Transverse (Betatron) Motion

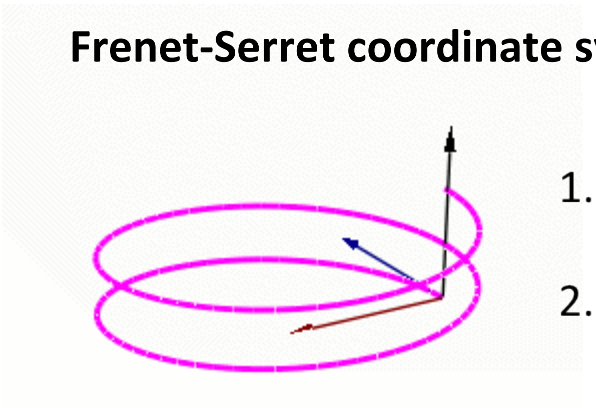
Linear betatron motion

Dispersion function of off momentum particle

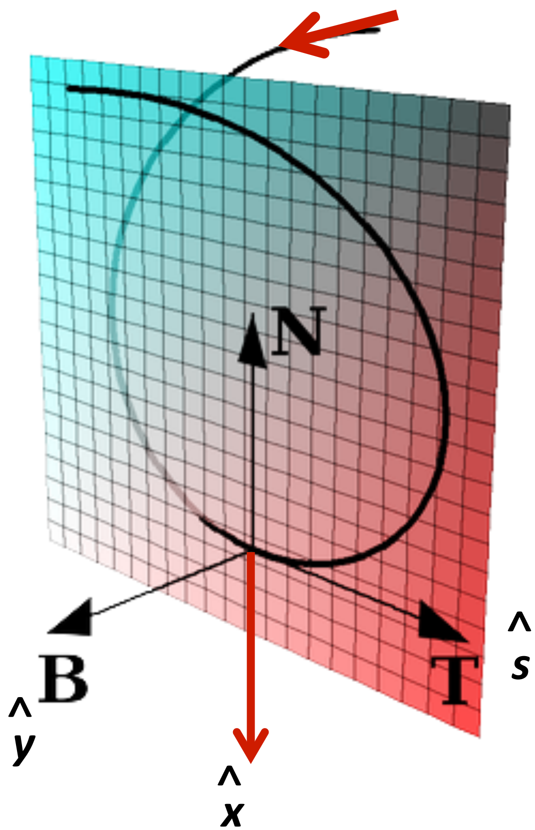
Simple Lattice design considerations

Nonlinearities

Frenet-Serret coordinate system:

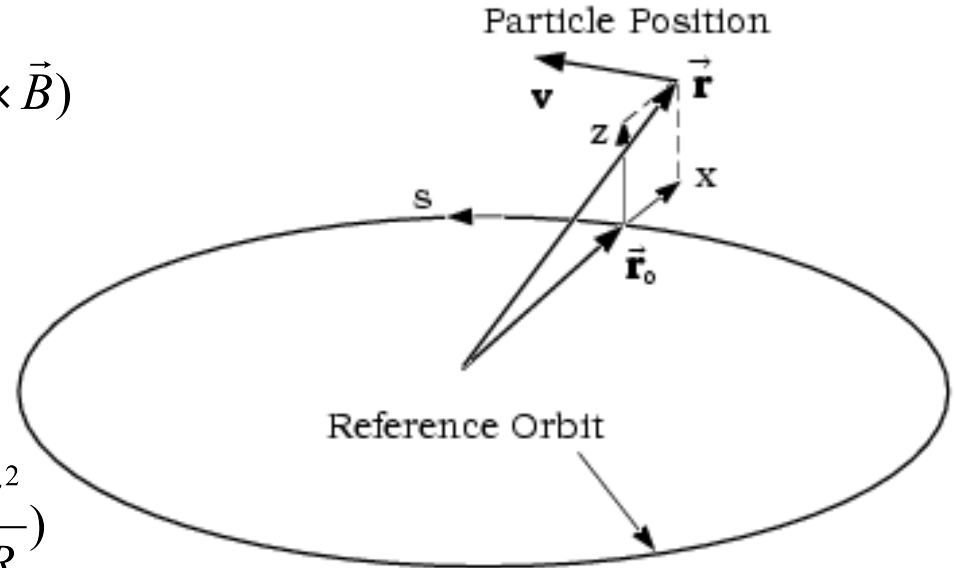


1. The tangential vector points toward the direction of beam motion
2. We define the normal vector points outward from the curve. Magnetic field direction y. So that $s=-x\times y$ (electron for this case)



http://en.wikipedia.org/wiki/Frenet-Serret

Lorentz force: $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$



$$F_x = \frac{d}{dt}(\gamma m \dot{x}) = \frac{\gamma m \beta^2 c^2}{R} = -q\beta c B_y$$

$$F_y = \frac{d}{dt}(\gamma m \dot{y}) = q\beta c B_x$$

$$\frac{d}{dt}(\gamma m \dot{x}) \approx \gamma m(\ddot{x} - \frac{v^2}{R}) = \gamma m(\frac{\beta^2 c^2}{R^2}\frac{d^2 x}{d\theta^2} - \frac{v^2}{R})$$

How to transform from the original coordinate system into the Frenet-Serret coordinate system?

$$\theta = \frac{s}{R} = \frac{\beta c t}{R}$$

$$n = -\frac{\rho}{B_0}(\frac{\partial B_y}{\partial x})_{x=0}$$ → Constant when $B_y \sim x^0$ or x^1

$$\frac{d^2 x}{d\theta^2} + (1 + \frac{\rho}{B_0}\frac{\partial B_y}{\partial x})x = 0 \qquad \frac{d^2 y}{d\theta^2} + (-\frac{\rho}{B_0}\frac{\partial B_y}{\partial x})y(1 + 2\frac{x}{\rho}) = 0$$

$$\frac{d^2 x}{d\theta^2} + (1 - n)x = 0 \qquad \frac{d^2 y}{d\theta^2} + ny = 0$$

$$\frac{dx}{d\theta} = \sqrt{1-n}(-A\sin\sqrt{1-n}\theta + B\cos\sqrt{1-n}\theta) \qquad \frac{dy}{d\theta} = \sqrt{n}(-C\sin\sqrt{n}\theta + D\cos\sqrt{n}\theta)$$

Number of transverse oscillation in one beam revolution ~ $\sqrt{1-n} / \sqrt{n}$, weak focusing!

For weak focusing, stable solution requires 0<n<1, which makes transverse (betatron) oscillation having a tune (number of circles per revolution) $Qx, Qy = \sqrt{1-n}, \sqrt{n}$ less than 1. The beam sizes in such machines scales with $C^{1/2}/(1-n)^{1/4}, C^{1/2}/n^{1/4}$

For a modern machine, especially light sources where smaller beam sizes are required to achieve higher beam brightness, strong focusing is required => external focusing magnets (quadrupoles) are often used.

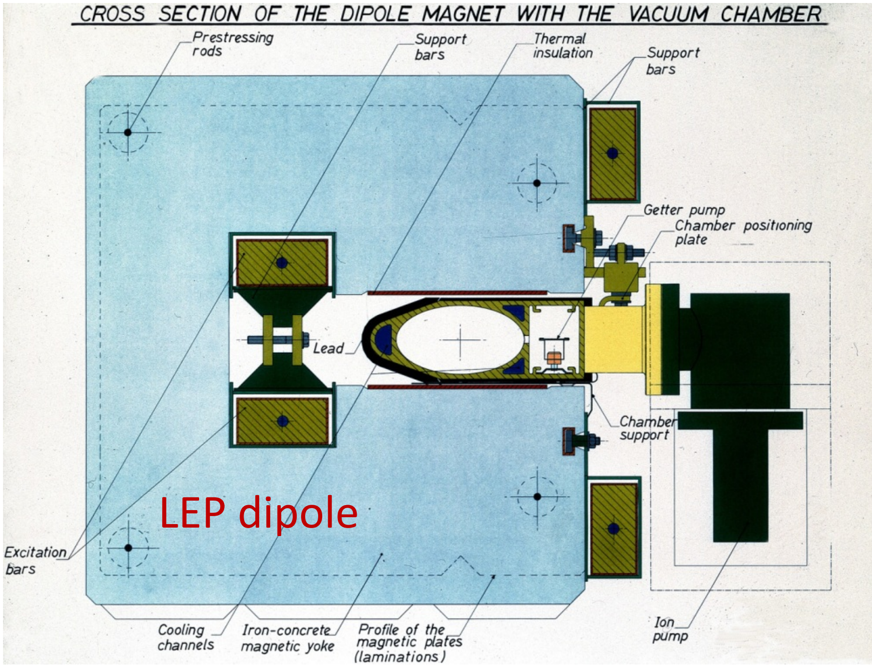
$$x'' + K_x(s)x = \pm \frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp \frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm \frac{B_1}{B\rho}$$

Higher order magnet, usually field errors

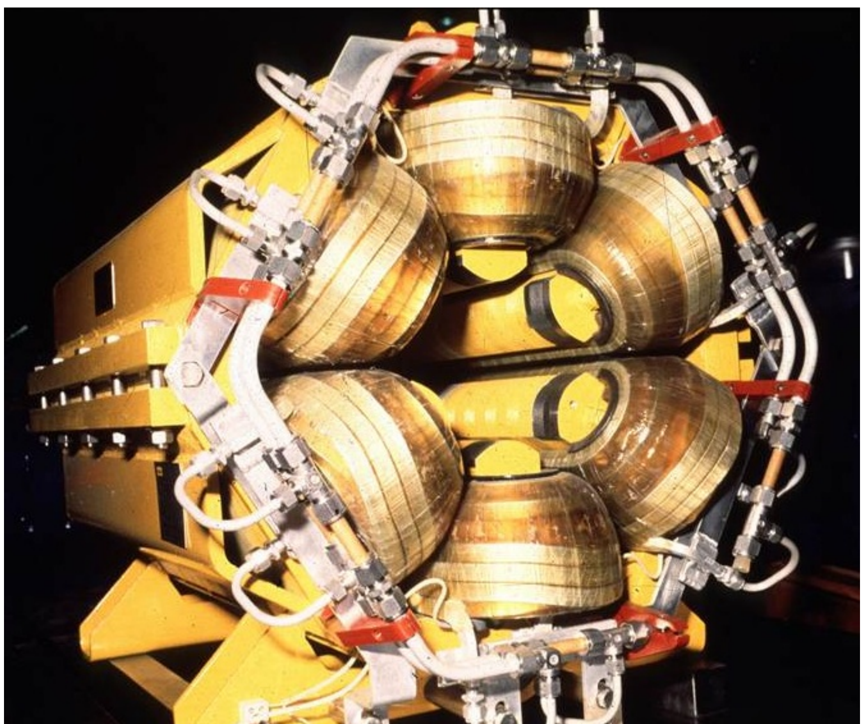
Natural focusing from dipoles

Focusing from quadrupoles



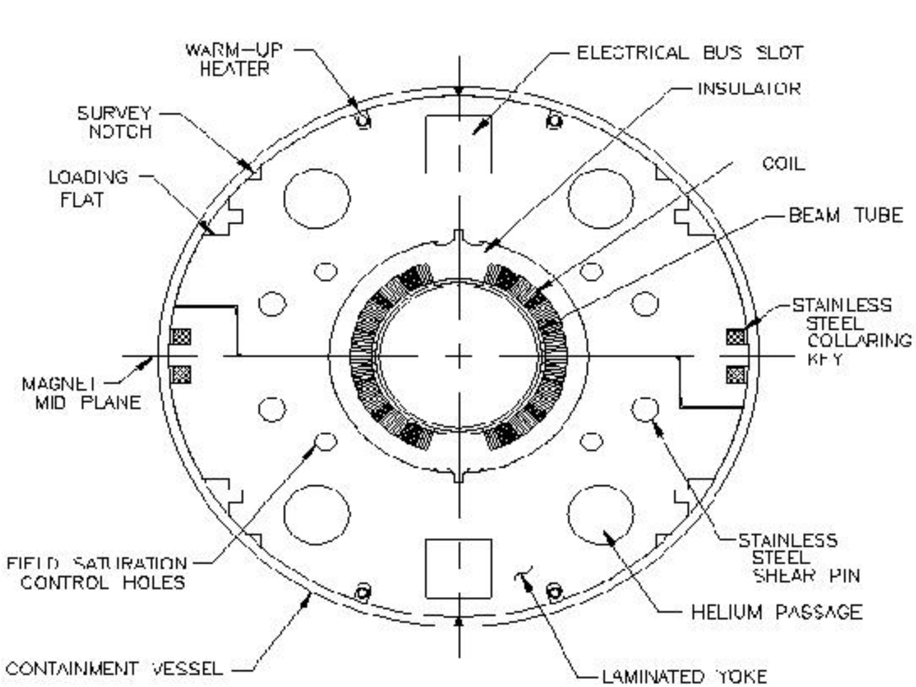
LEP dipole

ISR quadrupole

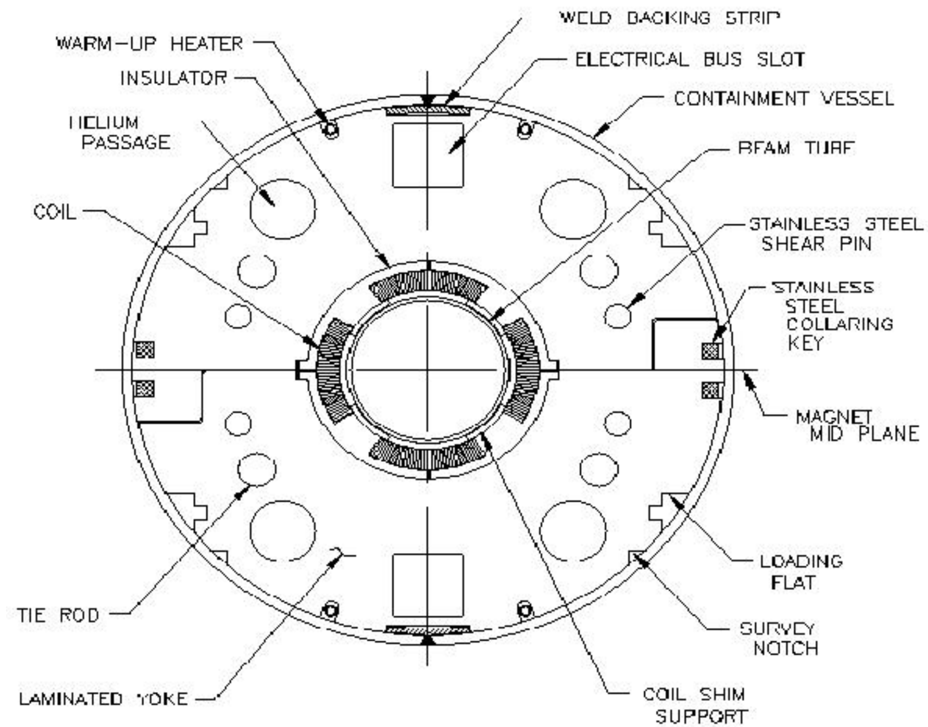


LEP Sextupole

RHIC ARC DIPOLE



RHIC ARC QUADRUPOLE



For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**.

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

$$B_x = -\frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial y} = -\frac{1}{h_s}\frac{\partial A_s}{\partial y}, B_y = \frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial x} = \frac{1}{h_s}\frac{\partial A_s}{\partial x}$$

For h_s=1 or ρ=∞, one obtains the multipole expansion:

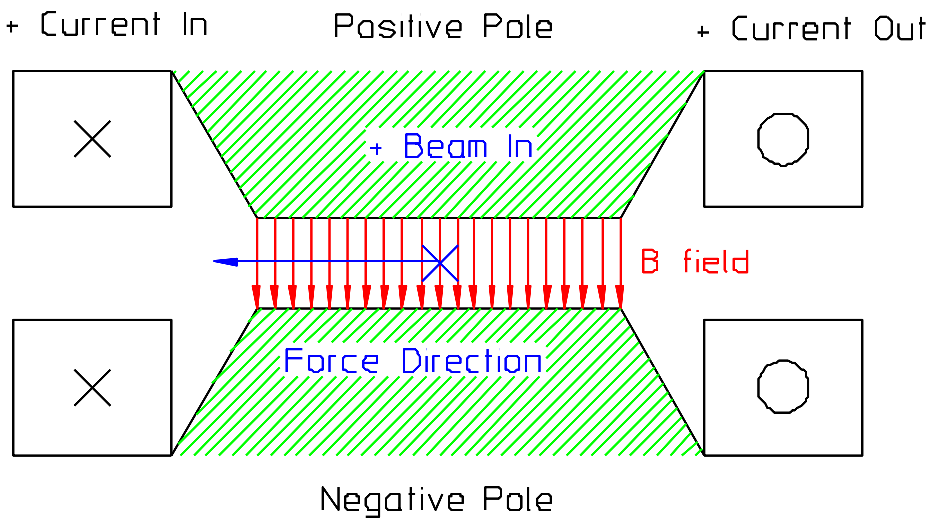
$$B_y + jB_x = B_0\sum_n (b_n + ja_n)(x+jy)^n, \qquad A_s = \mathrm{Re}\left\{B_0\sum_n \frac{1}{n+1}(b_n + ja_n)(x+jy)^{n+1}\right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_y = B_0 b_0$, $B_x = B_0 a_0$,

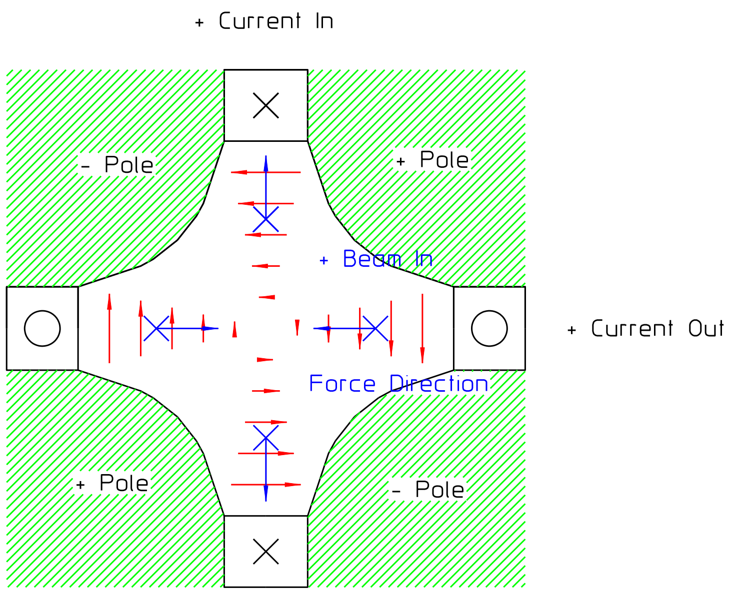
b_1 : quad, a_1 : skew quad; $B_y = B_0 b_1 x$, $B_x = B_0 b_1 y$, $B_y = -B_0 a_1 y$, $B_x = B_0 a_1 x$,

b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_y + jB_x) = \mp\frac{1}{\rho}\sum_n (b_n + ja_n)(x+jy)^n$$

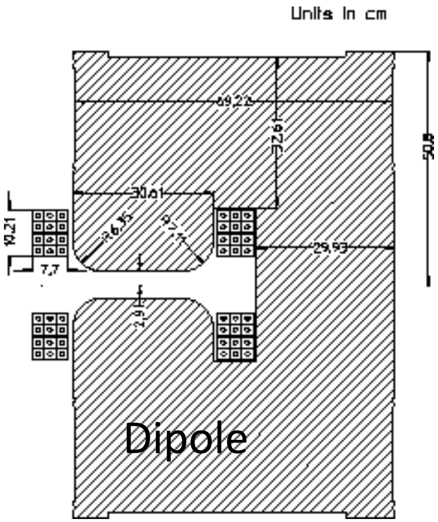


$$B_0 = \frac{\mu_0 N I}{g}, \quad A_s = B_0 x$$



$$B_1 = \frac{2\mu_0 N I}{a^2}, \quad A_s = \frac{B_1}{2}\left(x^2 - z^2\right)$$

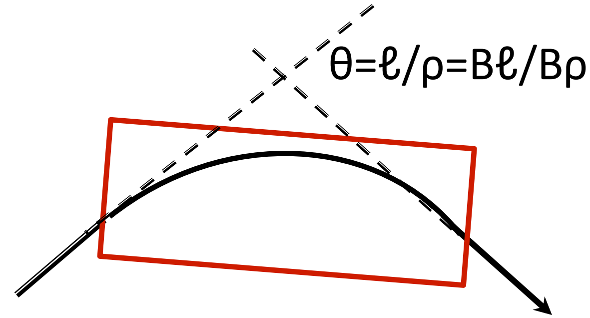
We will learn how to optimize the arrangement of beam control elements in achieving the wanted beam properties.



$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

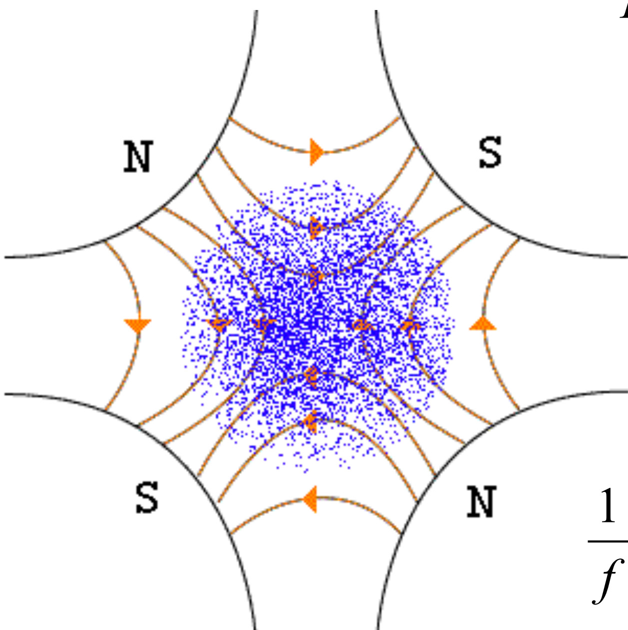
$$\gamma m \frac{\upsilon^2}{\rho} = q \upsilon B$$

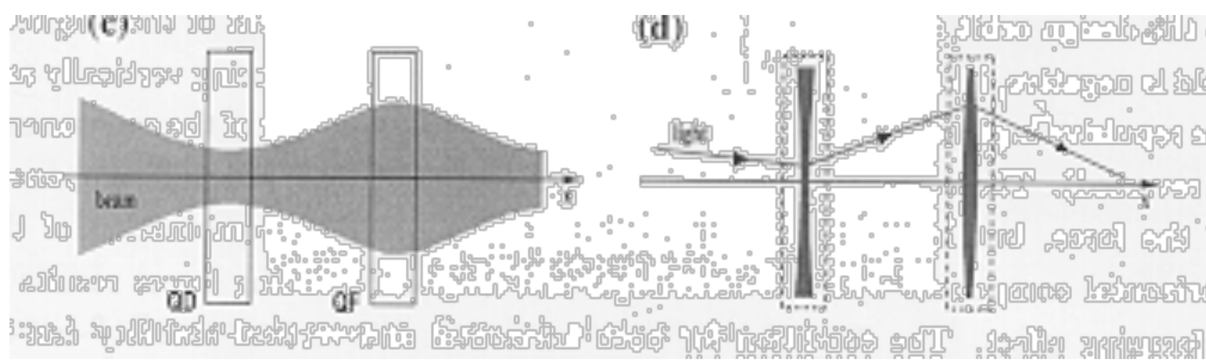
$$\rho = \frac{\gamma m v}{qB} = \frac{p}{qB}$$



quadrupole

$$B\rho[T-m] = \frac{p}{q} = \frac{A}{Z} \times 3.33564 \times p[GeV/c/u]$$



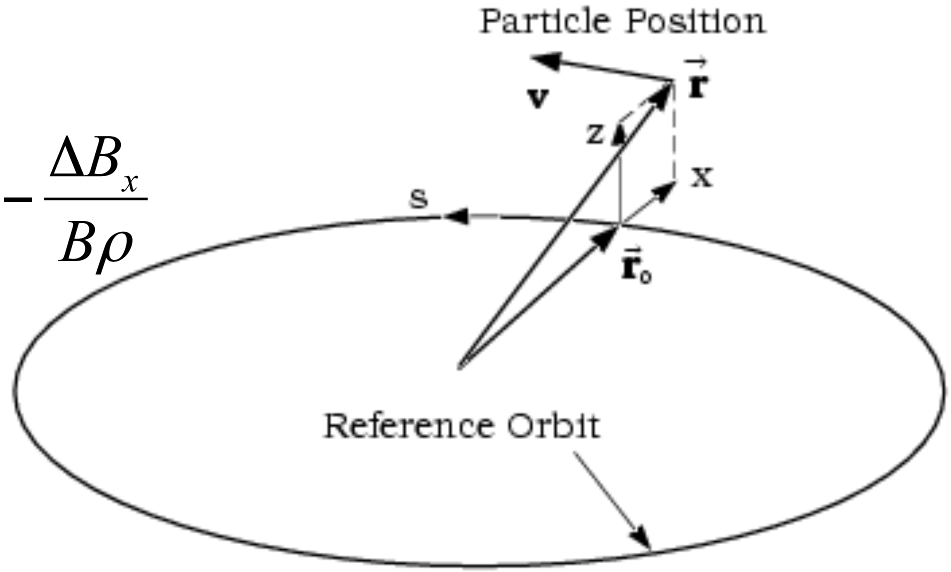


$$\frac{1}{f} = \mp \frac{B_1 \ell}{B\rho}$$

f>0, if focusing, f<0 if defocusing

How to solve the Hill's equation?

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$



Ideal Linear accelerator:

$$x'' + K_x(s)x = 0, \quad y'' + K_y(s)y = 0$$

Let X represent x or y:

$$X'' + K(s)X = 0,$$

$$X(s) = \begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s, s_0) \begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

M(s,s_0) is the betatron transfer matrix. For any two linearly independent solutions y_1, y_2 of Hill's equation, the Wronskian is independent of time

$$W(y_1, y_2, s) \equiv y_1 y_2' - y_1' y_2, \quad \frac{dW}{ds} = 0. \qquad W(s) = [\det M]W(s_0)$$

Det(M(s_2,s_1))=1

The focusing function is piecewise constant!

$$K(s) = \begin{cases} +K \\ -K \\ 0 \end{cases} \qquad y = \begin{cases} A\sin\sqrt{K}s + B\cos\sqrt{K}s \\ A\sinh\sqrt{K}s + B\cosh\sqrt{K}s \\ A + Bs \end{cases}$$

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}(s-s_0) & \frac{1}{\sqrt{K}}\sin\sqrt{K}(s-s_0) \\ -\sqrt{K}\sin\sqrt{K}(s-s_0) & \cos\sqrt{K}(s-s_0) \end{pmatrix}, \quad \ldots$$

$$\ell = s - s_0.$$

$$M(s|s_0) = \begin{cases} \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} & K > 0\text{: focusing quad.} \\ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} & K = 0\text{: drift space} \\ \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} & K < 0\text{: defocussing quad.} \end{cases}$$

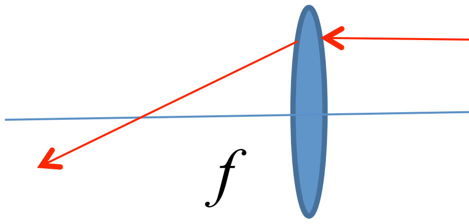
In thin-lens approximation with $\ell \to 0$, the transfer matrix for a quadrupole reduces to

$$M_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \quad M_{\text{defocussing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \qquad f = \lim_{\ell \to 0} \frac{1}{|K|\ell}$$

$$X'' + K(s)X = 0, \qquad K_x(s) = \frac{1}{\rho^2} \mp \frac{1}{B\rho}\frac{\partial B_z}{\partial x}, \quad K_z(s) = \pm \frac{1}{B\rho}\frac{\partial B_z}{\partial x},$$

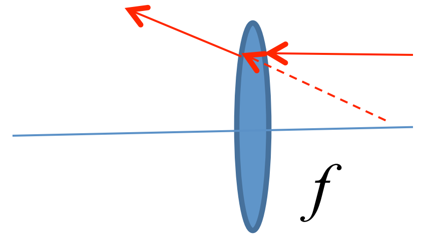
Thin lens approximation: Let $|K|\ell \to 1/f$ as $\ell \to 0$.

1. focusing quadrupole:



$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

2. de-focusing quadrupole:



$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

3. Dipole: $K_x(s)=1/\rho^2$.

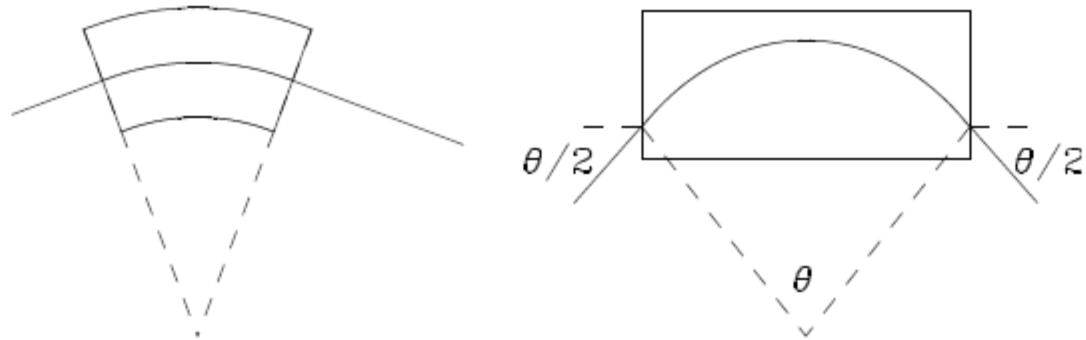
$$M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix} \to \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

4. Drift space: K=0

$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Sector dipole

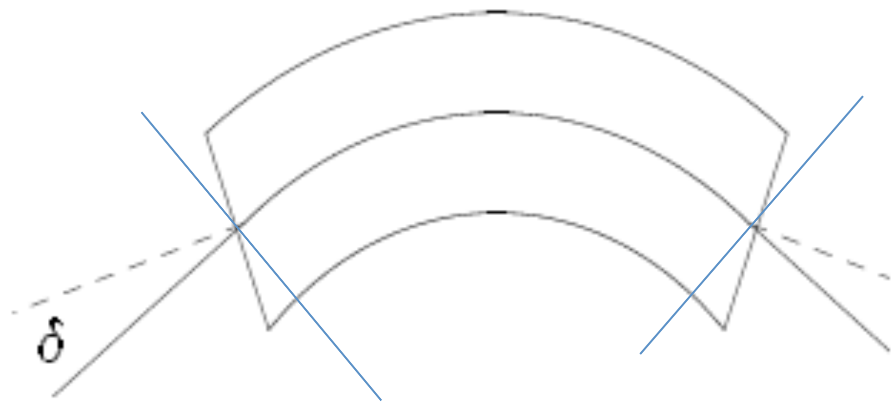
$$M(s, s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix}$$



(a) sector dipole (b) rectangular dipole

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{pmatrix} \quad M_z = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix}$$

where δ is the entrance or the exit angle of the particle with respect to the normal direction of the dipole edge. Thus the edge effect with $\delta > 0$ gives rise to horizontal defocussing and vertical focusing.



Using edge focusing, the zero-gradient synchrotron (ZGS) was designed and constructed in the 1960's at Argonne National Laboratory. The ZGS was made of 8 dipoles with a circumference of 172 m attaining the energy of 12.5 GeV. Its first proton beam was commissioned on Sept. 18, 1963. See L. Greenbaum, A Special Interest (Univ. of Michigan Press, Ann Arbor, 1971).

The most general representation of the matrix **M**(s) with **unit modulus** is given by the **Courant-Snyder** parameterization.

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

The ambiguity in the sign of sin can be resolved by requiring β to be a positive definite number if |Trace(M)| ≤ 2, and by requiring Im(sinΦ)>0 if |Trace(M)| > 2. The definition of the phase factor is still ambiguous up to an integral multiple of 2π. This ambiguity will be resolved when the matrix is tracked along the accelerator elements. Using the property of matrix J, we obtain the De Moivere's theorem:

$$\mathbf{M}^k = (\mathbf{I}\cos\Phi + \mathbf{J}\sin\Phi)^k = \mathbf{I}\cos k\Phi + \mathbf{J}\sin k\Phi,$$
$$\mathbf{M}^{-1} = \mathbf{I}\cos\Phi - \mathbf{J}\sin\Phi.$$

The necessary and sufficient condition for **stable orbital motion** is that all matrix elements of the matrix $[\mathbf{M}(s)]^{mP}$ remain bounded as m increases. Let λ_1, λ_2 be the eigenvalues and v_1, v_2 be the corresponding eigenvectors of the matrix **M**. Since M has a unit determinant, the eigenvalues are the reciprocals of each other, i.e. $\lambda_1=1/\lambda_2$, and $\lambda_1+\lambda_2=\text{Trace}(\mathbf{M})$. The eigenvalue satisfies the equation

$$\lambda^2 - Trace(M)\lambda + 1 = 0$$

Let Trace(**M**) = 2 cos(Φ), where Φ is real if Trace(**M**) ≤ 2, and complex if Trace(**M**) > 2. The eigenvalues are $\lambda_1=e^{j\Phi}$ and $\lambda_2=e^{-j\Phi}$, where Φ is the betatron phase advance of a periodic cell. Expressing the initial condition of beam coordinates (X_0, X'_0) as a linear superposition of the eigenvectors:

$$\begin{pmatrix} X_0 \\ X'_0 \end{pmatrix} = a v_1 + b v_2$$

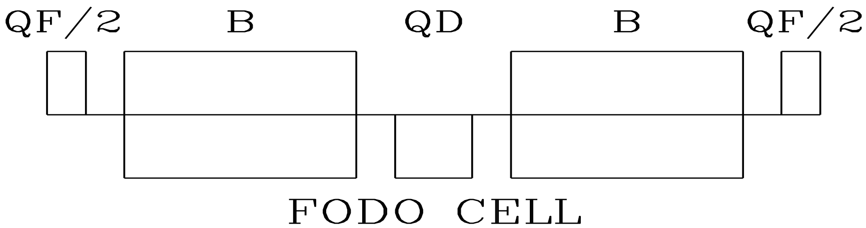
we find that the particle coordinate after the mth revolution becomes

$$\begin{pmatrix} X_m \\ X'_m \end{pmatrix} = M^m \begin{pmatrix} X_0 \\ X'_0 \end{pmatrix} = a\lambda_1^m v_1 + b\lambda_2^m v_2$$

The stability of particle motion requires that λ_1^m and λ_2^m not grow with m. Thus a necessary condition for orbit stability is to have a real betatron phase advance Φ, or

$$Trace(M) \le 2$$

Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

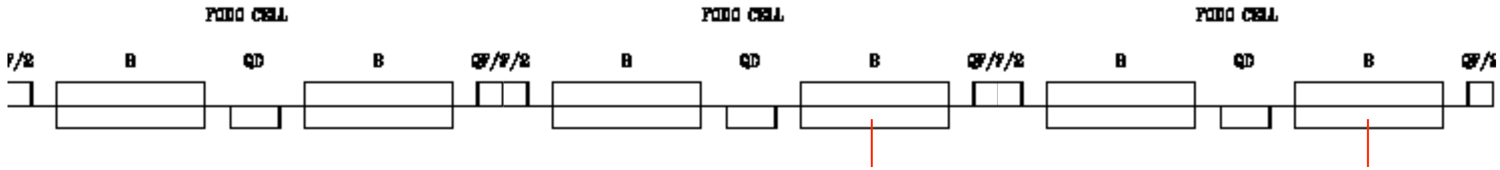
$$\cos\Phi = \tfrac{1}{2}\mathrm{Tr}(\mathrm{M})$$

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

$$\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

Example: FODO cell



$$\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

$$\begin{pmatrix}1 & \frac{L}{4}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{3L}{4}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

Questions:

1) Will Φ of these above matrix be identical?

2) Will α and β of these matrices be identical?

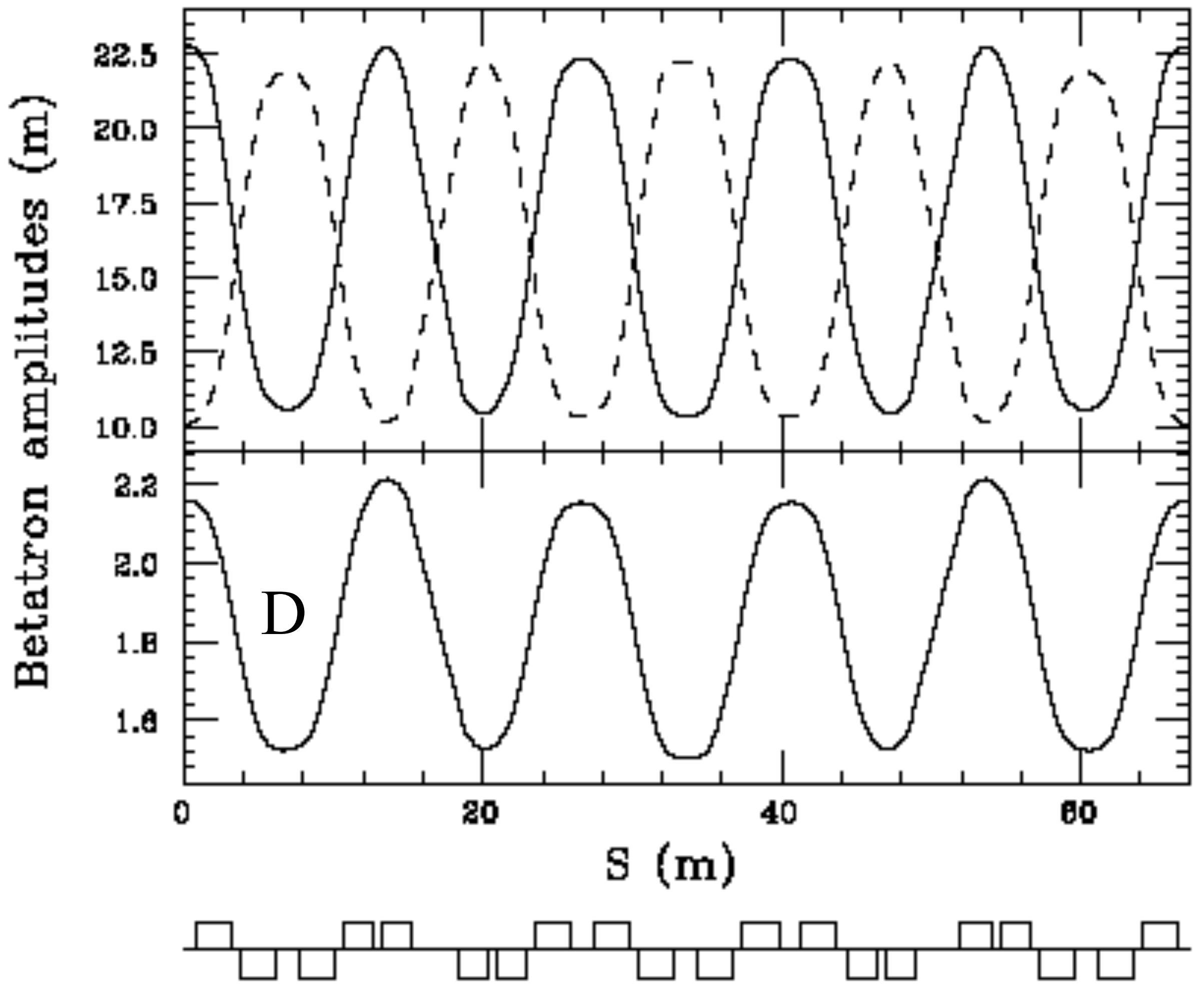
3) What are the meanings of these parameters?

AGS

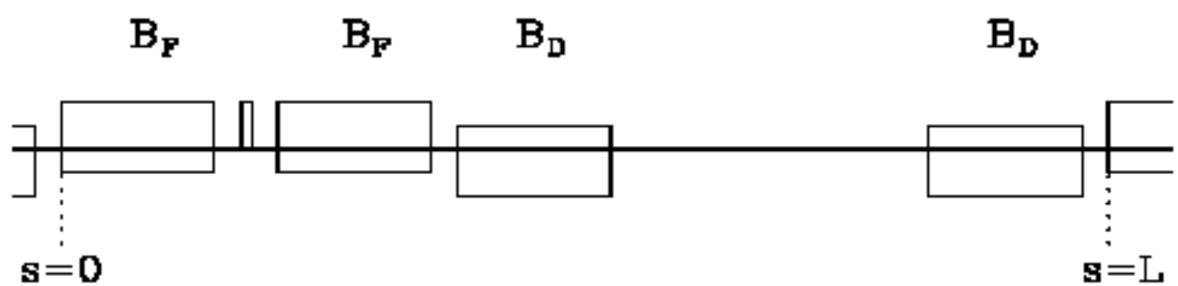
$$\beta = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

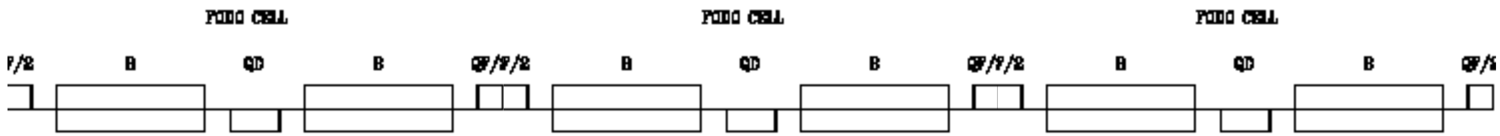
$$\sin\frac{\Phi}{2} = \frac{L_1}{2f}$$



Fermilab Booster



Stability of accelerator cells: FODO cell example



$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_x + \alpha_x \sin\Phi_x & \beta_x \sin\Phi_x \\ -\gamma_x \sin\Phi_x & \cos\Phi_x - \alpha_x \sin\Phi_x \end{pmatrix}$$

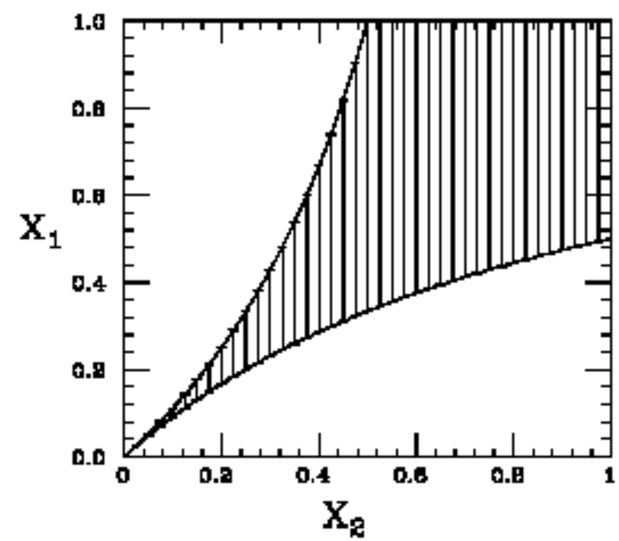
$$\begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_z + \alpha_z \sin\Phi_z & \beta_z \sin\Phi_z \\ -\gamma_z \sin\Phi_z & \cos\Phi_z - \alpha_z \sin\Phi_z \end{pmatrix}$$

$$\cos\Phi_x = 1 + \frac{L}{f_2} - \frac{L}{f_1} - \frac{L^2}{2f_1f_2} = 1 + 2X_2 - 2X_1 - 2X_1X_2$$

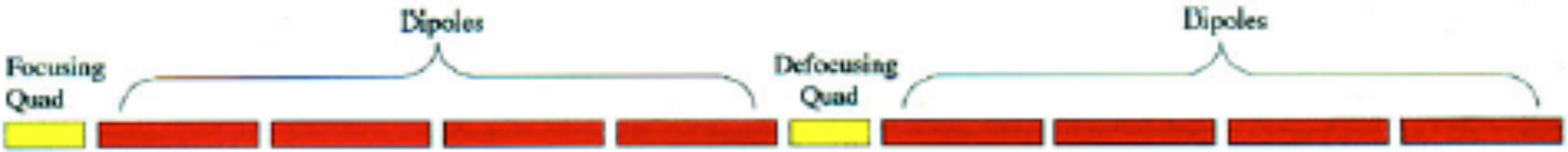
$$\cos\Phi_z = 1 - \frac{L}{f_2} + \frac{L}{f_1} - \frac{L^2}{2f_1f_2} = 1 - 2X_2 + 2X_1 - 2X_1X_2$$

Stability condition: (necktie diagram)

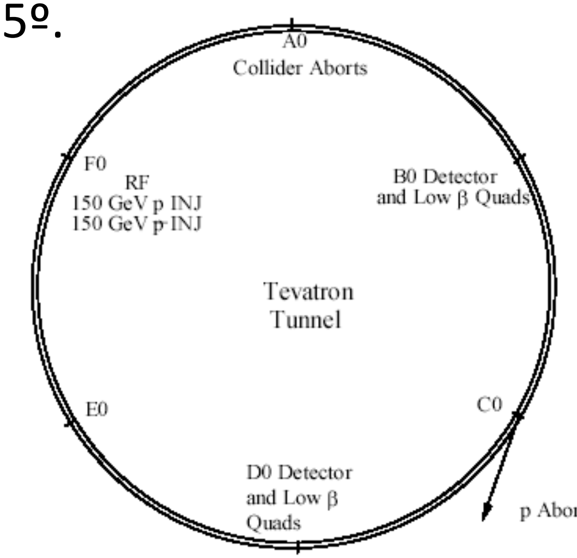
$$|\cos\Phi_x| \le 1, \quad |\cos\Phi_z| \le 1.$$

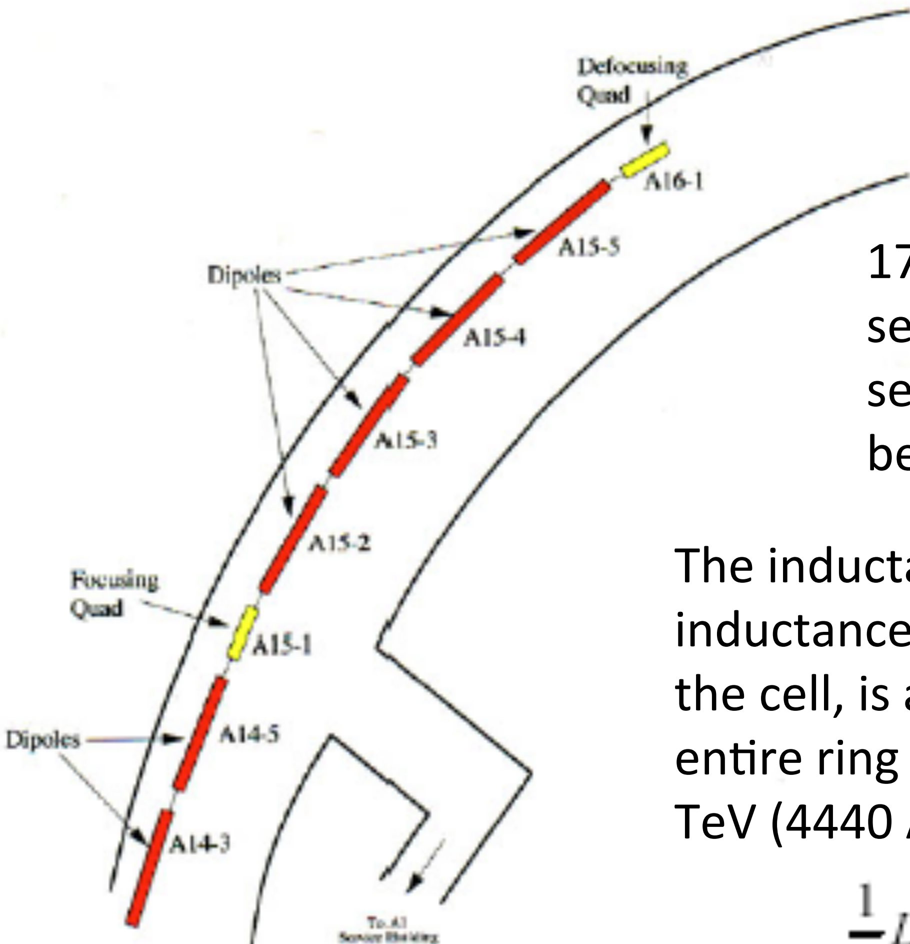



TEVATRON



A half-cell is composed of a quadrupole, a mini-straight section for correction coil spool piece, and 4 dipoles. A cell deflects the beam by 3.5º.





17 cells for each sector. Since each sector bends the beam 60 degrees.

The inductance of a typical “half cell”, that is the inductance of either the upper **or** lower bus through the cell, is about 0.18 H. The inductance for the entire ring of 36 H. The inductive stored energy at 1 TeV (4440 A) is 350MJ.

$$\frac{1}{2}LI^2 = \frac{1}{2}(36H)(4440A)^2 = 3.50x10^8\,Joules.$$