

**Home Work PHY 554 #8. Handed - October 1, 2018, Due - October 10, 2018**

**HW 1 (5 points):** RF cavity beam loading/unloading.

A short ultra-relativistic ( $1-v/c \ll 1$ ) bunch with charge of 5 nC is passing through a 0.3 meter long 500 MHz pillbox accelerating cavity operating at the fundamental  $TM_{010}$  with peak accelerating field of 5 MV/m.

(1) Find the change of the cavity voltage  $\Delta V/V$  (accelerating field) after the beam passes through it as function of the phase of the beam passing the cavity. What are the maximum and minimum  $\Delta V/V$ ?

(2) How the beam loading  $\Delta V/V$  depends on the accelerating field? At what level of accelerating it reaches  $\Delta V/V$  1%?

- (a) Assume that beam does not change velocity in the cavity;
- (b) Hint – use energy conservation law
- (c) Assume that relative change of the voltage  $\Delta V/V$  is small, e.g. the beam loading can be treated as a perturbation.

**Solution:**

First, we need to find the energy gain by each electron in the cavity operating at  $E=5$  MV/m using RF phase in the center as the reference:

$$dz \cong ct;$$

$$\Delta E = ecE \int_{-L/2c}^{L/2c} \cos(\omega t + \varphi) dt = eLE \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} \cos \varphi = eV_{RF} \cos \varphi$$

$$FF = \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} = 0.636179.. \quad V_{RF} = LE \cdot FF = 0.9543 \text{ MV}$$

It means that that energy take/given by the beam is

$$\Delta U = qV_{RF} \cos \varphi = \Delta U_o \cos \varphi$$

$$q = 5nC = 5 \cdot 10^{-9} C; V_{RF} = 0.9543 \cdot 10^6 V$$

$$\Delta U_o = \text{sign}[q] \cdot 4.77 \cdot 10^{-3} J$$

Naturally, when energy is taken by electron beam,  $q \cos \varphi > 0$ , RF voltage in the cavity drops (it is called beam loading) and with beam loses energy,  $q \cos \varphi < 0$ , RF voltage increases. To know the voltage change we need to know what EM energy is stored in the RF cavity. We should use the your favorite units system (SI)

$$W = \int \left( \epsilon_o \frac{\bar{\mathbf{E}}^2}{2} + \mu_o \frac{\bar{\mathbf{H}}^2}{2} \right) dV = \frac{\epsilon_o}{2} \int \bar{\mathbf{E}}_o^2 dV$$

or GSG

$$W = \frac{1}{8\pi} \int (\bar{\mathbf{E}}^2 + \bar{\mathbf{H}}^2) dV = \frac{1}{8\pi} \int \bar{\mathbf{E}}_o^2 dV$$

and the field pattern we derived for TM<sub>010</sub> mode

$$\mathbf{E}_o = \hat{z} \cdot E_o \cdot J_o \left( 2.405 \frac{r}{a} \right); E_o = 5 \cdot 10^6 \frac{V}{m} \equiv 166.7 \text{ Gs}$$

where radius of the cavity,  $a$ , is defined by its frequency:

$$J_o(ka) \equiv J_o \left( \frac{\omega}{c} a \right) = 0 : TM_{010} \rightarrow \frac{\omega}{c} a = 2.405$$

$$a = \frac{2.405c}{\omega} = \frac{2.405}{2\pi f} c = 0.2295 \text{ m}$$

and then integrate Bessel function over the radius of the cavity

$$\int J_o^2 r dr d\theta dz = 2\pi L \int_o^a J_o^2(kr) r dr = \frac{2\pi L}{k^2} \int_o^{x_o} J_o^2(x) x dx;$$

$$x_o = 2.404825557695773 \dots \int_o^{x_o} J_o^2(x) x dx = 0.779325$$

$$\frac{2\pi L}{k^2} \int_o^{x_o} J_o^2(x) x dx = 1.337 \cdot 10^{-2} \text{ m}^3 = 1.337 \cdot 10^4 \text{ cm}^3$$

Than using you favorite units we got identical

$$W = 1.48 \text{ J} = 1.48 \cdot 10^7 \text{ erg}$$

**Side note:** A smart RF engineer would use known value of  $R_{sh}/Q$

$$\frac{R_{sh}}{Q_o} = \frac{V_{RF}^2}{\omega_o W} \rightarrow W = \frac{V_{RF}^2}{\omega_o} \frac{R_{sh}}{Q_o}$$

for pillbox cavity of 196 Ohm (Slide 10, Lecture 11) to get the same:

$$\frac{R_{sh}}{Q_o} = 196; V_{RF} = 9.54E5 \text{ V}; \omega_o = 3.141593E9 \text{ Hz} \Rightarrow W = 1.48 \text{ J}$$

Finally we should notice that

$$V_{RF} \sim \sqrt{W} \rightarrow \frac{\Delta V_{RF}}{V_{RF}} = \frac{1}{2} \frac{\Delta W}{W}; \Delta W = -\Delta U$$

and maximum voltage drop in our case is

$$\Delta V_{RF} = -V_{RF} \frac{1}{2} \frac{\Delta U}{W} = -1.54 \text{ kV}; \quad \frac{\Delta V_{RF}}{V_{RF}} = -1.6 \cdot 10^{-3} = 0.16\%$$

Beam loading dependence on the accelerating field (RF voltage) is very simple to find from following

$$\Delta U = qV_{RF} \sim E_o; W \sim V_{RF}^2 \sim E_o^2 \Rightarrow \frac{\Delta V_{RF}}{V_{RF}} \sim \frac{1}{V_{RF}} \sim \frac{1}{E_o}$$

e.g. beam loading is inverse proportional to the accelerating field. Thus, to increase beam loading from 0.16% to 1% we should make the accelerating voltage to be  $0.16 E_o = 0.8 \text{ MV/m}$ .

**HW 2 (3 points):** Cavities filled with ferrite material are used for RF system requiring large frequency tuning range. The frequency is controlled by applying external magnetic field,  $B_{\text{ext}}$ , to the ferrite material and by doing so to change its permeability  $\mu(B_{\text{ext}})$ . A 300 m in circumference AGS synchrotron accelerates polarized protons from total energy of 2.5 GeV to 25 GeV.

- (a) Calculate the range of the beam revolution frequency in AGS;
- (b) Assuming 100% filling by ferrite, what should be ratio of  $\mu_{\text{max}}$  to  $\mu_{\text{min}}$ . Where  $\mu$  should have maximum value?

**Note:** RF systems operate on a fixed integer harmonic of the revolution frequency.

**Solution:**

(a) Rest energy of a proton is 0.938 GeV. It means that Lorentz factor changes from 2.66 to 26.6 and  $v/c$  changes from 0.9269 to 0.9993, e.g. revolution frequency increases 1.0781 fold during acceleration from 927.5 kHz to 999.987 kHz (e.g. 1 MHz!).

(b) Since the frequency of an RF cavity scales the same as speed of light in the media:

$$\omega_{\text{res}} = \frac{\omega_0}{\sqrt{\epsilon\mu}} \rightarrow \mu \propto \omega_{\text{res}}^{-2}$$

one should reduce  $\mu$  1.162-fold to accommodate necessary change in resonant frequency.

**HW 3 (2 points):** In RF cavity operating at 500 MHz, amplitude of the magnetic field at the part surface is 500 Gs or 500 Oe. Find power losses per square meter of the surface for:

- (a) Cu cavity\*
- (b) SRF cavity with surface resistance,  $R_s = 5 \cdot 10^{-9}$  Ohm.

How much water you can heat from 20 C° to 40 C° in one hour (3,600 second) by cooling such Cu cavity?

\*Hint: you may use the conductivity of Cu or scale  $R_s$  from results shown in Lecture 10. Thermal capacitance of water is 4,179 J/kg/ C°.

**Solution:**

We should use formula for surface losses for a good conductor (it is in SI units):

$$\frac{P_{loss}}{A} = \frac{1}{2} R_s |\vec{H}_{//}|^2$$

The most confusing is to transfer H from CGS (Gs = Oe) units to SI (A/m) with coefficient  $1000/4\pi$ :  $H=3.98 \cdot 10^4$  A/m: With  $A=1$  m<sup>2</sup> power lost is simply

$$P_{loss} = \frac{1}{2} R_s |\vec{H}_{//}|^2 A = \frac{1}{2} R_s |\vec{H}_{//}|^2$$

and for SRF cavity we would have 3.96 W losses per one square meter of the surface. For Cu surface impedance scales with the frequency

$$R_s = \sqrt{\frac{\omega\mu}{\sigma}} [\Omega]$$

In slide 8, lecture 11 we shown that for Cu  $R_s=10$  mOhm at frequency of 1.5 GHz, which is 3 time higher than in our case. Thus

$$R_s(Cu, 500Mhz) = \frac{10m\Omega}{\sqrt{3}} \approx 5.8m\Omega$$

and power loss density is 4.57 MW per m<sup>2</sup>. In one hour the EM field  $1.65 \cdot 10^{10}$  J. Heating one kg (e.g. one liter) of water by 20K requires 83.6 kJ: hence this power will heat from 20 C° to 40 C° 197 tons of water! e.g. a cube approximately 6m x 6 m x 6m.