**Homework 14. Due November 6**

**Problem 1. 5x3 points. Beam distribution in phase space.**

Consider a simplified 1D-Hamiltonian in terms of coordinates $p_x$ and $x$

$$H = 1/2(A\,{p_x}^2 + Bx^2)$$

the equilibrium beam profile is a function of the Hamiltonian. If we assume it to have a Gaussian distribution

$$\rho = C\exp(-\frac{1}{2{\sigma_{p_x}}^2}{p_x}^2 - \frac{1}{2{\sigma_x}^2}x^2)$$

a) Find the conditions where this Gaussian distribution satisfies Vlasov's equation.

b) Find the normalization factor C where

$$\int \rho dx dp_x = 1$$

c) Calculate the rms phase-space area of the beam

Solution:

**Problem 1.**

a) For the Hamiltonian, if we want to have a Gaussian beam, we can write the distribution in

$$\rho = \frac{\sqrt{AB}}{2\pi\sigma_x\sigma_{p_x}}\exp(-\frac{1}{2{\sigma_{p_x}}^2}{p_x}^2 - \frac{1}{2{\sigma_x}^2}x^2)$$

because this distribution needs to satisfies Vlasov's equation for an equilibrium solution, we have

$$\frac{d\rho}{ds} = 0 = \frac{\partial\rho}{\partial s} + \frac{\partial\rho}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\rho}{\partial p_x}\frac{\partial p_x}{\partial s}$$

from Hamilton's equations, it is easy to obtain

$$p_x' = -Bx \qquad\qquad x' = Ap_x$$

thus

$$\frac{\partial\rho}{\partial s} = \exp\left(-\frac{1}{2{\sigma_{p_x}}^2}{p_x}^2 - \frac{1}{2{\sigma_x}^2}x^2\right)(-\frac{p_x p_x'}{{\sigma_{p_x}}^2} - \frac{xx'}{{\sigma_x}^2})$$

$$\frac{d\rho}{ds} = 0 = \exp\left(-\frac{1}{2{\sigma_{p_x}}^2}{p_x}^2 - \frac{1}{2{\sigma_x}^2}x^2\right)\left(-\frac{p_x p_x'}{{\sigma_{p_x}}^2} - \frac{xx'}{{\sigma_x}^2} - \frac{Axp_x}{{\sigma_x}^2} + \frac{Bxp_x}{{\sigma_{p_x}}^2}\right)$$

$$= 2\exp\left(-\frac{1}{2{\sigma_{p_x}}^2}{p_x}^2 - \frac{1}{2{\sigma_x}^2}x^2\right)(-\frac{Axp_x}{{\sigma_x}^2} + \frac{Bxp_x}{{\sigma_{p_x}}^2})$$

the requirement reads

$$-\frac{A}{{\sigma_x}^2}+\frac{B}{{\sigma_{p_x}}^2}=0$$

b) Normalization factor as shown above is $\frac{\sqrt{AB}}{2\pi\sigma_x\sigma_{p_x}}$, where $\sigma_x = \sqrt{\frac{A}{B}}\sigma_{p_x}$

c) The rms area of the beam (being an ellipse) is $\pi\sigma_x\sigma_{p_x}$