HomeWorks 4-5 – catching up

Problem 1. 7 points. Long elements.

Prelude: Many elements of accelerators are straight – e.g. coordinate system is simply Cartesian (x,y,s=z). It allows you to forget about curvilinear coordinates and use simple div and curl and Laplacian… Many of them are DC - e.g. either with constant or nearly constant EM fields. Again, Maxwell equations without time derivatives – EM static. Furthermore, many of them are also long – e.g. have a constant cross-section with transverse size much smaller than the length of the element. It means that you can drop derivatives over z. Finally, all current and charges generating field are outside of the vacuum where particles propagate – e.g. Maxwell static equations are also homogeneous – charge and current densities are zero! It should come as no surprise – everybody like to have a solvable problem to rely upon.

(a) use electro-static equations for a long uniform electric element and show that

$$\vec{E} = \vec{\nabla}\,\mathrm{Re}\left[a_n\left(x+iy\right)^n\right] \tag{1}$$

satisfy static Maxwell equations with a_n being a complex number. Electric elements with real a_n call regular elements (they have plane symmetry!), element with imaginary a_n are called skew .

(b) use magneto-static equations for a long uniform magnetic element

$$\vec{B} = \vec{\nabla}\,\mathrm{Re}\left[b_n\left(x+iy\right)^n\right] \tag{2}$$

satisfy static Maxwell equations with *b* being a complex number. Magnetic elements with imaginary b_n call regular elements (they have plane symmetry!), element with real b_n are called skew.

(c) show that arbitrary combination of elements from (1) and (2) is also a solution of electrostatic equations.

Note: elements with various n have specific names: n=1 – dipole, n=2 – quadrupole, n=3 – sextupole, n=4 – octupole, …. Or 2n-pole element. Term "skew" is added as needed to names of quadrupole and higher order element. It also obvious that an arbitrary 2n-pole "element" can be constricted as combination a regular and a skew fields.

Solution: Most of Maxwell equations are satisfied automatically:

$$\vec{E} = \vec{\nabla}\phi_e = \vec{\nabla}\cdot\sum_{n=1}^{\infty}\phi_{en};\quad \vec{B} = \vec{\nabla}\phi_b = \vec{\nabla}\cdot\sum_{n=1}^{\infty}\phi_{bn}$$

$$\text{(a)}\qquad \phi_{en} = \mathrm{Re}\left[a_n\left(x+iy\right)^n\right];\phi_{bn} = \mathrm{Re}\left[b_n\left(x+iy\right)^n\right];$$

$$curl\vec{E} = curl\left(\vec{\nabla}\phi_e\right) \equiv 0;\quad curl\,\vec{B} = curl\left(\vec{\nabla}\phi_b\right) \equiv 0;$$

the only non-trivial equations remain are:

$$\text{(b)}\qquad \begin{array}{l} \phi_{en} = \mathrm{Re}\left[a_n\left(x+iy\right)^n\right];\phi_{bn} = \mathrm{Re}\left[b_n\left(x+iy\right)^n\right]; \\ div\vec{E} = \vec{\nabla}\cdot\left(\vec{\nabla}\phi_e\right) = \Delta\phi_e \equiv 0;\quad div\,\vec{B} = \vec{\nabla}\cdot\left(\vec{\nabla}\phi_b\right) = \Delta\phi_b \equiv 0; \end{array}$$

What we have to prove is trivial:

$$\Delta \operatorname{Re}\left[a_n(x+iy)^n\right] = \operatorname{Re}\left[a_n \cdot \Delta(x+iy)^n\right] = 0;$$

$$\text{(b)} \quad \Delta(x+iy)^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(x+iy)^n =$$

$$= n(n-1)(x+iy)^{n-2}(1+i^2) = 0$$

Needless to say, that we discussed that the one of most important features of EM fields is principle of superposition: if two fields are satisfying Maxwell equations, then their linear combinations also satisfy the equations.

What is really unusual is that we expressed magnetic field as a gradient of a scalar potential – it is only possible in the area where $curl\,\vec{B} = 0$, i.e. in the absence of currents and time dependent electric field! Do not try this for AC fields!

Problem 2. 10 points. Edge effects.

(a) We continue with Cartesian (x,y,s=z) coordinates for a straight element. But now we will suggest that field in this element depends on z;

$$\vec{E}, \vec{B} = \vec{\nabla} \operatorname{Re}\left[a_n(z)(x+iy)^n\right] \tag{3}$$

Show that such elements will generate terms in the field which are not a higher order multi-poles (1) or (2). Prove that a sum of higher order multi-poles with amplitudes dependent on z cannot be a solution for edge field.

(b) You proved that simple combination of field multipoles can not describe the edge of a magnet. You also learned that we can used Laplacian equation on effective field potential:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0$$

Let expand the potential in transverse direction while keeping arbitrary dependence along the beam propagating axis (s=z)

$$\varphi = \sum_{n+m=k}^{\infty} a_{nm}(z) x^n y^m$$

Derive the condition (connections) between functions $a_{nm}(z)$.

Solution:

(a) Similar to problem 1, there is only one not-trivial equation for E or B:

;

$$\Delta\left\{a_n(z)(x+iy)^n\right\} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left\{a_n(z)(x+iy)^n\right\} =$$

$$= \frac{\partial^2 a_n(z)}{\partial z^2}(x+iy)^n \neq 0$$

Since uniform $x\,,y$ polynomials of n-th order cannot be canceled by those of different order, this solution is invalid.

(b)

$$\varphi = \sum_{n+m=k}^{\infty} a_{nm}(z)x^n y^m$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\sum_{n+m=k}^{\infty} a_{nm}(z)x^n y^m =$$

$$\sum_{n+m=k}^{\infty}(a''_{nm}+(n+2)(n+1)a_{n+2,m}+(m+2)(m+1)a_{n,m+2})x^n y^m = 0$$

$$(n+2)(n+1)a_{n+2,m}+(m+2)(m+1)a_{n,m+2} = -a''_{nm}$$

It means that a "multipole" of kth order will generate terms $a_{n,k-n+2}$ where n=1,…k+2
No lower order terms are generated!

Problem 3. **8 points**. Prove what we discussed in class:

$$\det\left[I+\varepsilon A\right] = 1+\varepsilon\cdot Trace[A]+O\left(\varepsilon^2\right)$$

where I is unit *nxn* matrix, A is an arbitrary *nxn* matrix and ε is infinitesimally small real number. Term $O\left(\varepsilon^2\right)$ means that it contains second and higher orders of ε.

Hint: first, look on the diagonal elements $\prod_{m=1}^{n}\left(1+\varepsilon a_{mm}\right)$ first, then see what contribution to determinant comes from non-diagonal terms $a_{km}; k \neq m$.

Solution: The contribution to determinant from the diagonal elements is

$$\prod_{m=1}^{n}\left(1+\varepsilon a_{mm}\right) = 1+\varepsilon\sum_{m=1}^{n} a_{mm}+O\left(\varepsilon^2\right) = 1+\varepsilon\cdot Trace\left[A\right]+O\left(\varepsilon^2\right) \quad (1)$$

A generic term containing a non-diagonal element $a_{km}; k \neq m$, excludes from the product at least two diagonal elements $1+\varepsilon a_{mm}$ and $1+\varepsilon a_{kk}$.

$$\pm e_{m.....}e_{k.....}\varepsilon a_{m,k}\prod_{i\neq m;j\neq k}^{n} a_{i,j}\left(\delta_{ij}+\varepsilon a_{i,j}\right)$$

Since the total number of elements in the product is n, such term contains at least two non-diagonal elements, each of which contains ε. This proves that non-diagonal terms can contribute only second and higher order term into $O\left(\varepsilon^2\right)$. Combining it with (1) finishes the proof.