Transverse (Betatron) Motion

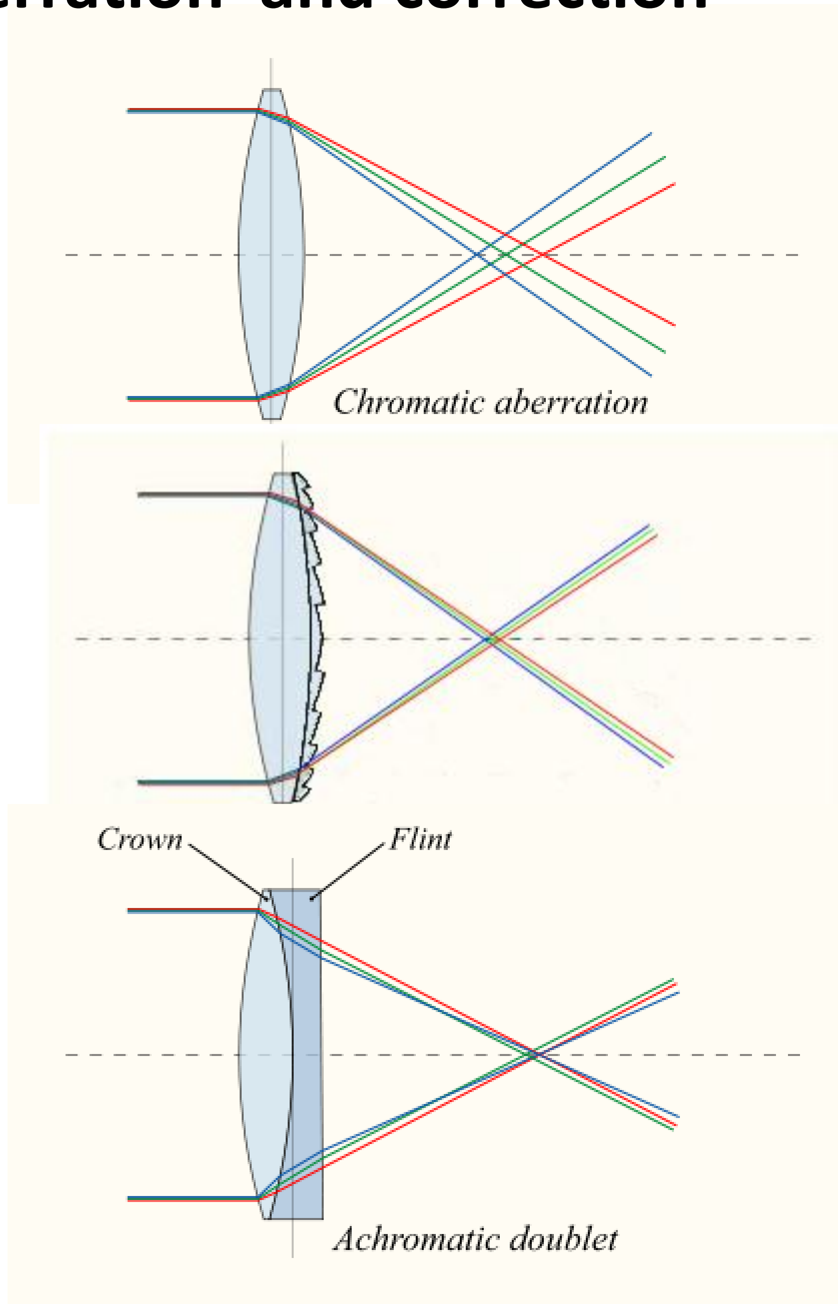
Linear betatron motion

Dispersion function of off momentum particle

Simple Lattice design considerations

Nonlinearities

Chromatic aberration and correction



Chromatic aberration in particle accelerators

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2, \qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

Inhomogeneous equation

$$p/p_0 = 1+\delta$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, \quad K(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_y}{\partial x}$$

$$x = x_\beta + D\delta \qquad D'' + \left(K_x(s) + \Delta K_x\right)D = \frac{1}{\rho} + O(\delta)$$

$$x''_\beta + \left(K_x(s) + \Delta K_x\right)x_\beta = 0, \quad y''_\beta + \left(K_y(s) + \Delta K_y\right)x_\beta = 0$$

$$K_x(s) = \frac{1}{\rho^2} - K(s), \qquad \Delta K_x(s) = [-\frac{2}{\rho^2} + K(s)]\delta \approx -K_x(s)\delta,$$

$$K_y(s) = +K(s), \qquad \Delta K_y(s) = [-K(s)]\delta = -K_y(s)\delta$$

Note that the betatron motion for off momentum particle is perturbed by a chromatic term. The betatron tunes must avoid half-integer resonances. But, the quadrupole error is proportional to the designed quadrupole field. They are called systematic chromatic aberration. It is an important topic in accelerator physics.

Tune shift, or tune spread, due to chromatic aberration:

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x/d\delta$$

$$\Delta\nu_y = \left[-\tfrac{1}{4\pi}\oint \beta_y(s)K_y(s)ds\right]\delta \equiv C_y\delta, \quad C_y = d\nu_y/d\delta$$

The chromaticity induced by quadrupole field error is called natural chromaticity. For a simple FODO cell, we find

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \approx -\frac{1}{4\pi}\sum\frac{\beta_{xi}}{f_i}\ \delta$$

$$C_{X,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{1}{4\pi}N\left(\frac{\beta_{\max}}{f}-\frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

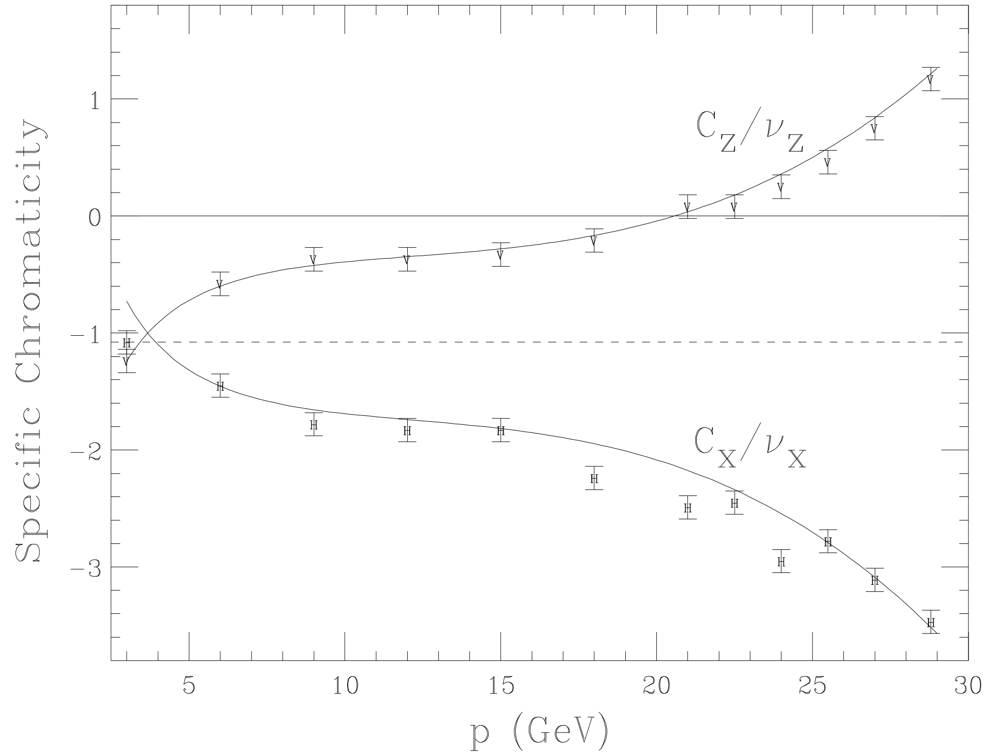
We define the specific chromaticity as $\xi_x = C_x/\nu_x, \quad \xi_y = C_y/\nu_y$

The **specific chromaticity is about −1 for FODO cells**, and can be as high as -4 for high luminosity colliders and high brightness electron storage rings.

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

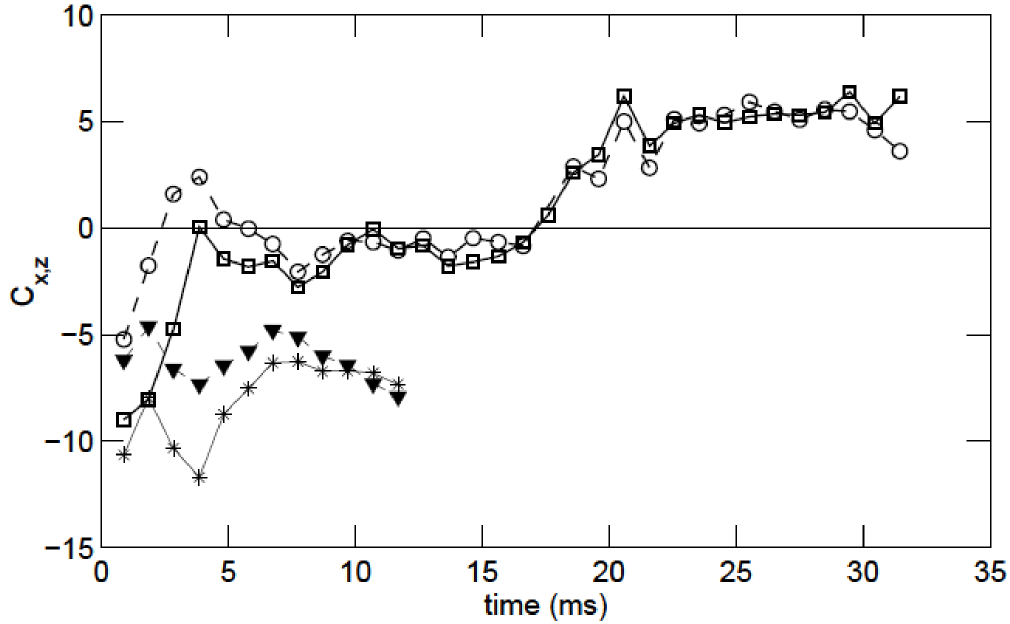
Examples:

BNL AGS (E. Blesser 1987): Chromaticities measured at the AGS.



$$C_{X,\text{nat}}^{\text{FODO}} = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

Fermilab Booster (X. Huang, Ph.D. thesis, IU 2005): The measured horizontal chromaticity C_x when SEXTS is on (triangles) or off (stars), and the measured vertical chromaticity C_y when SEXTS is on (dash, circles) or off (squares). The error bar is estimated to be 0.5. The natural chromaticities are $C_{nat,y}$=−7.1 and $C_{nat,x}$=−9.2 for the entire cycle. The betatron tunes are 6.7(x) and 6.8(y) respectively.



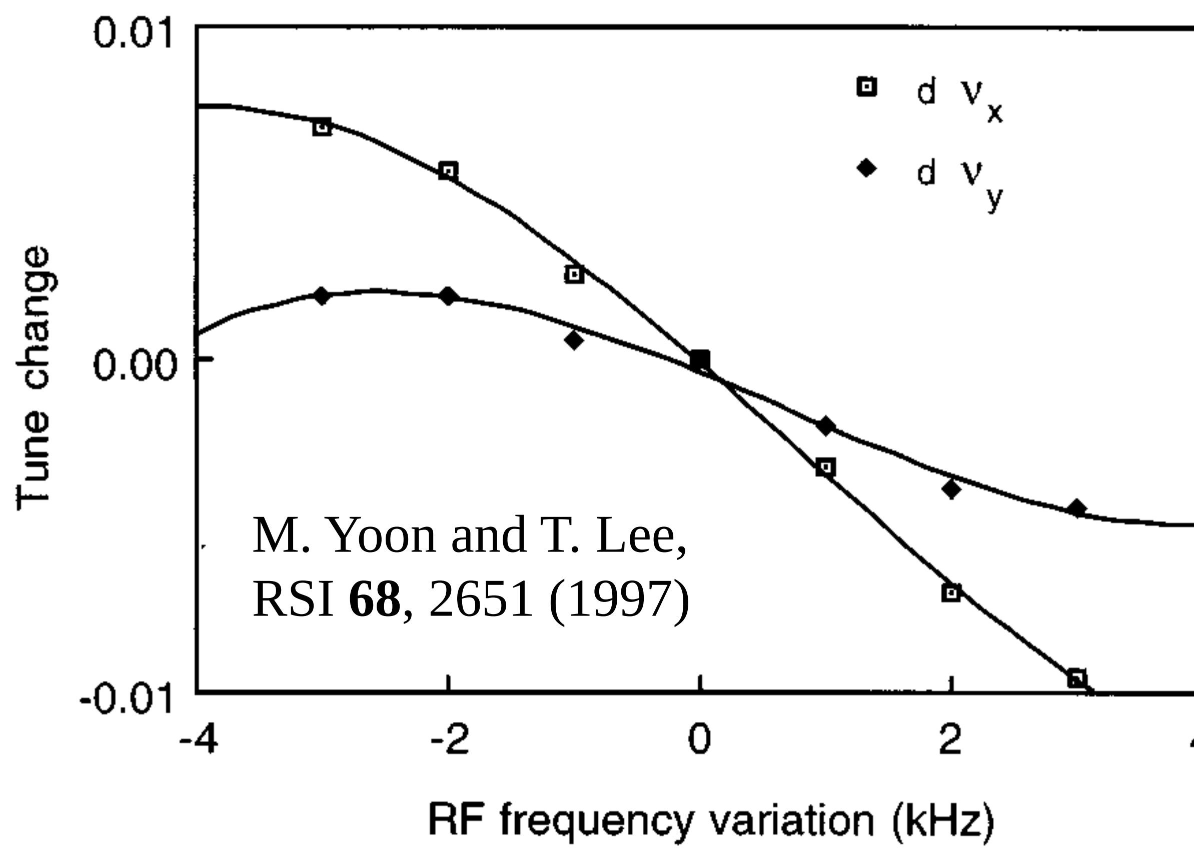
Chromaticity measurement:

The chromaticity can be measured by measuring the betatron tunes vs the rf frequency f, i.e.

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

$$\Delta f / f_0 = -\eta\delta,$$

$$C = \frac{d\nu}{dp/p} = -\eta f_{rf}\frac{d\nu}{df_{rf}}$$

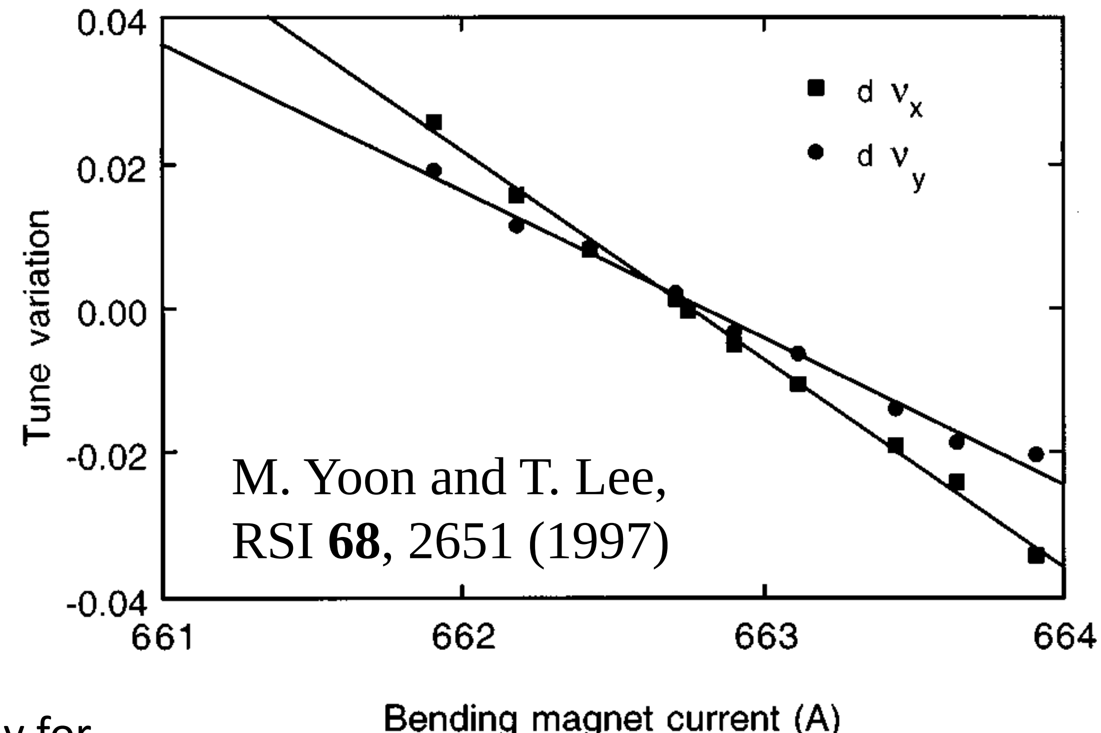


The chromaticites are $C_x = +2.9$, $C_y = +1.4$.

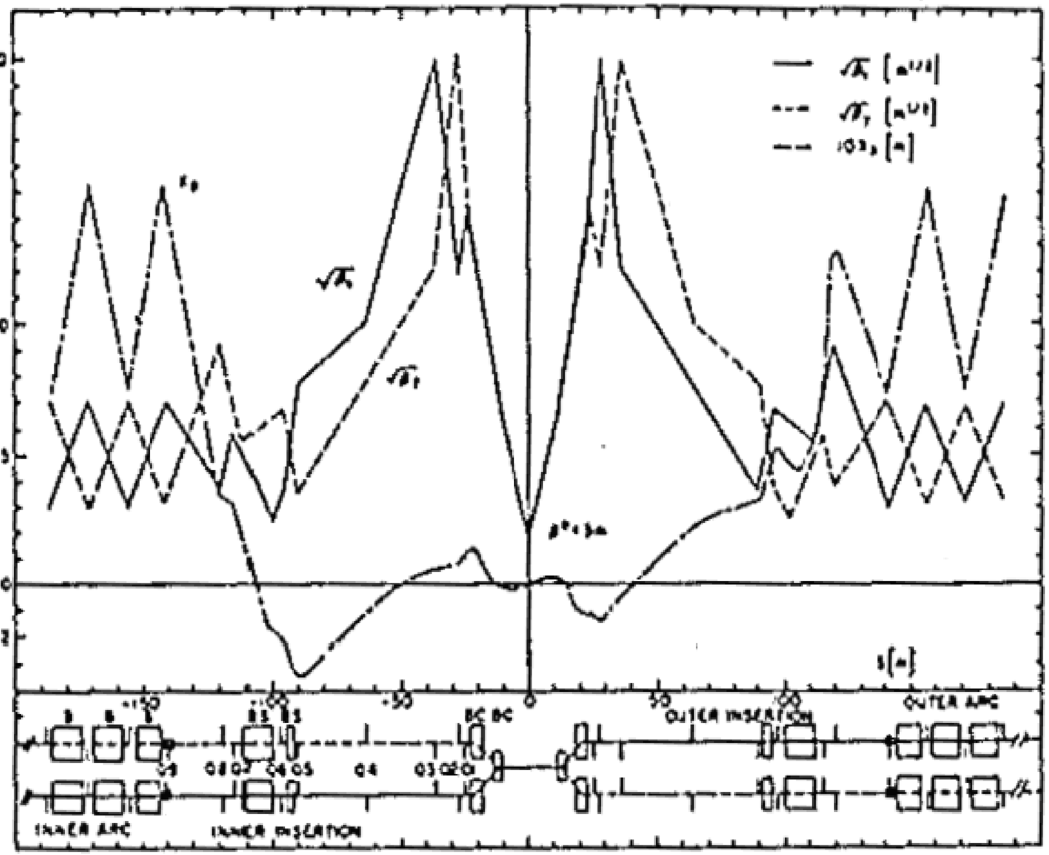
The **Natural chromaticity** can be obtained by measuring the tune variation vs the bending-magnet current at a **constant rf frequency**. Change of the bending-magnet current is equivalent to the change of the beam energy. Since the orbit is not changed, the effect of the sextupole magnets on the beam motion can be neglected. The Figure shows the horizontal and vertical tune vs the bending-magnet current in the PLS storage ring.

$$C = \frac{d\nu}{dp/p} = \frac{d\nu}{dB/B} = \frac{d\nu}{dI/I}$$

The data give $C_x = -18.96$, $C_y = -13.42$; vs theory: $C_x = -23.36$, $C_y = -16.19$.



Note that this method may not apply for combined function dipoles.

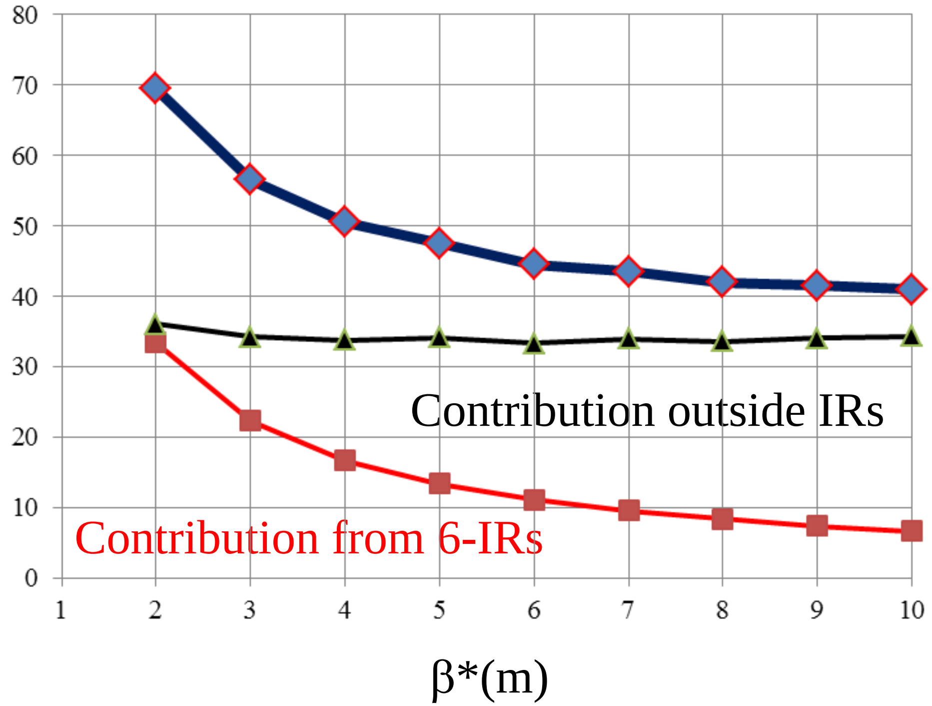


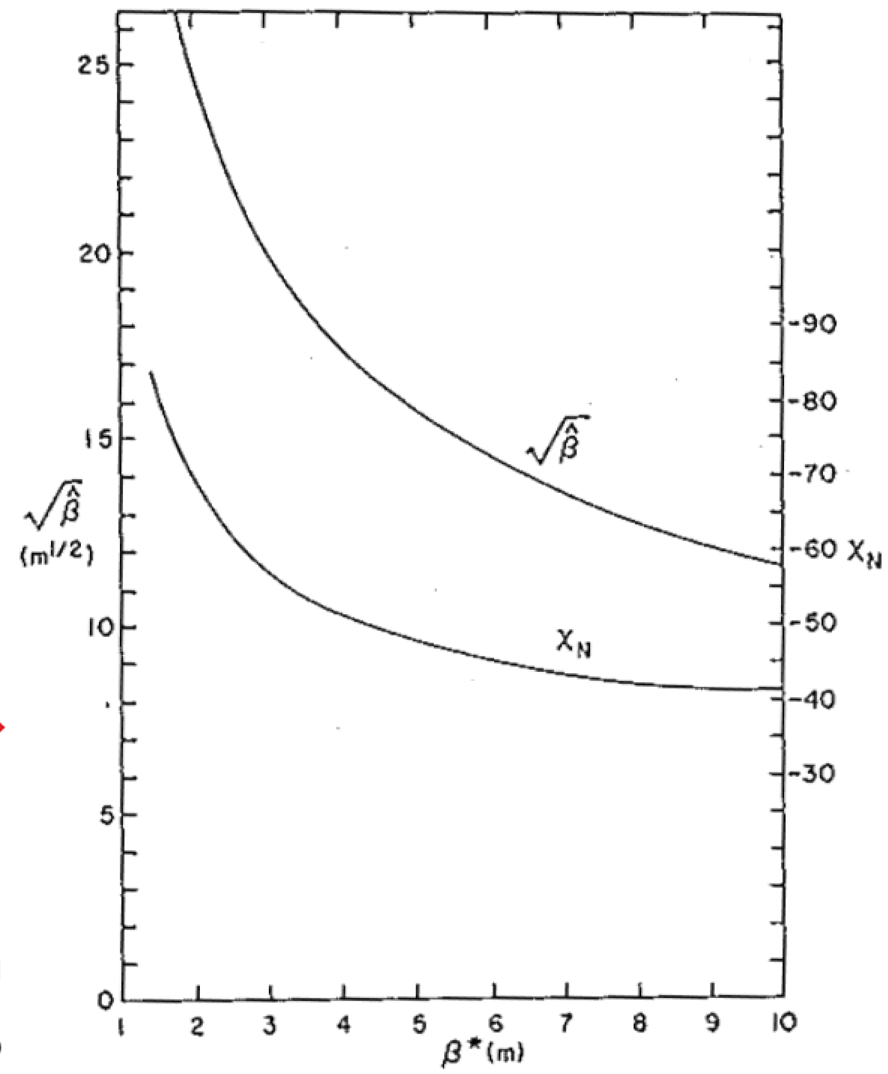
Contribution of low β triplets in an IR to the natural chromaticity is

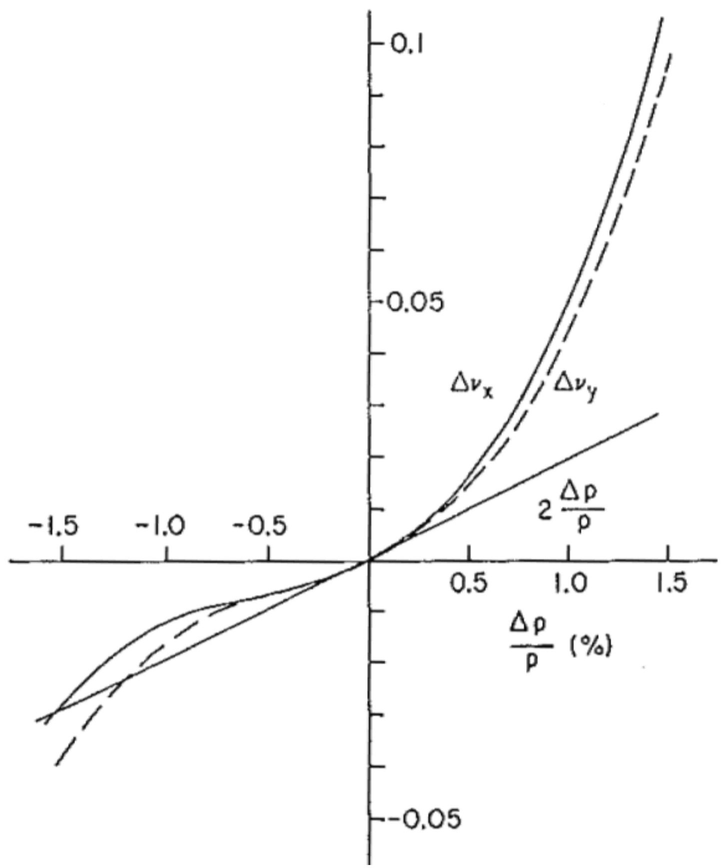
$$C_{total} = N_{IR} C_{IR} + C_{bare\,machine}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$

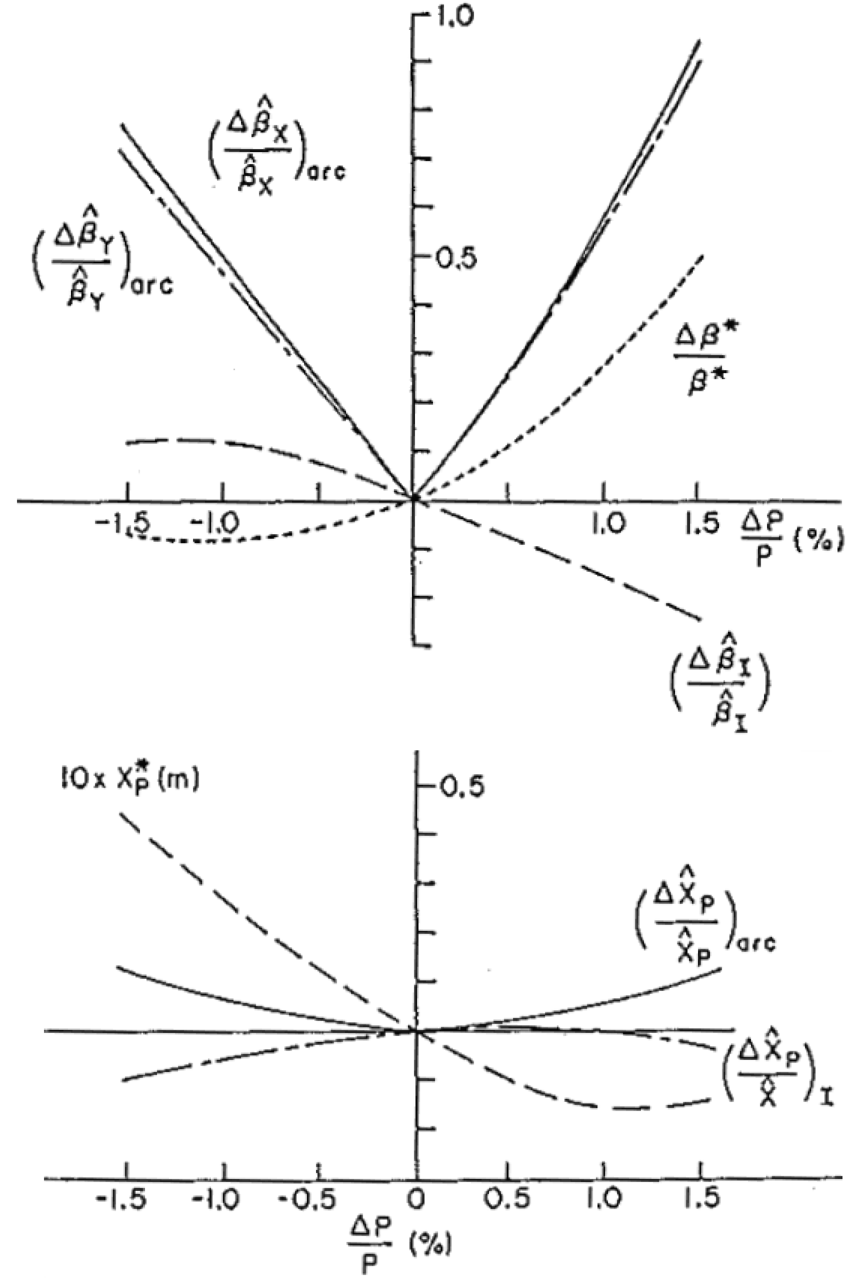
The total chromaticity is composed of contributions from the low β-quads and the rest of accelerators that is made of FODO cells. The decomposition to fit the data is **Δs≈35** m in RHIC.







ν vs Δp/p



β and D vs Δp/p

Chromaticity correction:

The chromaticity can cause tune spread to a beam with momentum spread $\Delta\nu = C\delta$. For a beam with C=-100, δ=0.005, $\Delta\nu$=0.5. The beam is not stable for most of the machine operation. Furthermore, there exists collective (head-tail) instabilities that requires positive chromaticity for stability! To correct chromaticity, we need to find magnetic field that provide stronger focusing for off-(higher)-momentum particles. We first try sextupole with

$$\Delta B_y + j\Delta B_x = B_0 b_2 (x + jy)^2, \qquad A_s = \tfrac{1}{3}\,\mathrm{Re}\left\{B_0 b_2 (x + jy)^3\right\}$$

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho} \qquad \begin{aligned} x &= x_\beta + D\delta \\ y &= y_\beta \end{aligned}$$

$$\Delta B_y = B_0 b_2 (x^2 - y^2) = B_0 b_2 (2x_\beta D\delta + D^2\delta^2 + x_\beta^2 - y_\beta^2)$$

$$\Delta B_x = B_0 b_2 2xy = B_0 b_2 2y_\beta D\delta + B_0 b_2 2x_\beta y_\beta$$

Let $K_2 = -2B_0 b_2/B\rho = -B_2/B\rho$, we obtain:

$$x''_\beta + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x''_\beta + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x = x_\beta + D\delta$$

$$\Delta K_x(s) = K_2(s)D(s)\delta, \quad \Delta K_y(s) = -K_2(s)D(s)\delta$$

$$C_x = -\tfrac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_y = -\tfrac{1}{4\pi}\oint \beta_y(s)[K_y(s) + K_2(s)D(s)]ds$$

- In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_y D_x$ are maximum.
- A large ratio of β_x/β_y for the focusing sextupole and a large ratio of β_y/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.
- The families of sextupoles should be arranged to minimize the systematic half-integer stopbands and the third-order betatron resonance strengths.

Revisit of half interger stopband intergral

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2\sin\Phi_0}\int_{\phi}^{\phi+2\pi} d\phi_1 \beta^2(\phi_1)k(\phi_1)\sin 2\nu_0(\pi+\phi-\phi_1)$$

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta(s)}{\beta(s)} + 4\nu_0^2\frac{\Delta\beta(s)}{\beta(s)} = -2\nu\beta^2 k(s)$$

$$\left[\nu_0\beta^2 k(s)\right] = \sum_{p=-\infty}^{\infty} J_p e^{jp\varphi},$$

$$J_p = \frac{1}{2\pi}\oint\left[\beta k(s)\right]e^{-jp\varphi}ds$$

Half integer stopband

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi}$$

What symmetry can do to stopbands?

Systematic chromatic half-integer stopband width

The effect of systematic chromatic gradient error on betatron amplitude modulation can be analyzed by using the chromatic half-integer stopband integrals

$$J_{p,x} = \frac{1}{2\pi}\oint \beta_x \Delta K_x e^{-jp\varphi_x} ds$$

$$J_{p,y} = \frac{1}{2\pi}\oint \beta_y \Delta K_y e^{-jp\varphi_y} ds$$

We consider a lattice made of P superperiods, where L is the length of a superperiod with K(s + L) = K(s), β(s + L) = β(s). Let C = PL be the circumference of the accelerator. The integral becomes

$$J_{p,X} = -\left\{\frac{\delta}{2\pi}\int_0^L \beta_X \Delta K_X e^{-jp\varphi} ds\right\}[1 + e^{-jp\frac{2\pi}{P}} + e^{-j2p\frac{2\pi}{P}} + \cdots + e^{-jp\frac{2\pi}{P}(P-1)}]$$

$$= -\left\{\frac{\delta}{2\pi}\int_0^L \beta_X \Delta K_X e^{-jp\varphi} ds\right\}P \quad when \quad p = 0\,(Mod\ P)$$

$$J_{p,X} = 0, \quad when \quad p \neq 0\,(Mod\ P)$$

When p = 0 (Mod P), the half-integer stopband integral increases by a factor of P, i.e. each superperiod contributes additively to the crhomatic stopband integral.

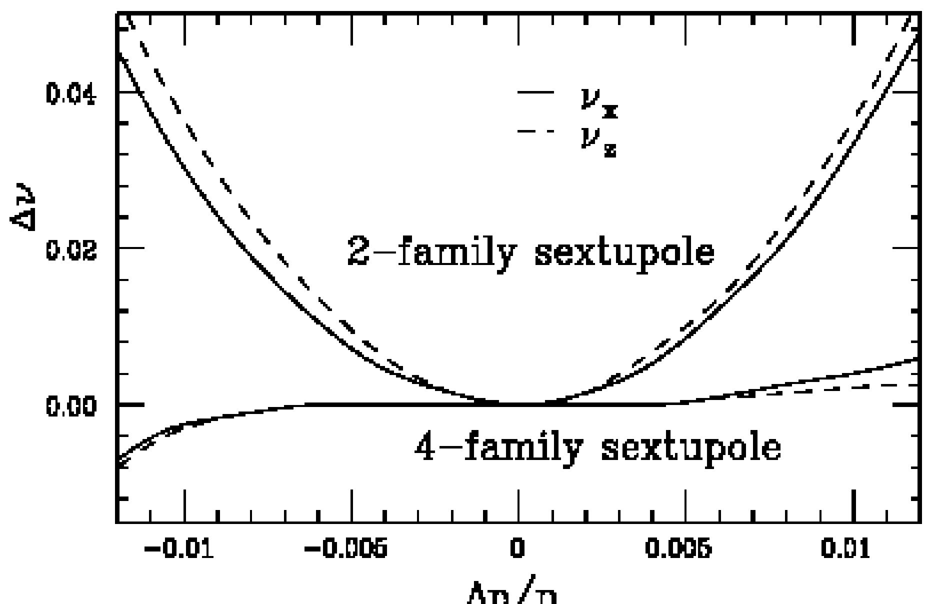
Effect of the chromatic stopbands on chromaticity

$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi} \approx -\frac{|J_p|\cos(p\phi)}{2(\nu_0 - p/2)}$$

$$\Delta\nu_X = C_X^{(1)}\delta + C_X^{(2)}\delta^2 + \cdots$$

$$C_X{}^{(1)} = -\tfrac{1}{4\pi}\oint \beta_X(s)[K_X(s) - K_2(s)D(s)]ds$$

$$C_X^{(2)} = -C_X^{(1)} - \frac{|J_{p,X}|^2}{4(\nu_X - p/2)\delta^2}$$



Effect of sextupoles on the chromatic stopband integrals

First we evaluate the stopband integral due to the chromatic sextupoles. Let S_F and S_D be the integrated sextupole strengths at QF and QD of FODO cells in the arc. The *p-th* harmonic stopband integral from these chromatic sextupoles is

$$J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P\times p\varphi/2\nu)}{\sin(p\varphi/2\nu)}[\beta_F S_F D_F + \beta_D S_D D_D e^{-jp\varphi/2\nu}]e^{-j(N-1)p\varphi/2\nu}$$

To change the stopband integral without perturbing the first order chromaticities, we group the sextupoles in four families, i.e. (S_{F1}, S_{D1}, S_{F2}, S_{D2}). By asking

$$S_{F1} \to S_{F1} + (\Delta S)_F, \qquad S_{D1} \to S_{D1} + (\Delta S)_D,$$

$$S_{F2} \to S_{F2} - (\Delta S)_F \qquad S_{D2} \to S_{D2} - (\Delta S)_D$$

$C^{(1)}$ stays same while the change in stopband integral (2^{nd}) gives

$$\Delta J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P\times(p\varphi/2\nu-\pi/2))}{\sin(p\varphi/2\nu-\pi/2)}[\beta_F(\Delta S)_F D_F + \beta_D(\Delta S)_D D_D e^{-jp\varphi/4\nu}]e^{-j(N-1)(p\varphi/2\nu-\pi/2)}$$

Under conditions $\quad p » 2n \qquad f » \frac{p}{2}$

$$\frac{\sin(P\times(p\varphi/2\nu-\pi/2))}{\sin(p\varphi/2\nu-\pi/2)} = P$$

$$C_X^{(1)} = -\frac{1}{4\pi}\oint \beta_X(s)[K_X(s) - K_2(s)D(s)]ds$$

$$C_X^{(2)} = -C_X^{(1)} - \frac{|J_{p,X}|^2}{4(\nu_X - p/2)\delta^2}$$

$$J_{p,sext} = \frac{\delta}{2\pi}\frac{\sin(P\times p\varphi/2\nu)}{\sin(p\varphi/2\nu)}[\beta_F S_F D_F + \beta_D S_D D_D e^{-jp\varphi/2\nu}]e^{-j(N-1)p\varphi/2\nu}$$

$$S_{F1} \rightarrow S_{F1} + (\Delta S)_F, \qquad S_{D1} \rightarrow S_{D1} + (\Delta S)_D, \qquad p \approx 2\nu$$

$$S_{F2} \rightarrow S_{F2} - (\Delta S)_F \qquad S_{D2} \rightarrow S_{D2} - (\Delta S)_D \qquad \varphi \approx \frac{\pi}{2}$$

$$\Delta J_{p,sext} = \frac{\delta}{2\pi}P[\beta_F(\Delta S)_F D_F + \beta_D(\Delta S)_D D_D e^{-j\pi/4}]$$

Every FODO cell contributes additively to the chromatic stopband. The resulting stopband width is proportional to $(\Delta S)_F$ and $(\Delta S)_D$ parameters. By adjusting $(\Delta S)_F$ and $(\Delta S)_D$ parameters, the betabeat and the second order Chromaticity can be minimized.

Similarly, a six-family sextupole scheme works for 60 degree phase advance FODO lattice, where The six-family scheme (S_{F1}, S_{D1}, S_{F2}, S_{D2} , S_{F3}, S_{D3}) has two additional parameters.