi_n(s).$$

The averaged Hamiltonian does not depend on s and is invariant of motion. Thus we can say that

$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{eU_{RF}(\tilde{\tau})}{mc} = \mathcal{H}_o; \quad \text{(M2-98)}$$

are equivalent to trajectories in the phase space of τ,π. Let's consider a single frequency RF – a traditional single frequency RF – well know pendulum equation:

$$\begin{gathered} \langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} = \mathcal{H}_o; \\ \frac{d\tau}{ds} = \eta_\tau \pi_\tau; \ \ \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right); \end{gathered} \quad \text{(M2-99)}$$

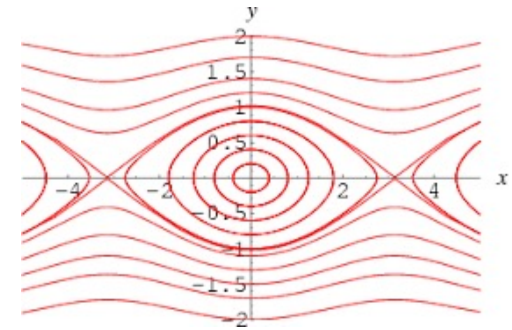


Plot of the equipotential for Hamiltonian (M2-99)– stable motion occurs around the zero or 180 degrees, depending on the relative sign of $eU_{RF}\eta_\tau$.

Stationary points are

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau = 0; \rightarrow \pi_\tau = 0; \ \frac{d\pi_\tau}{ds} = 0; \rightarrow \phi_o = k_o h_{rf}\tau = N\pi; \quad \text{(M2-100)}$$

Expanding around the stationary point we



$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\,2}}{2} - \frac{1}{C}\frac{eV_{RF}}{p_o c} k_o h_{rf} \frac{\tau^2}{2}\cos(\phi_o);$$

$$\Omega_s^{\,2} = \left| \eta_\tau h_{rf} \frac{k_o}{C} \frac{eV_{RF}}{p_o c} \right|; \ \cos(\phi_o) = \pm 1; \ Ck_o = 2\pi\beta_o$$

$$\Omega_s = \sqrt{\left| \frac{\eta_\tau k_o h_{rf}}{C} \frac{eV_{RF}}{p_o c} \right|} = \frac{1}{C}\sqrt{2\pi h_{rf} \left| \left\langle \left(\frac{mc}{p_o}\right)^3 + g_x \eta_x + g_y \eta_y \right\rangle \frac{eV_{RF}}{p_o c} \right|}; \qquad \text{(M2-101)}$$

$$\Omega_s = \sqrt{\left| \eta_\tau k_o h_{rf} \frac{eV_{RF}}{mc} \right|}; \mu_s = \Omega_s C = \sqrt{\left| 2\pi \eta_\tau h_{rf} \frac{eV_{RF}}{p_o c} \right|};$$

$$Q_s = \frac{\mu_s}{2\pi} = \sqrt{\left| \frac{\eta_{\tau o} h_{rf}}{2\pi} \frac{eV_{RF}}{p_o c} \right|}$$

Stable points are

$$-\eta_\tau eV_{RF} \cos(\phi_o) > 0; \Rightarrow \ \phi_s = 2N\pi; \qquad \eta_\tau eV_{RF} < 0;$$

$$\phi_s = (2N+1)\pi; \ \ \eta_\tau eV_{RF} > 0;$$

As we discussed during last class η_τ determines the sign on the longitudinal mass. When it is negative, not minima but maxima of the potential correspond to stable points.

Parameterization using real (non-complex) parameters. Since for a stable system eigen vectors are uni-modular complex numbers, eigen vectors are also complex and satisfy purely imaginary symplectic orthogonally conditions. Naturally matrix T can not be diagonalized using real matrices, but it can be brought to a block-diagonal form comprising simple 2x2 rotation matrices using following considerations:

$$\begin{aligned} &Y_k = R_k + iQ_k; Y_k^* = R_k - iQ_k; \mathbf{T}\cdot Y_k = e^{i\mu_k} Y_k; \mathbf{T}\cdot Y_k^* = e^{-i\mu_k} Y_k; \\ &\mathbf{T}\cdot R_k = R_k \cdot \cos\mu_k - Q_k \cdot \sin\mu_k; \mathbf{T}\cdot Q_k = Q_k \cdot \cos\mu_k + R_k \cdot \sin\mu_k; \end{aligned} \qquad \text{(M2-102)}$$

which is equivalent to

$$\mathbf{Q} = \left(R_1, Q_1, \ldots R_n, Q_n\right) \to \mathbf{T}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{O} \to \mathbf{T} = \mathbf{Q}\cdot\mathbf{O}\cdot\mathbf{Q}^{-1};$$

$$\mathbf{O} = \begin{pmatrix} O_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_n \end{pmatrix}; O_k = \begin{pmatrix} \cos\mu_k & \sin\mu_k \\ -\sin\mu_k & \cos\mu_k \end{pmatrix}; \mathbf{O}^T = \mathbf{O}^{-1} \qquad \text{(M2-103}$$

where by construction matrix **Q** is real. We can use a symbolic form of expressing block diagonal shape of **O** by writing

$$\mathbf{O} = \begin{pmatrix} O_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_n \end{pmatrix} = \sum_{k\oplus} O_k; \qquad \text{(M2-104)}$$

It is also symplectic, which is result of simple observation that follows from symplectic orthogonally of R_k, Q_k pairs:

$$Y_k^{*T}\mathbf{S}Y_{m\neq k} = 0 \rightarrow R_k^T\mathbf{S}R_m = 0; R_k^T\mathbf{S}Q_{m\neq k} = 0; Q_k^T\mathbf{S}Q_m = 0;$$

$$Y_k^{*T}SY_k = \left(R_k - iQ_k\right)^T \mathbf{S}\left(R_k + iQ_k\right) = \left(-iQ_k\right)iR_k^T\mathbf{S}Q_k - iQ_k^T\mathbf{S}R_k = 2iR_k^T\mathbf{S}Q_k = 2i;$$

$$R_k^T\mathbf{S}Q_k = -Q_k^T\mathbf{S}R_k = 1;$$

$$\mathbf{Q}^T\mathbf{S}\mathbf{Q} = \left(...R_k, Q_k...\right)^T \mathbf{S}\left(...R_k, Q_k...\right)\left(...R_k Q_k...\right)^T\left(...\mathbf{S}R_k, \mathbf{S}Q_k...\right) =$$

$$\begin{pmatrix} \begin{pmatrix} R_1^T\mathbf{S}R_1 & R_1^T\mathbf{S}Q_1 \\ Q_1^T\mathbf{S}R_1 & Q_1^T\mathbf{S}Q_1 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} R_n^T\mathbf{S}R_n & R_n^T\mathbf{S}Q_n \\ Q_n^T\mathbf{S}R_n & Q_n^T\mathbf{S}Q_n \end{pmatrix} \end{pmatrix} = \qquad \text{(M2-105)}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} = \mathbf{S}\#$$

There is one to one connection between real matrix **Q** and complex matrix **U**

$$\mathbf{U}=\mathbf{Q}\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{pmatrix};\mathbf{Q}=\frac{\mathbf{U}}{2}\begin{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \end{pmatrix} \qquad \text{(M2-106)}$$

which means that putting matrix **Q** in motion is

$$\tilde{\mathbf{Q}}(s_1)=\mathbf{M}(s|s_1)\mathbf{Q}(s)=\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1);$$

$$\tilde{\mathbf{O}}(s|s_1)=\sum_{\oplus}\begin{pmatrix} \cos\left(\psi_k(s_1)-\psi_k(s)\right) & \sin\left(\psi_k(s_1)-\psi_k(s)\right) \\ -\sin\left(\psi_k(s_1)-\psi_k(s)\right) & \cos\left(\psi_k(s_1)-\psi_k(s)\right) \end{pmatrix} \qquad \text{(M2-107)}$$

Again, it gives us connection between transport matrices and parametrization:

$$\mathbf{M}(s|s_1)=\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1)\mathbf{Q}(s)^{-1}=-\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1)\mathbf{S}\mathbf{Q}^T(s)\mathbf{S} \qquad \text{(M2-108)}$$

Interesting is application of this expression for full period matrix:

$$\mathbf{T}=\mathbf{Q}\cdot\mathbf{O}\cdot\mathbf{Q}^{-1}=\sum_{\oplus}\mathbf{Q}\cdot\left[O_k\right]\cdot\mathbf{Q}^{-1};\ \left[O_k\right]=\begin{pmatrix} 0 & & & & & & 0 \\ & & \dots & 0 & \dots & & \\ \dots & & 0 & O_k & 0 & & \dots \\ & & \dots & 0 & \dots & & \\ 0 & & & \dots. & & & 0 \end{pmatrix} \tag{M2-109}$$

$$O_k=\begin{pmatrix} \cos\mu_k & \sin\mu_k \\ -\sin\mu_k & \cos\mu_k \end{pmatrix}=\cos\mu_k I_k+\sin\mu_k\sigma_k;\sigma=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where I_k,σ_k are block diagonal 2x2 matrices with non-zero block in k-position on the diagonal. Now we will extract constants and expand one-turn transport matrix though eigen matrices:

$$\mathbf{T}=\sum_{k\oplus}\left(\mathbf{E}_k\cos\mu_k+\mathbf{J}_k\sin\mu_k\right);$$
$$\mathbf{E}_k=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1};\mathbf{J}_k=\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1};\ \mathbf{Q}^{-1}=-\mathbf{S}\mathbf{Q}^T\mathbf{S}. \tag{M2-110}$$

These matrices have very nice features of n mutually orthogonal pair of *1 and i*:

$$\left[I_k\right]^2=\left[I_k\right]\to\mathbf{E}_k^2=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}=\mathbf{E}_k;$$
$$\left[\sigma_k\right]^2=-\left[I_k\right]\to\mathbf{J}_k^2=\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1}.\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1}=-\mathbf{E}_k \tag{M2-110}$$
$$\left[I_k\right]\left[\sigma_k\right]=\left[I_k\right]\left[\sigma_k\right]=\left[\sigma_k\right]\to\mathbf{E}_k\mathbf{J}_k=\mathbf{J}_k\mathbf{E}_k=\mathbf{J}_k;$$
$$\left[I_k\right]\left[I_{m\neq k}\right]=\left[\sigma_k\right]\left[I_{m\neq k}\right]=\left[\sigma_k\right]\left[\sigma_{m\neq k}\right]\equiv 0\to\mathbf{E}_k\mathbf{E}_{m\neq k}=\mathbf{E}_k\mathbf{J}_{m\neq k}=\mathbf{J}_k\mathbf{J}_{m\neq k}\equiv 0$$

which result in trivial adding phase advance in equation (M2-109):

$$\mathbf{T}^n=\sum_{k\oplus}\left(E_k\cos n\mu_k+J_k\sin n\mu_k\right). \tag{M2-112}$$

This expression is especially beautiful for 1D case when because matrix is just a 2x2 block itself:

$$[I_k] = \mathbf{I}; [\sigma_k] = \mathbf{S};$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{E} = \mathbf{Q}\cdot\mathbf{I}\cdot\mathbf{Q}^{-1} = \mathbf{I}; \mathbf{J} = -\mathbf{Q}\cdot\mathbf{S}\cdot\mathbf{S}\mathbf{Q}^T\mathbf{S} = \mathbf{Q}\cdot\mathbf{Q}^T\cdot\mathbf{S}$$

where we can use specific expression for **Q**

$$\mathbf{Q} = [\mathrm{Re}\,Y, \mathrm{Im}\,Y] = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & \frac{1}{\mathrm{w}} \end{bmatrix}; \mathbf{Q}\cdot\mathbf{Q}^T = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & \frac{1}{\mathrm{w}} \end{bmatrix}\begin{bmatrix} \mathrm{w} & \mathrm{w}' \\ 0 & \frac{1}{\mathrm{w}} \end{bmatrix} = \begin{bmatrix} \mathrm{w}^2 & \mathrm{ww}' \\ \mathrm{ww}' & \frac{1}{\mathrm{w}^2} + \mathrm{w}'^2 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}\cdot\mathbf{Q}^T\cdot\mathbf{S} = \begin{bmatrix} \mathrm{w}^2 & \mathrm{ww}' \\ \mathrm{ww}' & \frac{1}{\mathrm{w}^2} + \mathrm{w}'^2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{ww}' & \mathrm{w}^2 \\ -\left(\frac{1}{\mathrm{w}^2} + \mathrm{w}'^2\right) & \mathrm{ww}' \end{bmatrix}$$

and you can directly show that $\mathbf{J}^2 = -\mathbf{I}$. Using traditional definitions of α,β,γ functions introduced by Courant and Snider we can rewrite (M2-110) in form you would find in standard accelerator books:

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;\ \mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \mathbf{J}^2 = -\mathbf{I}; \tag{M2-111}$$

$$\beta = \mathrm{w}^2;\ \alpha = -\mathrm{ww}' = -\beta'/2; \gamma = \mathrm{w}'^2 + \mathrm{w}^{-2} = \frac{1+\alpha^2}{\beta}.$$

Now we are ready to make use of our parameterization:

$$X_o = \mathrm{Re}\sum_{k=1}^{n} a_k Y_k(s) e^{i\psi_k(s)} \equiv$$

$$\sum_{k=1}^{n} |a_k| \left(R_k(s)\cos\left(\psi_k(s)+\varphi_k\right) - Q_k(s)\sin\left(\psi_k(s)+\varphi_k\right)\right); \quad a_k = |a_k| e^{i\varphi_k}; \tag{M2-114}$$

with 2n constants of motion coming in pairs of amplitude and phased of ociallator $\{a_k, \varphi_k\}, k = 1,...,n$. Starting from this point we will use real amplitudes $a_k \to |a_k|$ and separate phase explicitly: $a_k \to a_k e^{i\varphi_k}$.

Action-angle variables. A very important transformation (not-only!) in accelerator physics is the transformation to the action-angle variables $\left\{\varphi_k, I_k = \frac{a_k^{\,2}}{2}\right\}$. Usually this requires two steps: The first is to Canonically transfer to Canonical conjugate oscillators (you may remember them from quantum mechanics?):

$$\left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\}. \tag{M2-116}$$

$$X^T \equiv \{..q_k, p_k...\} \Leftrightarrow A_{qo}^{\;\;T} \equiv \left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\};$$
$$X^T = \mathbf{V}A_{qo}; \quad \mathbf{V} = \frac{1}{\sqrt{2}}\left[Y_1, iY^*_{\;1}......\right] \Rightarrow \mathbf{V}^T\mathbf{S}\mathbf{V} = \mathbf{S} \;\# \tag{M2-117}$$

The second step is very simple since it is well known from classical theory of harmonic oscillators. A generation function transformation making this Canonical transformation happening is very simple to construct:

$$\left\{q_k = \varphi_k; p_k \equiv I_k = \frac{a_k^{\,2}}{2}\right\} \Leftrightarrow \left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\}$$

$$F(q,\tilde{q}) = -\sum_{k=1}^{n} i\frac{\tilde{q}_k^{\,2}}{2}e^{-2i\varphi_k}; \frac{\partial F}{\partial s} = 0 \rightarrow \tilde{H} = H \tag{M2-118}$$

$$I_k = \frac{\partial F}{\partial q_k} \equiv \frac{\partial F}{\partial \varphi_k} = \tilde{q}_k^2 e^{-2i\varphi_k} = \frac{a_k^{\,2}}{2}; \; \tilde{p}_k = -\frac{\partial F}{\partial \tilde{q}_k} = i\tilde{q}_k e^{-2i\varphi_k} i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}.$$

Similarly, we can make transformation for pairs of real oscillator components:

$$\left\{\tilde{q}_k = a_k \cos\varphi_k, \tilde{p}_k = -a_k \sin\varphi_k\right\}. \tag{M2-119}$$

with obvious symplectic transformation

$$\begin{gathered} A^T{}_{osc} = \left\{...q_k, p_k...\right\} = \left\{...a_k \cos\varphi_k, -a_k \sin\varphi_k...\right\} \\ X = \mathbf{Q}\cdot A_{osc} \to A_{osc} = \mathbf{Q}^{-1}X;\ \mathbf{Q}^{-1T}\mathbf{S}\mathbf{Q}^{-1} = \mathbf{S}. \end{gathered} \tag{M2-120}$$

Again, the generation function transformation making this Canonical transformation happening is very simple to construct:

$$\left\{q_k = \varphi_k; p_k \equiv I_k = \frac{a_k{}^2}{2}\right\} \Leftrightarrow \left\{\tilde{q}_k = a_k \cos\varphi_k, \tilde{p}_k = -a_k \sin\varphi_k\right\}$$

$$F(q,\tilde{q}) = \sum_{k=1}^{n} \frac{\tilde{q}_k{}^2}{2} \tan\varphi_k; \frac{\partial F}{\partial s} = 0 \to \tilde{H} = H \tag{M2-121}$$

$$I_k = \frac{\partial F}{\partial q_k} \equiv \frac{\partial F}{\partial \varphi_k} = \frac{\tilde{q}_k^2}{2\cos^2\varphi} = \frac{a_k{}^2}{2};\ \tilde{p}_k = -\frac{\partial F}{\partial \tilde{q}_k} = -\tilde{q}_k \tan\varphi_k = -a_k \sin\varphi_k.$$

This result (even though expected) has long-lasting consequences – the trivial (linear) part in the Hamiltonian can be removed from equations of motion, so allowing one to use this in perturbation theory or at least to focus only on non-trivial part of the motion. But by design for a linear Hamiltonian system,

$$H_L = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s) x_i x_j \equiv \frac{1}{2} X^T \cdot \mathbf{H}(s) \cdot X \qquad \text{(M2-118)}$$

A^T = const. It means that

$$\frac{\partial F\left(q,\tilde{q},s\right)}{\partial s} = -H_L \qquad \text{(M2-119)}$$

It means that equation of motion for a linear *s*-dependent Hamiltonian system are reduced to a set of constant: amplitudes and phases of oscillations:

$$\varphi_k = const; I_k = \frac{{a_k}^2}{2} = const;\ k = 1,2...,n \qquad \text{(M2-120)}$$

What it important to note that I_k is an adiabatic invariant of an oscillator, e.g. is the phase space area of the covered by oscillator divided by π. We can call it "particle's emittance" in the k-th mode.

Thus, if we are applying transformation of the action-angle Canonical variables of an arbitrary (in general case, nonlinear) Hamiltonian system

$$H(X,s) = H_L(X,s) + H_1(X,s) \tag{M2-121}$$

we will come to the reduced equations of motion with the Hamiltonian:

$$\begin{gathered} \tilde{H} = H + \frac{\partial F}{\partial s} = H - H_L = H_1(X,s); \\ \tilde{H}(A,s) = H_1(X(A,s),s). \end{gathered} \tag{M2-122}$$

where we eliminated "boring" oscillating part of the motion.

Since next step of transformation to the action-angle variables (41) does not change the Hamiltonian, we finally get:

$$\begin{gathered} \tilde{H}(\varphi_k, I_k, s) = H_1(X(\varphi_k, I_k, s), s); \\ \frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}; \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}. \end{gathered} \tag{M2-123}$$

These "reduced" equations of motion can be very useful when H_1 can be treated as perturbation or in studies of a non-linear map. We will return to them again and again.

To conclude

- We learned about parametrization of linearized stable particle motion in storage rings
- The is described by the set of three periodic eigen vectors, with each particles assigned individual amplitudes (actions) and phases for each mode of oscillations
- We arrived to reduced equations of motion for actions and phases
- We spent some time to focus on longitudinal (energy and arrival time) oscillations, which are usually very slow. An approximate set of the eigen vectors could be used in this case using components of dispersion functions