1. For an electron of 3GeV energy circulating in a storage ring with bending radius of 9 meters, its energy loss per turn due to synchrotron oscillation is

$$U_0 = \frac{e^2\beta^3\gamma^4}{3\varepsilon_0\rho} = 0.7962MeV \ .$$

The critical angular frequency of the radiation is

$$\omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho} = 1.011\times 10^{19}\, rad/s \quad ,$$

and hence the critical energy of the photons is

$$E_c = \hbar\omega_c = 6.655KeV \ .$$

The synchrotron radiation power due to 400 mA of electron current is

$$P_w = U_0\frac{\Delta N_e}{\Delta T} = U_0\frac{I_e}{e} = 318.5KW \ .$$

The spectral density of synchrotron radiation energy from a single electron is (slide #22 of lecture 12)

$$\left.\frac{d^2I}{d\omega d\Omega}\right|_{\theta=0,\omega=\omega_c} = \frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{4\pi^2 c}K_{2/3}^2\left(\frac{1}{2}\right) = 1.822\times 10^{-11}\, eV\cdot s \ .$$

The relative spectral density of photon flux due to 500 mA is given by

$$\left.\frac{d^2F}{(d\omega/\omega)d\Omega}\right|_{\theta=0,\omega=\omega_c} = \left.\frac{d^2I}{(d\omega/\omega)d\Omega}\right|_{\theta=0,\omega=\omega_c}\cdot\frac{\Delta N_e}{\Delta T} = \frac{1}{\hbar\omega_c}\cdot\omega_c\left.\frac{d^2I}{d\omega d\Omega}\right|_{\theta=0,\omega=\omega_c}\cdot\frac{I_e}{e} = 6.94\times 10^{22}\, s^{-1} \ .$$

We assume that the cross-section of the radiation is equal to that of the electron beam. For a Gaussian profile, we should take the area as

$$A_e = \left(\sqrt{2\pi}\sigma_x\right)\left(\sqrt{2\pi}\sigma_y\right) = 2\pi\sqrt{\varepsilon_x\beta_x}\cdot\sqrt{\varepsilon_y\beta_y} = 7.695\times 10^{-10}\, m^2 \ .$$

Thus the spectral brightness is given by

$$B = \frac{1}{A}\left.\frac{d^2F}{(d\omega/\omega)d\Omega}\right|_{\theta=0,\omega=\omega_c} = 9.018\times 10^{31}\, m^{-2}s^{-1} = 9.018\times 10^{16}\,\frac{1}{s\cdot mm^2\cdot mrad^2\,(0.1\%BW)} \ .$$

(You can also use the practical formula from slide #15 of Lecture 16 to get the answer.)

2. (a) The undulator period can be derived from the undulator equation with $\theta = 0$:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$
$$\Rightarrow \lambda_u = \lambda \frac{2\gamma^2}{1+\frac{K^2}{2}} = 2.3cm \ .$$

(b) The power radiated into the central cone is (slide #28, Lecture 16)

$$P_{cen} = \frac{\pi e \gamma^2 I_e}{\varepsilon_0 \lambda_u} \frac{K^2}{\left(1+\frac{K^2}{2}\right)^2} f(K) = 12.554W \ ,$$

where $f(K) = \left[J_0\left(\frac{K^2}{4\left(1+\frac{K^2}{2}\right)}\right) - J_1\left(\frac{K^2}{4\left(1+\frac{K^2}{2}\right)}\right)\right]^2$. The photon flux is then

$$F_{cen} = \frac{P_{cen}}{\hbar \omega_0} = 3.16\times 10^{16} s^{-1} \ ,$$

with $\omega_0 = 2\pi c/\lambda = 3.767\times 10^{18} rad/s$. The spectral brightness is given by (slide #33 in Lecture 16)

$$B_{cen} = \frac{F_{cen}}{\Delta A \cdot \Delta\Omega \cdot N^{-1}} = \frac{F_{cen}}{2\pi \sigma_x \sigma_y \pi \theta_{Tx} \theta_{Ty} N^{-1}} = 2.991\times 10^{35} m^{-2} s^{-1} = 2.991\times 10^{20} \frac{1}{s\cdot mm^2 mrad^2 (0.1\%BW)}$$

with $\theta_{Tx} = \sqrt{\theta_{cen}^2 + \sigma_{x'}^2} = 33.55\mu rad$, $\theta_{Ty} = \sqrt{\theta_{cen}^2 + \sigma_{x'}^2} = 27.08\mu rad$, $\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}} = 26.93\mu rad$,

$\gamma^* = \frac{\gamma}{\sqrt{1+K^2/2}}$, $\sigma_{x'} = \sqrt{\varepsilon_x/\beta_x} = 20\mu rad$, $\sigma_{y'} = \sqrt{\varepsilon_y/\beta_y} = 2.83\mu rad$, $\sigma_x = \sqrt{\varepsilon_x \beta_x} = 50\mu m$, and $\sigma_y = \sqrt{\varepsilon_y \beta_y} = 7.07\mu m$. One can also use the practical formula in slide #33 of Lecture 16 to get the answer.