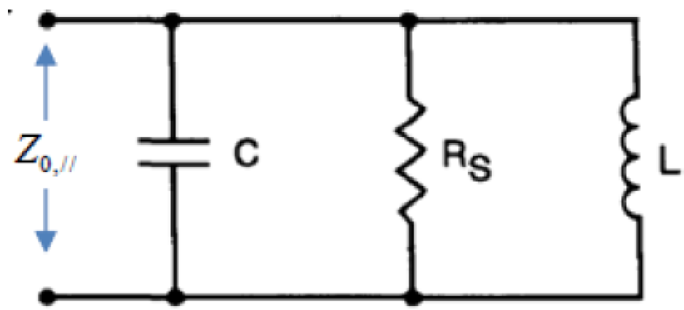
Solutions for HW # 20

1.



The impedance is determined by

$$\begin{aligned}
\frac{1}{Z_{0,//}} &= \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} \\
&= \frac{1}{R_s} + \frac{1}{j\omega L} + j\omega C \\
&= \frac{1 + jR_s\sqrt{\frac{C}{L}}\left(\omega\sqrt{LC} - \frac{1}{\omega\sqrt{LC}}\right)}{R_s}, \qquad (1) \\
&= \frac{1 + jQ\left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right)}{R_s} \\
&= \frac{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}{R_s}
\end{aligned}$$

i.e.

$$Z_{0,//} = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}$$

where $j = -i$, $Q \equiv R_s\sqrt{\frac{C}{L}}$ and $\omega_R \equiv \frac{1}{\sqrt{LC}}$.

2.

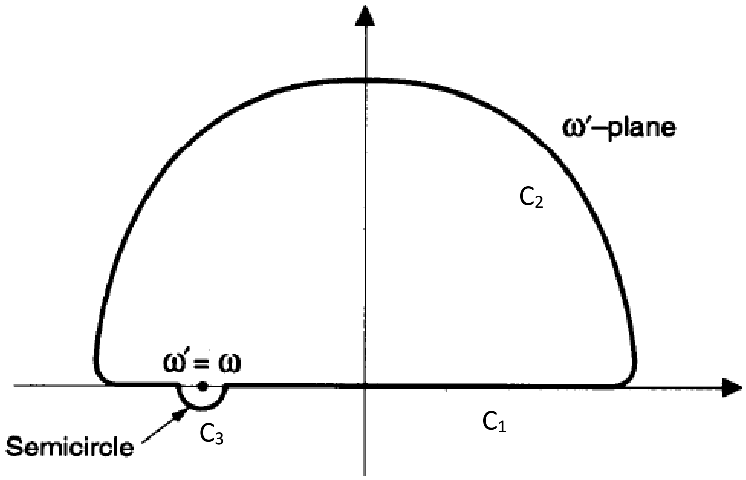


Figure 1: Integration contour in complex ω' plane.

From Cauchy residue theorem, the contour integral can be calculated as

$$\int_C \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' = 2\pi i Z_{//}(\omega) \ . \tag{2}$$

The LHS of (2) can be split into the following form

$$\begin{aligned}\int_C \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' &= \int_{C_1} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + \int_{C_2} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + \int_{C_3} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \\ &= P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + 0 + \int_{e^{i\pi}}^{e^{i2\pi}} \frac{Z_{//}(\omega)}{e^{i\theta}} de^{i\theta} \qquad , \\ &= P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + 0 + i\pi Z_{//}(\omega)\end{aligned} \tag{3}$$

where the integral along C₂ vanishes since we assume $Z_{//}(\omega')$ is well behaved at large $|\omega'|$. From eq. (2) and (3), it follows

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \ . \tag{4}$$

Splitting eq. (4) into the real and imaginary part leads to

$$\mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega'-\omega} d\omega' \ , \tag{5}$$

and

$$\mathrm{Im}\left[Z_{//}\left(\omega\right)\right]=-\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left[Z_{//}\left(\omega'\right)\right]}{\omega'-\omega}d\omega' \quad . \tag{6}$$