20.4 Solutions of Exercises of Chapter 9: Weak Focusing Synchrotron

9.1

Construct Saturne I. Spin Resonances

A photo of Saturne I synchrotron can be found in Fig. 9.3. A schematic layout of the ring and 90 deg cell is given in Fig. 9.23. This figure as well as Tab. 9.2 which lists the parameters of the synchrotron, will be referred to in building the Saturne I ring in the following.

(a) A model of Saturne I synchrotron.

DIPOLE is used to simulate the 90^o cell dipole. It is necessary to have Fig. 20.55 at hand (in addition to the users' guide), when filling up the data list under DIPOLE. Some comments regarding these data:

- DIPOLE is defined in a cylindrical coordinate system.
- The bending sector is 90 degrees, however the field region extent AT has to encompass the fringe fields, at both ends of the 90 deg sector. A large 5 deg extension is taken, for a total AT=100 deg which ensures absence of truncation of the fringe fields at the AT sector boundaries, over the all radial excursion of the beam.
- RM is given the curvature radius value, $RM = B\rho/B = 0.274426548_{[\mathrm{T\,m}]}/0.03259493_{[\mathrm{T}]} =$ 8.4193 m, this makes magnet positioning and closed orbit checks easier (see below). Note that the field and reference rigidity are, as for the simulation input data, for 3.6 MeV, the injection energy, this is an arbitrary choice.
- ACENT=50 deg is the reference azimuth, for the positioning of the entrance and exit effective field boundaries (EFB). It is taken in the middle of the AT range, an arbitrary choice.
- The entrance radius in the AT sector is $RE = RM/\cos(AT - \omega^+) = RM/\cos(5^o)$, with $\omega^+ = 45$ deg the positioning of the entrance EFB with respect to ACENT. And similarly for the positioning of the exit reference frame, $RS = RM/\cos(AT - (ACENT - \omega^-)) = RM/\cos(5^o)$ with $\omega^- = -45$ deg the positioning of the exit EFB. Note that $\omega^+ - \omega^- = 90^o$, the value of the bend angle.
- The entrance angle TE identifies with the extension to the 90 deg sector, namely, TE=5 deg. And similarly for the positioning of exit frame, 5 deg downstream of the exit EFB, thus TS=5 deg.

In order to build the cell it is a good idea to proceed by steps:

(i) first build a 90 deg deviation sector in the hard edge model (Tab. 20.46). FAISCEAU located next to DIPOLE indicates that a trajectory entering DIPOLE at radius R=RM, normally to the EFB (thus, $Y_0 = 0$ and $T_0 = 0$ in OBJET) exits with Y=0 and T=0. Data validation at this stage can be performed by comparing DIPOLE's transport matrix computed with MATRIX, and the theoretical expectation (after Eq. **??**, Sec. 19.5, numerical values truncated to 4th decimal)

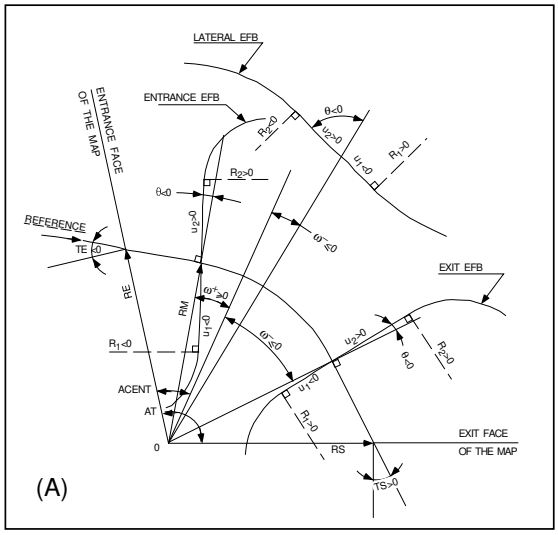

Fig. 20.55 This figure gives a representation of the data that define a dipole magnet, using DIPOLE [1]

$$T = \begin{pmatrix} \cos\sqrt{1-n}\alpha & \frac{\rho}{\sqrt{1-n}}\sin\sqrt{1-n}\alpha & 0 & 0 & 0 & \frac{\rho}{1-n}(1-\cos\sqrt{1-n}\alpha) \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{1-n}\alpha & \cos\sqrt{1-n}\alpha & 0 & 0 & 0 & \frac{1}{\sqrt{1-n}}\sin\sqrt{1-n}\alpha \\ 0 & 0 & \cos\sqrt{n}\alpha & \frac{\rho}{\sqrt{n}}\sin\sqrt{n}\alpha & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{n}}{\rho}\sin\sqrt{n}\alpha & \cos\sqrt{n}\alpha & 0 & 0 \\ \frac{1}{\sqrt{1-n}}\sin\sqrt{1-n}\alpha & \frac{\rho}{1-n}(1-\cos\sqrt{1-n}\alpha) & 0 & 0 & 1 & \frac{\rho}{(1-n)^{3/2}}(\sqrt{1-n}\alpha-\sin\sqrt{1-n}\alpha) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\rho=8.4193\ n=0.6]{\alpha=\pi/2,} \begin{pmatrix} 0.545794 & 11.15444 & 0 & 0 & 0 & 9.560222 \\ 0.062944 & 0.545794 & 0 & 0 & 0 & 1.324865 \\ 0 & 0 & 0.346711 & 10.19506 & 0 & 0 \\ 0 & 0 & -0.086295 & 0.346711 & 0 & 0 \\ 1.324865 & 9.560222 & 0 & 0 & 1 & 5.17640 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20.13}$$

MATRIX computation outcomes from raytracing can be found for comparison in Tab. 20.47.

It is leisurable, at this point, to choose to add fringe fields. This is the case in Exercise 9.2, however the hard edge model will be carried on with, here.

(ii) next, assemble a cell including this dipole with a pair of half-drifts at each ends, 2 m each (Fig. 9.23 and Tab. 20.48).

Table 20.46 Simulation input data file: a 45 degree sector bend in the hard edge model. The magnet is spilt into two identical halves, this is in order to allow access to particle coordinates at the center of the dipole, via FAISTORE or (here) FAISCEAU. The reference trajectory has equal entrance and exit position, and opposite sign angles. It coincides with the arc R=RM. MATRIX computes the transport matrix of the dipole (bottom of this Table), for comparison with the fringe field model, and possible comparison with matrix codes outcomes. This input data file is named SatI_DIP.inc and defines the Saturne I cell sequence segment S_SatI_DIP to E_SatI_DIP, for use with INCLUDE in subsequent exercises

```
File name: SatI_DIP.inc
Saturne I. Hard edge dipole model. Transport matrix.
'OBJET'
0.274426548e3                                                    ! Reference Brho: 3.6 MeV proton.
5                                          ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                                ! Coordinate sampling.
0. 0. 0. 0. 0. 1. 'o'      ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1

'MARKER' S_SatI_DIP                                                  ! Cell dipole begins here.

'DIPOLE'  upstream_half                                    ! Analytical modeling of a dipole magnet.
 0                     ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
45. 841.93                    ! Field region angle=90; reference radius set to curvature radius value.
22.5 0.3259493638 -0.6 0. 0.   ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
.0  0.                                                          ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                ! Enge coefficients.
22.5 0.  1.E6  -1.E6  1.E6  1.E6                       ! Angle to ACENT; face angle; face is straight.
.0  0.                                                                                ! EFB 2.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 0.  1.E6  -1.E6  1.E6  1.E6
0. 0.                                                                                 ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1                                      ! Degree of interpolation polynomial; flying grid sizing.
2.                                      ! Integration step size. It can be large in uniform field.
2 841.93 0. 841.93 0.                                  ! Positioning of entrance and exit frames.
'MARKER' half-dipole !.plt              ! Uncomment LABEL_2='.plt' (may go with IL=2 under DIPOLE) to
                                                                 ! log particle data in zgoubi.plt.
'FAISCEAU'       ! Provides local coordinates, and ellipse parameters, at center of Saturne I dipole.
'DIPOLE'  downstream_half                                  ! Analytical modeling of a dipole magnet.
 0                     ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
45. 841.93                    ! Field region angle=90; reference radius set to curvature radius value.
22.5 0.3259493638 -0.6 0. 0.   ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
.0  0.                                                          ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                ! Enge coefficients.
22.5 0.  1.E6  -1.E6  1.E6  1.E6                       ! Angle to ACENT; face angle; face is straight.
.0  0.                                                                                ! EFB 2.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 0.  1.E6  -1.E6  1.E6  1.E6
0. 0.                                                                                 ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1                                      ! Degree of interpolation polynomial; flying grid sizing.
2.                                      ! Integration step size. It can be large in uniform field.
2 841.93 0. 841.93 0.                                  ! Positioning of entrance and exit frames.

'MARKER' E_SatI_DIP                                                  ! Cell dipole ends here.

'FAISCEAU'                                                        ! Local particle coordinates.
'MATRIX'                                    ! Compute transport matrix, from trajectory coordinates.
1 0
'END'
```

Table 20.47 Outcomes of the simulation file of Tab. 20.46

An excerpt from zgoubi.res. Coordinates of the first particle (the reference trajectory) and its path length under FAISCEAU, at OBJET on the left hand side below, locally on the right hand side:

```
         3  Keyword, label(s) :  FAISCEAU                                                                    IPASS= 1
                                          TRACE DU FAISCEAU
                                       (follows element #      2)
                                            11 TRAJECTOIRES
                                 OBJET                                          FAISCEAU
            D       Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)      D-1     Y(cm)    T(mr)    Z(cm)    P(mr)       S(cm)
   0  1   1.0000    0.000     0.000     0.000     0.000      0.0000    0.0000    0.000    0.000    0.000    0.000   1.322501E+03
```

Transport matrix of Saturne I 90 *degree sector bend, in the hard edge model, two difference cases of integration step size, namely,* 4 *cm and* 1 *m* (*an excerpt of MATRIX computation, from zgoubi.res*). *It can be checked against matrix transport expectations. The "first order symplectic conditions" are very small in the* 4 *cm step size case, which is an indication of accurate numerical integration of the trajectories across DIPOLE*; *the reference trajectory* (*first one*) *exits better aligned* (*reference coordinates, before change of frame for MATRIX computation, are closer to zero*):

- *Case of* 4 *cm step size*:

```
         4  Keyword, label(s) :  MATRIX                                                                     IPASS= 1

      Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
       0.00000000E+00   4.53054326E-07   6.27843350E-07   0.00000000E+00   0.00000000E+00   1.32250055E+03   4.41138700E-02

                     TRANSFER  MATRIX  ORDRE  1  (MKSA units)
              0.545795         11.1544          0.00000          0.00000          0.00000          9.56022
             -6.294423E-02     0.545795         0.00000          0.00000          0.00000          1.32487
              0.00000          0.00000          0.346711         10.1951          0.00000          0.00000
              0.00000          0.00000         -8.629576E-02     0.346711         0.00000          0.00000
              1.32487          9.56022          0.00000          0.00000          1.00000          5.17640
              0.00000          0.00000          0.00000          0.00000          0.00000          1.00000

              DetY-1 =        0.0000000278,    DetZ-1 =        0.00000000045
              R12=0 at   -20.44     m,         R34=0 at   -29.41     m
         First order symplectic conditions (expected values = 0) :
            2.7767E-08    4.4762E-09     0.000          0.000          0.000          0.000
```

- *Case of* 1 *m step size*:

```
         4  Keyword, label(s) :  MATRIX                                                                     IPASS= 1

      Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
       0.00000000E+00  -7.54923113E-03  -1.08904867E-02   0.00000000E+00   0.00000000E+00   1.32249873E+03   4.41138091E-02

                     TRANSFER  MATRIX  ORDRE  1  (MKSA units)
              0.545757         11.1567          0.00000          0.00000          0.00000          9.56154
             -6.295274E-02     0.546125         0.00000          0.00000          0.00000          1.32517
              0.00000          0.00000          0.346697         10.1954          0.00000          0.00000
              0.00000          0.00000         -8.629900E-02     0.346750         0.00000          0.00000
              1.32486          9.56148          0.00000          0.00000          1.00000          5.17692
              0.00000          0.00000          0.00000          0.00000          0.00000          1.00000

              DetY-1 =        0.0003978566,    DetZ-1 =        0.0000685588
              R12=0 at   -20.43     m,         R34=0 at   -29.40     m
         First order symplectic conditions (expected values = 0) :
            3.9786E-04    6.8559E-05     0.000          0.000          0.000          0.000
```

Table 20.48 Simulation input data file: Saturne I cell, obtained by assembling DIPOLE taken from Tab. 20.46 together with two half-drifts. This input data file is named SatI_cell.inc and defines the Saturne I cell sequence segment S_SatI_cell to E_SatI_cell, for possible use in INCLUDE statements in subsequent exercises

```
File name: SatI_cell.inc.
! Saturne I, one cell of the 4-period ring.
'MARKER'   SatICellMATRIX_S                                          ! Just for edition purposes.
'OBJET'
0.274426548e3                                                  ! Reference Brho: 3.6 MeV proton.
5                                        ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                          ! Coordinate sampling.
0. 0. 0. 0. 0. 1.          ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1

'MARKER'  S_SatI_cell
'DRIFT' half_drift
200.
'INCLUDE'
1
./SatI_DIP.inc[S_SatI_DIP:E_SatI_DIP]
'DRIFT' half_drift
200.
'MARKER'  E_SatI_cell
'FAISCEAU'                                                     ! Local particle coordinates.
'TWISS'
2  1. 1.
'MARKER'   SatICellMATRIX_E                                          ! Just for edition purposes.
'END'
```

Lattice parameters

The TWISS command down the sequence (Tab. 20.48) produces the periodic beam matrix results shown in Tab. 20.49

Table 20.49 Results obtained running the simulation input data file of Tab. 20.48, Saturne I cell - an excerpt from zgoubi.res

```
   14  Keyword, label(s) :  TWISS                                                                                IPASS= 4

Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
 0.00000000E+00   6.02895730E-07   6.54169939E-07   0.00000000E+00   0.00000000E+00   1.72250055E+03   6.57784696E-01

     Beam  matrix  (beta/-alpha/-alpha/gamma) and  periodic  dispersion  (MKSA units)
        14.418595     0.000000     0.000000     0.000000     0.000000    21.048250
         0.000000     0.069355     0.000000     0.000000     0.000000     0.000000
         0.000000     0.000000    11.411041     0.000000     0.000000    -0.000000
         0.000000     0.000000     0.000000     0.087634     0.000000     0.000000
         0.000000     0.000000     0.000000     0.000000     0.000000     0.000000
         0.000000     0.000000     0.000000     0.000000     0.000000     0.000000

                          Betatron  tunes  (Q1 Q2 modes)
                 NU_Y =  0.18103144         NU_Z =  0.22214599

                          dL/L / dp/p =   1.9194487
  (dp =  0.000000E+00     L(0)   =    1.72250E+03 cm,  L(0)-L(-dp) =    3.30606E-01 cm,  L(0)-L(+dp) =   -3.30645E-01 cm)

                               Transition gamma  =  7.21791469E-01

                               Chromaticities :
                 dNu_y / dp/p = -0.60221729                 dNu_z / dp/p =  0.38005442
```

The TWISS command also produces a zgoubi.TWISS.out file which details the optical functions along the sequence (at the downstream end of the optical elements. The header of that file details the optical parameters of the structure (Tab. 20.50).

Table 20.50 An excerpt of zgoubi.TWISS.out file resulting from the execution of the Saturne I cell simulation input data file of Tab. 20.48. Note that the ring (4-period) wave numbers are 4 times the cell values Q1, Q2 displayed here. Optical functions (betatron function and derivative, orbit, phase advance, etc.) along the optical sequence are listed as part of zgoubi.TWISS.out following th header. The top part and last line of that listing are given below

```
@ LENGTH          %le     17.22500552
@ ALFA            %le     1.919448707
@ ORBIT5          %le                -0
@ GAMMATR         %le    0.7217914685
@ Q1              %le    0.1810314404      [fractional]
@ Q2              %le    0.2221459901      [fractional]
@ DQ1             %le   -0.6022172911
@ DQ2             %le    0.3800544183
@ DXMAX           %le     2.10586311E+01    @ DXMIN           %le     2.10482503E+01
@ DYMAX           %le     0.00000000E+00    @ DYMIN           %le     0.00000000E+00
@ XCOMAX          %le     2.10528899E-01    @ XCOMIN          %le     0.00000000E+00
@ YCOMAX          %le     0.00000000E+00    @ YCOMIN          %le     0.00000000E+00
@ BETXMAX         %le     1.57006971E+01    @ BETXMIN         %le     1.44132839E+01
@ BETYMAX         %le     1.30884296E+01    @ BETYMIN         %le     1.14110171E+01
@ XCORMS          %le     6.05227342E-04
@ YCORMS          %le    0.    not computed
@ DXRMS           %le     2.98427468E-03
@ DYRMS           %le     0.00000000E+00
```

Top and bottom four lines (*truncated*) *of zgoubi.TWISS.out optical functions listing, including the periodic* β_x, β_y (β_Y, β_Z *in zgoubi notations*) *and* D_x (η_Y *in zgoubi notations*) *values at cell ends*:

```
 # alfx          btx             alfy            bty             alfl            btl             Dx              Dxp
 3.1789380E-009  1.4426805E+001 -8.1670406E-009  1.1411067E+001  0.0000000E+000  0.0000000E+000  2.1058631E+001  1.1261478E-003
 1.3369744E-008  1.4426805E+001 -1.7451004E-009  1.1411067E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725930E-009
 1.3369744E-008  1.4426805E+001 -1.7451004E-009  1.1411067E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725930E-009
-1.3863082E-001  1.4704066E+001 -1.7526845E-001  1.1761604E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725930E-009
 ......................................
 1.3919473E-001  1.4692541E+001  1.7526999E-001  1.1761559E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725925E-009
 1.3919473E-001  1.4692541E+001  1.7526999E-001  1.1761559E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725925E-009
 4.3382070E-004  1.4413284E+001  7.7156422E-007  1.1411017E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725925E-009
 4.3382070E-004  1.4413284E+001  7.7156422E-007  1.1411017E+001  0.0000000E+000  0.0000000E+000  2.1048250E+001  1.6725925E-009
```

6561 *Tune scan*

6562 A simulation is given in Tab. 20.51, derived from Tab. 20.48: TWISS has been
6563 replaced by MATRIX, a REBELOTE do loop repeatedly changes n. A plot of the
6564 scan is given in Fig. 20.56, some detailed values are given in Tab. 20.52.

Table 20.51 Simulation input data file: tune scan, using REBELOTE to repeatedly change n. Beam matrix and wave numbers are computed by MATRIX, from the coordinates of the 11 particle sample generated by OBJET[KOBJ=5]

```
Saturne I, tune scan.
'MARKER'   SatI_Qscan_S                                              ! Just for edition purposes.
'OBJET'
0.274426548e3                                                  ! Reference Brho: 3.6 MeV proton.
5                                        ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                                ! Coordinate sampling.
0. 0. 0. 0. 0. 1.          ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1

'MARKER'  S_SatI_cell
'DRIFT' half_drift
200.
'INCLUDE'
1
./SatI_DIP.inc[S_SatI_DIP:E_SatI_DIP]
'DRIFT' half_drift
200.
'MARKER'  E_SatI_cell
'FAISCEAU'                                                          ! Local particle coordinates.
'MATRIX'
1 11  PRINT   ! Comoute a 10+4 period transport matrix, and tunes. Save outcomes to zgoubi.MATRIX.out.

'REBELOTE'                   ! A do loop: repeat the section above commencing at the top of the file,
10  1.1  0 1                                                                          ! 10 times.
1
DIPOLE 6  -0.757:-0.5   ! Change the value of parameter 30 (namely, n) in DIPOLE (prior to repeating).
                                                                  ! in any DIPOLE in the sequence.
'SYSTEM'
1
gnuplot <./gnuplot_MATRIX_Qxy.gnu                                          ! Plot tunes vs index.
'MARKER'   SatI_Qscan_E                                           ! Just for edition purposes.
'END'
```

gnuplot script to obtain Fig. 20.56:

```
# ./gnuplot_MATRIX_Qxy.gnu
set xlabel "index n";set ylabel "{/Symbol n}_x,     ({/Symbol n}_x^2+{/Symbol n}_y^2)^{1/2}"
set y2label "{/Symbol n}_y"; set xtics; set ytics nomirror; set y2tics nomirror; ncell=4
set key t l; set key maxrow 2; set yrange [:1.3]; set y2range [:1.06]
n1 = -0.757; dn=(.757-.5)/10.; R=10.9658; rho=8.4193
plot \
"zgoubi.MATRIX.out" u (n1+($61-1)*dn): \
($61>1? $56 *ncell :1/0) w p pt 5 lt 1 lw .5 lc rgb "red"               tit "{/Symbol n}_x " ,\
"zgoubi.MATRIX.out" u (n1+($61-1)*dn):($61>1? sqrt((1+(n1+($61-1)*dn))*R/rho): \
1/0) w l lt 1  lc rgb "red" tit "theor. " ,\
"zgoubi.MATRIX.out" u (n1+($61-1)*dn): \
($61>1? $57 *ncell :1/0) axes x1y2 w p pt 6 lt 3 lw .5 lc rgb "blue"   tit "{/Symbol n}_y " ,\
"zgoubi.MATRIX.out" u (n1+($61-1)*dn): \
($61>1? sqrt((-(n1+($61-1)*dn))*R/rho):1/0) axes x2y2 w l lt 3  lc rgb "blue" tit "theor. " ,\
"zgoubi.MATRIX.out" u (n1+($61-1)*dn): \
($61>1? sqrt($56**2+$57**2) *ncell :1/0) w p pt 7 lt 1 lc rgb "black" tit "({/Symbol n}_x^2+{/Symbol n}_y^2)^{1/2}" ,\
"zgoubi.MATRIX.out" u (n1+($61-1)*dn):($61>1? sqrt(R/rho):1/0)  w l lt 1  lc rgb "black" tit "theor. "
pause 1
```

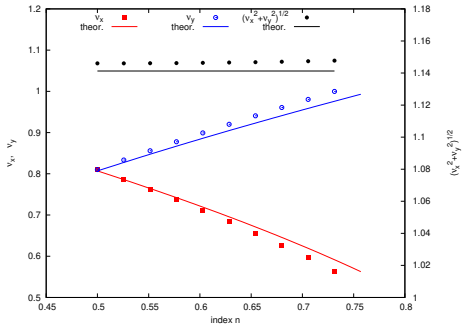

Fig. 20.56 A scan of the wave numbers, and of $\sqrt{\nu_Y^2 + \nu_Z^2} \approx \sqrt{R/\rho_0} = 1.141$, in Saturne I for $0.5 \leq n \leq 0.757$

Table 20.52 Dependence of wave numbers on index n, from numerical raytracing (columns denoted "ray-tr.") and from theory

	ν_Y		ν_Z	
n	ray-tr.	$\sqrt{(1-n)\frac{R}{\rho_0}}$	ray-tr.	$\sqrt{n\frac{R}{\rho_0}}$
0.5	0.810353	0.806987	0.810353	0.806987
0.6	0.724125	0.721791	0.888583	0.884010
0.7	0.626561	0.625089	0.960806	0.954840
0.757	0.563635	0.562580	0.999804	0.992955

(b) Betatron functions of Saturne I cell.

Among the various ways to produce the betatron functions along the sequence (and throughout the DIPOLEs), here are two possibilities, based on the storage of particle coordinates in zgoubi.plt during stepwise raytracing:

1. a direct way consists in using OBJET[KOBJ=5.1] and transport the 11-particle set so obtained across the sequence. Then, betaFromPlt from zgoubi toolbox [2] can be used to compute the transport matrix, step by step across the sequence, from the coordinate values logged in zgoubi.plt during the stepwise integration;
2. an indirect way consists in launching a few particles on a common invariant (horizontal and/or vertical) and subsequently plot the s-dependent quantities $\hat{Y}^2(s)/\epsilon_Y$ and/or $\hat{Z}^2(s)/\epsilon_Z$. The maximum value of the latter, a function of the distance s, is the betatron function along the sequence, $\beta_{Y,Z}(s)$.

The second method is used here (this is an arbitrary choice. Exercises may be found in the various Chapters, that use the first method and may be referred to, if desired).

The input data file to derive the betatron function following method (2) above is given in Tab. 20.53. The initial ellipse parameters (under OBJET) are the periodic values, namely, $\alpha_Y = \alpha_Z = 0$, $\beta_Y = 14.426\,\text{m}$, $\beta_Z = 11.411\,\text{m}$, they are a subproduct of the TWISS procedure performed in (a), to be found in zgoubi.TWISS.out (Tab. 20.50). The resulting envelopes and their squared value are shown in Fig. 20.57. Note that this raytracing also provides the coordinates of the 60 particles on their common upright invariant

$$x^2/\beta_x + \beta_x x'^2 = \epsilon_x/\pi$$

at start and at the end of the cell (with x standing for either Y or Z, and $\epsilon_{Y,Z}/\pi = 10^{-4}$, here). This allows checking that the initial ellipse parameters (under OBJET, Tab. 20.53) are effectively periodic values, and that the raytracing went correctly, namely by observing that the initial and final ellipses do superimpose (Fig. 20.58).

Table 20.53 Simulation input data file: raytrace 60 particles across Saturne I cell to generate beam envelopes. Store particle data in zgoubi.plt, along DRIFTs and DIPOLEs. The INCLUDE file and segments are defined in Tab. 20.48

```
SatI envelopes.
'OBJET'
0.274426548e3                                                    ! Reference Brho: 3.6 MeV proton.
8                                ! Create a set of 60 particles evenly distributed on the same invariant;
1 60 1    ! case of 60 particles on a vertical invariant; use 60 1 1 instead for horizontal invariant.
0. 0. 0. 0. 0. 1.
0.  14.426 1e-4
0.  11.411 1e-4
0. 1. 0.

'FAISTORE'                              ! This logs the coordinates of the particle to zgoubi.fai,
zgoubi.fai  S_SatI_cell  E_SatI_cell                          ! at the two LABEL1s as indicated.
1

'MARKER'  S_SatI_cell                                            ! Saturne I cell begins here.
'DRIFT' half_Drift                                  ! Option 'split' devides the drift in 10 pieces,
200. split 10  2                                 ! 'IL=2' causes log of particle data to zgoubi.plt.

'INCLUDE'
1
./SatI_DIP.inc[S_SatI_DIP:E_SatI_DIP]

'DRIFT' half_Drift                                  ! Option 'split' devides the drift in 10 pieces,
200. split 10  2                                 ! 'IL=2' causes log of particle data to zgoubi.plt.

'MARKER'  E_SatI_cell                                            ! Saturne I cell ends here.
'FAISCEAU'
'END'
```

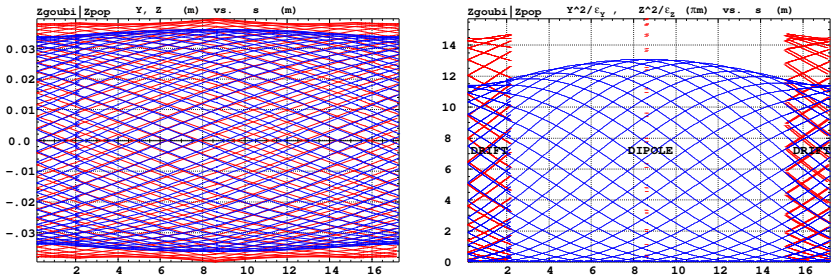

Fig. 20.57 Left: horizontal and vertical envelopes as generated by plotting the coordinates Y(s) (greater excursion, red, along the drifts and at dipole center) or Z(s) (smaller excursion, blue) across the Saturne I cell, of 60 particles evenly distributed on a common $10^{-4}\ \pi\mu$m invariant, either horizontal or vertical (while the other invariant is zero). Right: a plot of $Y^2(s)/\epsilon_Y$ and $Y^2(s)/\epsilon_Y$: the extrema identify with $\beta_Y(s)$ and $\beta_Z(s)$, respectively. The extrema extremorum values are $\hat{\beta}_Y = 14.4$ m and $\hat{\beta}_Z = 15.7$ m, respectively. These plots are obtained using zpop, which reads stepwise particle data from zgoubi.plt

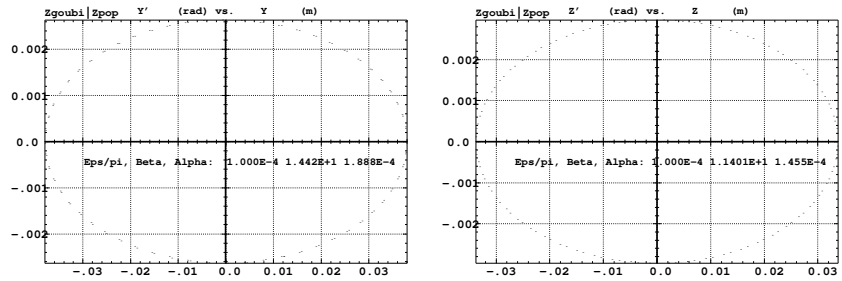

Fig. 20.58 Sixty particles evenly distributed on a common periodic invariant (of value either $\epsilon_Y = 10^{-4}\pi\mu$m and $\epsilon_Z = 0$, left plot, or the reverse, right plot) have been tracked from start to end of the cell. These periodic invariants are defined assuming the periodic ellipse parameters determined from prior TWISS, given in Tab. 20.50; values resulting from an *rms* match of the coordinates are given in the figure, and do agree with those TWISS data. The figure shows the good superposition of the start and end invariants (the start and end *rms* match ellipse parameters show negligible difference), which confirms the correct value of the periodic ellipse parameters, namely, left graph: horizontal phase space at start (crosses) and end (dots) of the cell; right graph: vertical phase space at start (crosses) and end (dots) of the cell

Dispersion function

Raytracing off-momentum particles on their chromatic closed orbit provides the periodic dispersion function. In order to do so, the input data file of Tab. 20.53 can be used, it just requires changing OBJET to the following:

```
'OBJET'
0.274426548e3                                                    ! Reference Brho: 3.6 MeV proton.
2                                                                 ! Create particles individually.
3 1                                                                          ! Three particles.
+.21056 0. 0. 0. 0. 1.0001 'p'  ! Chromatic orbit coordinates Y0 and T0 for D=1.001 relative rigidity.
0. 0. 0. 0. 0. 1.  'o'                                                       ! On-momentum orbit.
-.21056 0. 0. 0. 0. 0.9999 'm'  ! Chromatic orbit coordinates Y0 and T0 for D=0.999 relative rigidity.
1 1 1
```

The position and angle of the chromatic particles, which are offset by $\Delta p/p = \pm 10^{-4}$, are drawn from the value of the periodic dispersion $\eta_Y = 21.05\,\mathrm{m}$ and its derivative $\eta'_Y \approx 0$ (Tab. 20.50), namely, $Y_0 = \eta_Y \Delta p/p = \pm 0.2105\,\mathrm{cm}$ and $T_0 = \eta_Y \Delta p/p = 0$.

Running Tab. 20.53 simulation file with this new OBJET produces the following coordinates at FAISCEAU, located at the end of the sequence (an excerpt from zgoubi.res):

```
          13  Keyword, label(s) :  FAISCEAU                                                    IPASS= 1
                                          TRACE DU FAISCEAU
                                       (follows element #     12)
                                            3 TRAJECTOIRES
                                 OBJET                                  FAISCEAU
              D       Y(cm)     T(mr)     Z(cm)     P(mr)      S(cm)      D-1      Y(cm)     T(mr)     Z(cm)     P(mr)      S(cm)
   p  1   1.0001     0.211     0.000     0.000     0.000     0.0000   0.0001     0.211     0.000     0.000     0.000  1.722831E+03     1
   o  1   1.0000     0.000     0.000     0.000     0.000     0.0000   0.0000     0.000     0.000     0.000     0.000  1.722501E+03     2
   m  1   0.9999    -0.211     0.000     0.000     0.000     0.0000  -0.0001    -0.210     0.000     0.000     0.000  1.722170E+03     3
```

The local coordinates Y, T (under FAISCEAU, right hand side) and initial coordinates Y_0, T_0 (under OBJET, left hand side) are identical (to better than 5 μm, 0.5 μrad accuracy, respectively - zgoubi.fai can be consulted for greater precision on these values), confirming the periodicity of these chromatic trajectories. Figure 20.59 shows the particle trajectories through the cell DIPOLE, they appear to be at constant radius as expected.

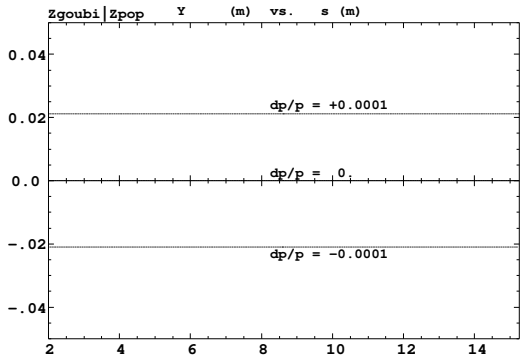

Fig. 20.59 A plot of the radial excursion, across DIPOLE body (namely, AT=90^o extent,(Tab. 20.48), of an on-momentum particle (its radial position along the dipole body is $R_0 \approx 8.4193\,\mathrm{m}$, corresponding to Y=0 in this graph) and two particles at respectively $dp/p = \pm 10^{-3}$. A graph obtained using zpop, which reads stepwise particle data from zgoubi.plt: menu 7; 1/1 to open zgoubi.plt; 2/[6,2] to select Y versus distance; 7 to plot

(c) Some verifications regarding the model.

The field along large excursion orbits can be logged in zgoubi.plt, using option IL=2 (or 20, or 200, etc. for printout every 10, or 100, etc. integration step) under DIPOLE.

The simulation file of Tab. 20.53 is used to raytrace five particles, with OBJET changed to the following:

```
'OBJET'
0.274426548e3                                                  ! Reference Brho: 3.6 MeV proton.
2                                                               ! Create particles individually,
5 1                                                                          ! five particles.
+0.21056 0. 0. 0. 0. 1.01  'p'          ! Chromatic orbit coordinates for D=1.01 relative rigidity.
0. 0. 0. 0. 0. 1.  '0'                                                ! On-momentum closed orbit.
-0.21056 0. 0. 0. 0. 0.99  'm'          ! Chromatic orbit coordinates for D=0.99 relative rigidity.
0. 0. 5. 0. 0.  1.   'm'                    ! Initial vertial excursion is Z0= 5 cm off-mid-plane.
0. 0. 20. 0. 0.  1.  'm'                    ! Initial vertial excursion is Z0=20 cm off-mid-plane.
1 1 1 1 1
```

Apart from the on-momentum particle (2nd in the list) this OBJET defines two particles on $\Delta p/p = \pm 1\%$ chromatic orbit (1st and 3rd in the list), this is an excursion of a few tens of centimeters, large as requested, as $\Delta x \approx 38 \times dp/p$. OBJET also defines 2 particles launched into the cell at respectively $Z_0 = 5\,\text{cm}$ and $Z_0 = 20\,\text{cm}$.

The magnetic field as a function of the azimuthal angle in DIPOLE frame, along these trajectories across the upstream DIPOLE of the cell, is shown in Fig. 20.60. The field curves for the first four trajectories essentially superimpose except for the fringe field regions (Fig. 20.60), due to the wedge angles. This behaves as expected. Detail inspection is possible, from the detailed particle coordinate and field data in zgoubi.plt - this is out of the scope of the present question.

The field along the 5th particle trajectory features overshoots (Fig. 20.60), this is due to the very large vertical excursion ($Z \approx 20\,\text{cm}$ in the entrance fringe field region). It looks reasonable, however it may be an artifact in the case that the high order derivatives of the field in that region are large, resulting from the truncated Taylor series method used for off mid-plane field extrapolation [1, Sec. 1.3.3].

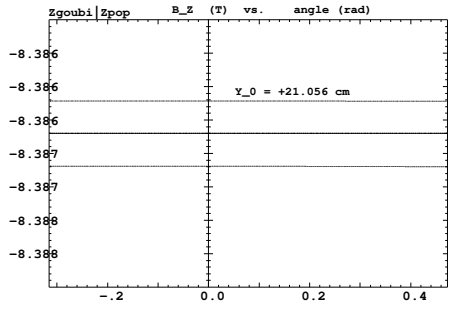

Fig. 20.60 Magnetic field along 5 different trajectories across the upstream DIPOLE, including four large horizontal and vertical excursion cases

(d) Sinusoidal approximation of the betatron motion.

The approximation

$$y(\theta) = A\ \cos(\nu_Z \theta + \phi)$$

is checked here considering the vertical motion (considering the horizontal motion leads to similar conclusions). The value of the various parameters in that expression are determined as follows:

- the particle raytraced for comparison is launched with an initial excursion $Z_0(\theta = 0) = 5\,\text{cm}$ (4th particle in OBJET, above). At the launch point (middle of the drift) the beam ellipse is upright (Fig. 20.58), whereas phase space motion is clockwise, thus take

$$\text{A} = 5\,\text{cm} \quad \text{and} \quad \phi = \pi/2$$

- the vertical betatron of the 4-cell ring tune is (Tab. 20.50)

$$\nu_\text{Z} = 4 \times 0.222146 = 0.888284$$

- $\theta = s/R$ and $R = \oint ds/2\pi$ with (Tab. 20.50)

$$2\pi\text{R} = \text{circumference} = 2\pi \times 10.9658 = 68.9\,\text{m}$$

The comparison with a trajectory obtained from raytracing is given in Fig. 20.61 and confirms the validity of the sinusoidal approximation.

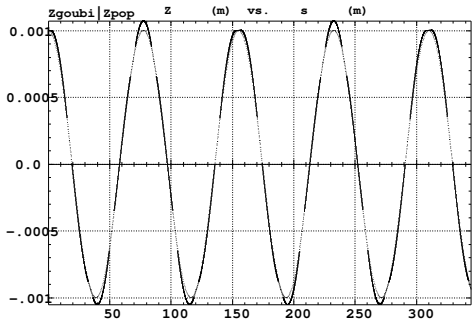

Fig. 20.61 Vertical betatron motion, five turns around Saturne I ring, from raytracing (modulated oscillation), and sine approximation, superimposed

(e) An acceleration cycle. Symplecticity checks.

Eleven particles are launched for a 30,000 turn tracking at a rate of

$$\Delta W = q\hat{V}\,\cos\phi_s = 200 \times \sin 150^0 = 100\,\text{keV/turn}$$

($E : 3.6 \rightarrow 3.0036\,\text{GeV}$), all evenly distributed on the same initial vertical invariant

$$Z^2/\beta_Z + \beta_Z Z'^2 = \epsilon_Z/\pi \tag{20.14}$$

with $\epsilon_Z/\pi = 10^{-4}$ m, or, normalized, $\beta\gamma\epsilon_Z/\pi = 0.08768 \times 10^{-4}$ m.

The simulation file is given in Tab. 20.54. CAVITE[IOPT=3] is used, it provides an RF phase independent boost

$$\Delta W = q\hat{V}\sin\phi_s$$

as including synchrotron motion is not necessary here, even better, this ensures constant depolarizing resonance crossing speed, so precluding any possibility of multiple crossing (it can be referred to [3] regarding that effect).

Table 20.54 Simulation input data file: track 11 particles launched on the same vertical invariant, with quasi-zero horizontal invariant. The INCLUDE adds the Saturne I cell four times, the latter is defined in Tab. 20.48 and Fig. 9.23

```
Saturne I ring. Polarization landscape.
'MARKER'   SatIPolarLand_S                                          ! Just for edition purposes.
'OBJET'
0.274426548e3                                                   ! Reference Brho: 3.6 MeV proton.
8                                ! Create a set of 60 particles evenly distributed on the same invariant;
1 11 1    ! case of 60 particles on a vertical invariant; use 60 1 1 instead for horizontal invariant.
0. 0. 0. 0. 0. 1.
0.  14.426 1e-4                    ! Periodic optical functions and invariant value, horizontal and
0.  11.411 1e-4                                                                          ! vertical.
0. 1. 0.                                                                      ! No momentum spread.

!'MCOBJET'                                                                              ! Commented.
!1.035270 36749193e3                                                 ! Reference Brho: 50 MeV proton.
!3                                      ! Create an 11 particle set, proper for MATRIX computation.
!200
!2 2 2 2 2 2
!0. 0. 0. 0. 0. 1.
!0. 14.426 25e-6 3                                 ! Periodic alpha_Y, beta_Y, and invariant value;
!0. 11.411 10e-6 3                                 ! Periodic alpha_Z, beta_Z, and invariant value.
!0. 1. 1.e-8 3
!123456 234567 345678

'PARTICUL'
PROTON               ! Necessary data in order to allow  (i) spin trackingand, and  (ii) acceleration.
'SPNTRK'                                                                        ! Switch on spin tracking,
3                                                                            ! all initial spins vertical.
'FAISCEAU'
'FAISTORE'
b_polarLand.fai                    ! Log particle data in b_polarLand.fai, turn-by-turn; "b_" imposes
7                                                    ! binary write, which results in faster i/o.

'SCALING'
1 1
DIPOLE
-1                                    ! Causes field increase in DIPOLE, in correlation to particle
1.                                                                  ! rigidity increase by CAVITE.
1

! 4 cells follow.
'INCLUDE'
1
4* ./SatI_cell.inc[S_SatI_cell:E_SatI_cell]

'CAVITE'
3
0 0
200e3  0.523598775598                                  ! Acceleration rate is 200*0.5=100keV/turn.
! 20e3  0.523598775598                       ! Commented: an acceleration rate of 20*0.5=10keV/turn.

'REBELOTE'
30000 0.3 99                             ! Case of 100 keV/turn: ~30,000 turns from 3.6 MeV to 3 GeV.
! 30000 0.3 99                ! Commented: case of 10 keV/turn: ~300,000 turns from 3.6 MeV to 3 GeV.

'FAISCEAU'
'MARKER'   SatIPolarLand_E                                          ! Just for edition purposes.
'SPNPRT'

'END'
```

Betatron damping

Figure 20.62 shows the damped vertical motion of the individual particles, over the acceleration range, together with the initial and final distributions of the 11 particles on elliptical invariants. Departure from the matching ellipse at the end of the acceleration cycle, 3 GeV (Eq. 20.14 with $\epsilon_Z/\pi = 1.0745 \times 10^{-6}$ m), is marginal.

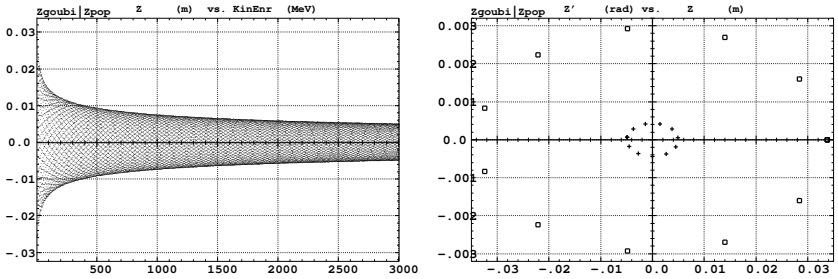

Fig. 20.62 Left: damped vertical motion, from 3.6 MeV to 3.004 GeV in 30,000 turns. Right: the initial coordinates of the 11 particles (squares) are taken on a common invariant $\epsilon_Z(0) = 10^{-4}\,\pi$m (at 3.6 MeV, $\beta\gamma = 0.0877$, thus $\beta\gamma\epsilon_Z(0) = 8.77 \times 10^{-6}\,\pi$m); the final coordinates after 30,000 turns (crosses) appear to still be (with negligible departure) on a common invariant of value $\epsilon_Z(\text{final}) = 2.149 \times 10^{-6}\,\pi$m (at 3.004 GeV, $\beta\gamma = 4.08045$) thus $\beta\gamma\epsilon_Z(\text{final}) = 8.77 \times 10^{-6}\,\pi$m, , equal to the initial value

Degree of non-symplecticity of the numerical integration

The degree of non-symplecticity as a function of integration step size is illustrated in Fig. 20.63. The initial motion is taken paraxial, vertical motion is considered as it resorts to off-mid plane Taylor expansion of fields [1, DIPOLE Sec.], a stringent test as the latter is expected to deteriorate further the non-symplecticity inherent to the Lorentz equation integration method (a truncated Taylor series method [1, Eq. 1.2.4]).

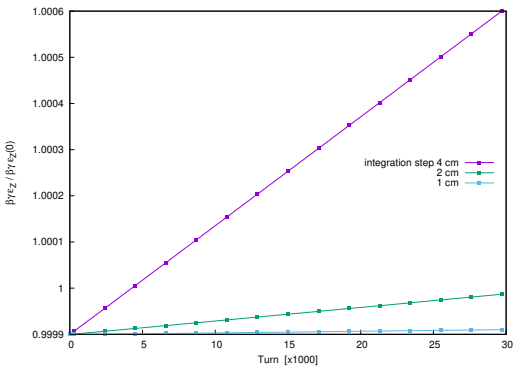

Fig. 20.63 Turn-by-turn evolution of the normalized invariant, $\beta\gamma\epsilon_Z(\text{turn})/\beta\gamma\epsilon_Z(0)$ (initial $\epsilon_Z(0)$ taken paraxial), for four different integration step size values: 1, 2 and 4 cm

Evolution of the wave numbers

The Fortran tool tunesFromFai_iterate can be used to computes tunes as a function of turn number or energy, it reads turn by turn particle data from zgoubi.fai and computes a discrete Fourier transform over so many turns (a few tens, 100 here for instance), every so many turns (300, here) [4]. Typical results are displayed in Fig. 20.64, tunes have the expected values: $\nu_Y = 0.7241$, $\nu_Z = 0.8885$. In acceleration rate of 100 keV/turn has been taken (namely, $\hat{V} = 200$ kV and still $\phi_s = 150^0$), to save on computing time. SCALING with option NTIM=-1 causes the magnet field to strictly follow the momentum boost by CAVITE.

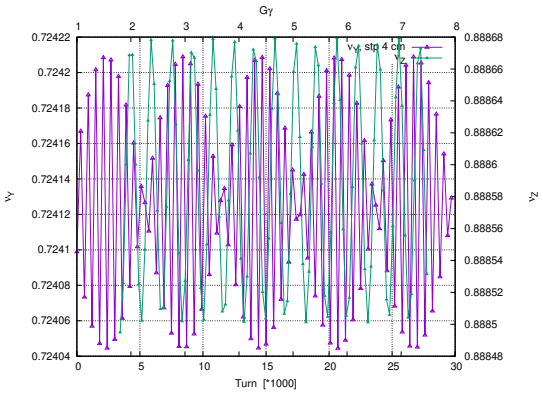

Fig. 20.64 Horizontal ring tune (left vertical axis), $\nu_Y \approx 0.7241$, and vertical ring tune (right vertical axis), $\nu_Z \approx 0.8885$, as a function of turn number, over 30,000 turns (E : 0.0036 $\rightarrow$ 3 GeV at a rate of 100 keV/turn)

(f) Spin tracking. Bunch polarization.

Spin depolarizing resonances in Saturne I synchrotron are located at (Figs. 20.65, 20.66)

$$G\gamma_R = k \pm \nu_Z = k \pm 0.888284 \quad \equiv 4 - 0.888284,\ 4 + 0.888284,\ 8 - 0.888284$$

where ν_Z has been taken from Tab. 20.50, or from Fig. 20.64. $G\gamma_R$ is bounded by $G\gamma(3\,GeV) = 7.525238 < 8 + \nu_Z$

The simulation data file to track through these resonances is the same as in question (e), Tab. 20.54, except for the following substitutions:

- substitute MCOBJET (to be uncommented) to OBJET (to be commented),
- under CAVITE substitute a peak voltage $V = 20\,\text{kV}$ to $V = 200\,\text{kV}$,
- under REBELOTE, request a 300,000 turn cycle rather than 30,000.

MCOBJET creates a 200 particle bunch with Gaussian transverse and longitudinal densities, with the following *rms* values at 3.6 MeV:

$$\epsilon_Y/\pi = 25\,\mu m, \quad \epsilon_Z/\pi = 10\,\mu m, \quad \frac{dp}{p} = 10^{-4}$$

CAVITE accelerates that bunch from 3.6 MeV to 3 GeV at a rate of $q\hat{V}\sin(\phi_s) = 10\,\text{keV/turn}$ ($\hat{V} = 20\,kV$, $\phi_s = 30^o$), in 300,000 turns.

Figure 20.65 shows sample S_Z spin components of a few particles taken among the 200 tracked. Figure 20.66 displays $\langle S_Z \rangle$, the vertical polarization component of the 200 particle set. A gnuplot script is used, given in Tab. 20.55.

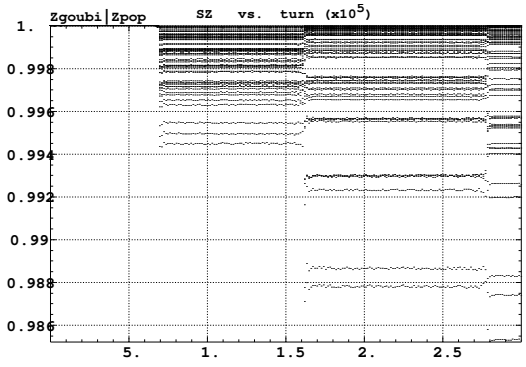

Fig. 20.65 Vertical spin component of a few particles accelerated from 3.6 MeV to 3 GeV. This plot is obtained using zpop, which reads data from [b_]zgobui.fai: menu 7; 1/2 to open b_zgoubi.fai; 2/[20,23] to select S_Z versus turn; 7 to plot

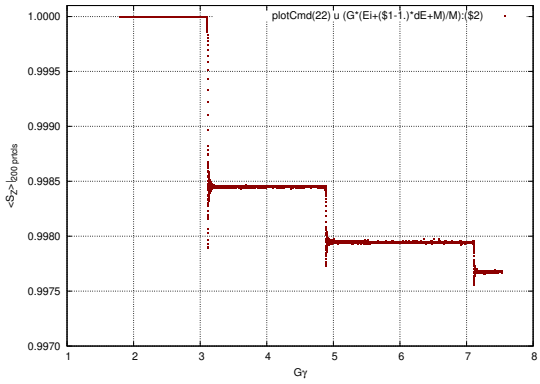

Fig. 20.66 Average vertical spin component of a 200 particle bunch, accelerated from 3.6 MeV to 3 GeV

Table 20.55 A gnuplot script to plot the average vertical spin component of the 200 particle set, along the acceleration ramp (Fig. 20.66). The average is prior computed by an awk script, which reads the necessary data from zgoubi.fai.

```
# gnuplot_avrgFromFai.gnu
fName = 'zgoubi.fai'; plotCmd(col_num)=sprintf('< gawk -f average.awk -v col_num=%d %s', col_num, fName)
set xtics; set ytics; set xlabel "G{/Symbol g}"; set ylabel "<S_Z>|_{200 prtcls}"
set format y '%0.4f'; set grid; set xr [:]; set yr [.997:1.0001]
Qy=0.888248;
do for [intgr=1:2] {   set arrow nohead from 4*intgr-Qy, 0.997 to 4*intgr-Qy, 1.0001 lw 1 dt 2
                       set arrow nohead from 4*intgr+Qy, 0.997 to 4*intgr+Qy, 1.0001 lw 1 dt 2  }
M=938.27208; Ei = 3.6; G = 1.79284735; Qy = 0.888284; dE = 0.01  # MeV/turn
plot  plotCmd(22) u (G*(Ei+($1-1.)*dE+M)/M):($2) w p pt 5 ps .4 lc rgb 'dark-red'; pause 1
```

average.awk script to compute $\langle S_Z \rangle$ [5]:

```
function average(x, data){
    n = 0;mean = 0;
    val_min = 0;val_max = 0;
    for(val in data){
        n += 1;
        delta = val - mean;
        mean += delta/n;
        val_min = (n == 1)?val:((val < val_min)?val:val_min);
        val_max = (n == 1)?val:((val > val_max)?val:val_max);
    }
    if(n > 0){
        print x, mean, val_min, val_max;
    }}
{    curr = $38;
    yval = $(col_num);

    if(NR==1 || prev != curr){
        average(prev, data);
        delete data;
        prev = curr;   }
    data[yval] = 1;  }
END{
    average(curr, data);  }
```

(g) Crossing an isolated intrinsic depolarizing resonance.

The simulation uses the file given in Tab. 20.54, with the following changes:

- Under OBJET:
  - 1st line, change the reference rigidity to $BORO = 4.08807740024\,\mathrm{T\,m}$ which corresponds to an initial $G\gamma = 2.65$, 448.604769 MeV (in lieu of 0.274426548 T m, 3.6 MeV), which is conveniently upstream of $G\gamma_R = 4 - \nu_Z \approx 3.1$,
  - 3rd line, request a single particle ("1 1 1", in lieu of "1 11 1" which distributes 11 particles on the vertical invariant),
  - 6th line, set the invariant ϵ_Z/π to the desired value,
- change the dipole field accordingly under DIPOLE, to 4.85560248505 T (in lieu of 0.3259493638 T), which maintains the expected curvature radius $\rho_0 = BORO/B = 8.4193\,\mathrm{m}$ (Tab. 9.2),
- under REBELOTE, set the number of turns. A total of 15,000, about 7,500 turns upstream and as many downstream of the resonance, is convenient for the present peak voltage $\hat{V} = 20\,\mathrm{kV}$ (acceleration rate 20 keV/turn).

Changing the particle invariant value

Particle spin motion through the isolated resonance for five different invariant values, $\epsilon_Z/\pi = 2,\ 10,\ 20,\ 40,\ 200\,\mu$m, is displayed in Fig.20.67.

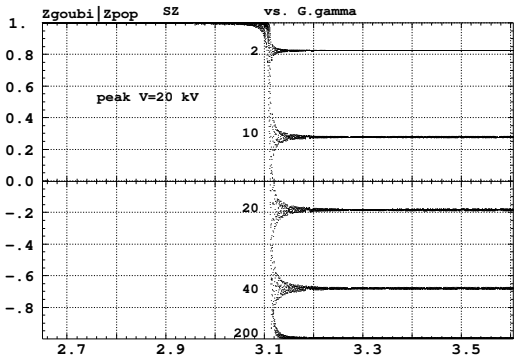

Fig. 20.67 Turn by turn spin motion through the isolated resonance $G\gamma_R = 4 - \nu_Z$, observed at the beginning of the optical sequence (FAISTORE location, Tab. 20.54) for six different values of the particle invariant from $2\,\mu$m to $200\,\mu$m where full spin flip occurs. A graph obtained using zpop, which reads particle data from b_polarLand.fai (as specified under FAISTORE): menu 7; 1/8 to choose b_polarLand.fai; 2/[59,23] to select S_Z versus $G\gamma$; 7 to plot

The intrinsic resonance strength satisfies $\epsilon_R^2 = A\,\epsilon_Z$, with A a factor which characterizes the lattice. On the other hand, from the Froissart-Stora formula (Eq. 9.45) one gets

$$\epsilon_R^2 = \frac{2\alpha}{\pi} \ln\left(\frac{2}{1 + S_{Z,f}/S_{Z,i}}\right) \overset{\frac{S_{Z,f}}{S_{Z,i}} \ll 1}{\longrightarrow} \frac{\alpha}{\pi}\left(1 - \frac{S_{Z,f}}{S_{Z,i}}\right) \tag{20.15}$$

with α, crossing speed, a constant. Thus one expects to find $\ln\left(\frac{2}{1+S_{Z,f}/S_{Z,i}}\right)$ / ϵ_Z constant. This property appears to be satisfied by the tracking outcomes, Tab. 20.56. Note that numerical results slowly depart from that rule if $P_f/P_I \rightarrow 1$ or when approaching full flip, explaining this variation requires closer inspection of the theory and dedicated simulations.

Calculation of the resonance strength from the P_f/P_i tracking results, using Eq. 20.15, requires the crossing speed, namely,

$$\alpha = \frac{1}{2\pi}\frac{\Delta E}{M} = \frac{1}{2\pi}\frac{20\times 10^3 \times \sin 30^o\,[\text{eV/turn}]}{938.27208\times 10^6[\text{eV}]} = 1.696\times 10^{-6} \qquad (20.16)$$

Table 20.56, rightmost column, displays the ratio $|\epsilon_R|/\sqrt{\epsilon_Z/\pi}$ so obtained. The resonance strength $|\epsilon_R|$ appears to be proportional to $\sqrt{\epsilon_Z/\pi}$, this is the expected result [6].

Table 20.56 Relationship between the invariant value ϵ_Z/π and the quantity $\ln\left(\frac{2}{1+S_{Z,f}/S_{Z,i}}\right) \propto |\epsilon_R|$. $\hat{V} = 20\,\text{kV}$, here. $S_{Z,f}/S_{Z,i}$ (col. 2) is from raytracing, ϵ_R^2 results, using Eq. 20.15 with crossing speed $\alpha = 1.696\times 10^{-6}$ (Eq. 20.16). The resulting ratio ϵ_R^2 / ϵ_Z/π, rightmost column, appears to be about constant, this is the expected result (Eq. 20.15)

ϵ_Z/π ($\times 10^{-4}\,\mu$m)	$\frac{S_{Z,f}}{S_{Z,i}} \equiv S_{Z,f}$	$\ln\frac{2}{1+S_{Z,f}}$	$\frac{\ln\frac{2}{1+S_{Z,f}}}{\epsilon_Z/\pi}$	$\frac{\epsilon_R^2}{\epsilon_Z/\pi}$ ($\times 10^{-8}$)
1	+0.9	0.0513	0.0513	5.5
2	+0.822	0.0932	0.0466	5.0
10	+0.27	0.454	0.0454	4.9
20	-0.19	0.904	0.0452	4.9
40	-0.68	1.833	0.0458	4.9
50	-0.775	2.19	0.438	4.7
80	-0.963	3.297	0.0412	4.5

Changing the crossing speed

A crossing case using $\hat{V} = 10\,\text{kV}$ is displayed in Fig. 20.68. an acceleration rate/resonance crossing speed twice as slow compared to the previous ones.

Comparison with the case $\hat{V} = 20\,\text{kV}$ is given in Tab. 20.57. From what precedes, with the crossing speed $\alpha \propto \hat{V} \propto \Delta E/\text{turn}$, one expects to find $\frac{\hat{V}}{\epsilon_Z/\pi} \times \ln\left(\frac{2}{1+S_{Z,f}/S_{Z,i}}\right)$ =constant. This property appears to be satisfied by tracking outcomes, Tab. 20.57.

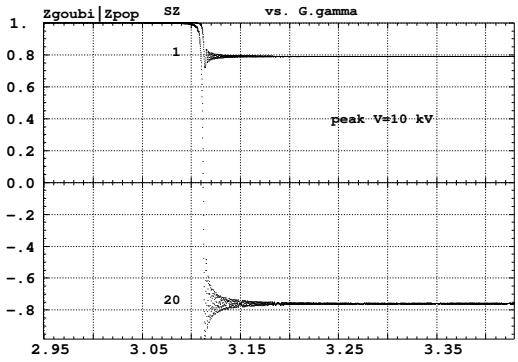

Fig. 20.68 Turn by turn spin motion through the isolated resonance $G\gamma_R = 4 - \nu_Z$, observed at the beginning of the optical sequence (FAISTORE location, Tab. 20.54) for $\epsilon_Z/\pi = 1\,\mu$m and $20\,\mu$m. $\hat{V} = 20$ kV. A graph obtained using zpop, which reads particle data from b_polarLand.fai (as specified under FAISTORE): menu 7; 1/8 to choose b_polarLand.fai; 2/[59,23] to select S_Z versus $G\gamma$; 7 to plot

Table 20.57 Relationship between the acceleration rate $\Delta E \propto \hat{V}$ and the quantity $\ln\left(\frac{2}{1+S_{Z,f}/S_{Z,i}}\right)$. Normalized to ϵ_Z/π, their product (rightmost column) appears about constant, this is the expected result. Explaining the observed $\approx 10\%$ variation requires closer inspection of the theory and dedicated simulations

ϵ_Z/π ($\times 10^{-4}\,\mu$m)	$\hat{V}$ (kV)	$\frac{S_{Z,f}}{S_{Z,i}} \equiv S_{Z,f}$	$\ln\frac{2}{1+S_{Z,f}}$	$\frac{\hat{V}}{\epsilon_Z/\pi} \times \ln\frac{2}{1+S_{Z,f}}$
1	10	+0.79	0.111	1.1
20	10	-0.763	2.133	1.07
1	20	+.9	0.051	1.0
2	20	+0.82	0.094	0.94

(h) Spin motion across a weak depolarizing resonance.

The goal is to check numerical outcomes against the Fresnel integral model (Eq. 9.51). A weak resonance is obtained using small amplitude vertical motion and fast crossing.

A single particle is raytraced, in the following conditions:

- resonance to be crossed: $G\gamma_R = 4 - \nu_y \approx 3.1115$,
- acceleration: peak voltage $\hat{V} = 100\,\text{kV}$, synchronous phase $\phi_s = 30^o$,
- particle invariant $\epsilon_Z/\pi = 10^{-6}\,\mu\text{m}$.

The initial rigidity is taken a few hundred turns upstream of the resonance, namely, $B\rho_{\text{ref}} = 4.0880774\,\text{T m}$, 605226550 MeV, $G\gamma = 2.94931241$, a distance to $G\gamma_R$ of $4 - \nu_Z - 2.949312415 \approx 0.16223$. Tracking extends a few thousand turns beyond $G\gamma_R$ so that S_Z reaches its asymptotic value, from which the resonance strength $|\epsilon_R|$ can be calculated, using Eq. 20.15.

The simulation file is given in Tab. 20.58. Note the new setting of the SCALING factor SCL: DIPOLE field was set for a curvature radius $\rho_0 = 8.4193\,\text{m}$, given a reference rigidity $B\rho_{\text{ref}} \equiv BORO = 0.274426548\,\text{Tm}$ (Tab. 20.46). However the reference rigidity is now changed to $B\rho_{\text{ref}} = 4.0880774\,\text{T m}$, thus maintaining ρ_0 requires scaling the field in DIPOLE by $4.0880774/0.274426548 = 14.8968$ at turn 1: this is the new factor, $SCL = 14.8968$, under SCALING (Tab. 20.58). Option NT=-1 under SCALING ensures that the scaling factor will automatically follow, turn-by-turn, the rigidity boost by CAVITE so preserving constant curvature radius $\rho_0 = 8.4193\,\text{m}$.

The resulting turn-by-turn spin motion is displayed in Fig. 20.69. The Fresnel integral model (Eq. 9.51) has been superimposed. Parameters in the latter are as follows:

- crossing speed $\alpha = \dfrac{1}{2\pi}\dfrac{\Delta E}{M} = \dfrac{1}{2\pi}\dfrac{10^5 \times \sin 30^o\,[\text{eV/turn}]}{938.27208 \times 10^6[\text{eV}]} = 8.4812 \times 10^{-6}$,
- asymptotic $S_{Z,f} = 0.999780$, whereas initial $S_{Z,i} = 1$, thus (Eq. 20.15)

$$\epsilon_R^2 = 5.939 \times 10^{-10}$$

- orbital angle origin set at the location of $G\gamma_R$, which is turn 1699.

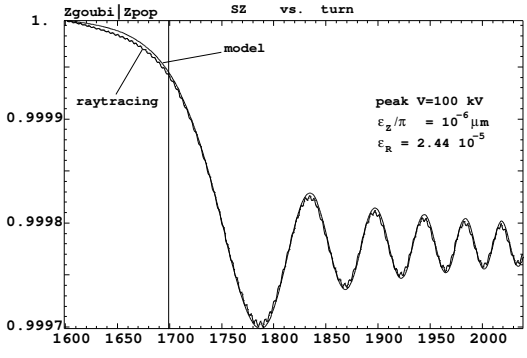

Fig. 20.69 Turn by turn spin motion through the isolated resonance $G\gamma_R = 4 - \nu_Z$, case of weak resonance strength. Modulated curve: from ray-tracing. Smooth curve: from the Fresnel integral model

Table 20.58 Simulation input data file: track a particle launched on a vertical invariant $\epsilon_y/\pi = 10^{-6}\,\mu$m, with horizontal motion indifferent, taken zero here. The INCLUDE adds the Saturne I cell four times, the latter is defined in Tab. 20.48 and Fig. 9.23

```
Saturne I ring. Crossing Ggamma=4-nu_Z, weak resonance case (small vertical invariant)
'MARKER'   SatIWeakXing_S                                              ! Just for edition purposes.
'OBJET'
4.08807740024e3                                          ! Reference Brho: 605226550 MeV proton.
8                                          ! Create a (set of) particle(s) on a given invariant.
1  1  1                                                    ! create a single particle.
0. 0. 0. 0. 0. 1.
0.  14.426 0                                              ! Horizontal invariant is null.
0. 11.411   1e-6                                 ! Periodic alpha_Z, beta_Z, and invariant value.
0. 1. 0.                                                     ! No momentum spread.
'PARTICUL'
PROTON               ! Necessary data in order to allow  (i) spin trackingand, and  (ii) acceleration.
'SPNTRK'                                                        ! Switch on spin tracking,
3                                                               ! nitial spin vertical.
'FAISCEAU'
'FAISTORE'
xing4-Qy.fai                                      ! Log particle data in xing.fai, turn-by-turn.
1

'SCALING'
1 1
DIPOLE
-1                          ! Causes field increase in DIPOLE to follow rigidity increase by CAVITE.
14.8968                                                    ! Relative rigidities at turn 1.
1

! 4 cells follow.
'INCLUDE'
1
4* ./SatI_cell.inc[S_SatI_cell:E_SatI_cell]

'CAVITE'
3
0 0
100e3  0.523598775598                                ! Acceleration rate is 200*0.5=100keV/turn.

'REBELOTE'
3999 0.3 99                                                ! A total of 3999+1=4000 turns.

'FAISCEAU'
'MARKER'   SatIWeakXing_E                                     ! Just for edition purposes.
'SPNPRT'

'END'
```

(i) Static spin motion near a resonance

The simulation input data file of Tab. 20.58 can be used for these fixed energy trials, however OBJET[KOBJ=8] has to be changed, as follows:

- OBJET[KOBJ=1] is used instead, as it allows to define a set of particles with sampled momentum offset,

- the reference rigidity in OBJET is set closest to the resonance, $G\gamma_R = 4 - \nu_y \approx 3.1115$ (considering that $\nu_Z \approx 0.8885$ (Fig. 20.64), thus $BORO \equiv B\rho_{\text{ref}} = 4.439362178$. Note that $G\gamma_R = 4 - \nu_Z$ is only known at the accuracy that ν_Z is, from prior first order mapping computation, or Fourier analysis; finding the resonant $G\gamma_R$ from spin motion will bring its value

- the SCALING factor is set to $SCL = 1$ and concurrently the DIPOLE field is set to $B_0 = 5.27283999437$,

- since half an oscillation of $S_Z(turn)$ is enough to determine $\langle S_Z \rangle$, a number of turns $IPASS \approx$ a few thousand, under REBELOTE, is enough considering the vertical motion amplitude considered. considering the initial Z=3 cm in this simulations (thus $\epsilon_Z/\pi = Z^2/\beta_Z = 79\,10^{-6}$ with $\beta_Z = 11.411$ (Tab. 20.49)), $IPASS = 3000$ is enough.

To conclude on this updating of the input data file of Tab. 20.58:
- substitute the following to OBJET:

```
'OBJET'
4.4393621786553803e3      ! BORO taken as close to resonant G.gamma as prior knowledge of nu_Z allows.
1                                                                  ! Create a set of particles.
1 1 1 1 1 41                                                         ! 41 particles sampling a
0.  0.  0.  0.  0.  .00001                          ! momentum offset, in -20*1e-5< D-1 < 20*1e-5.
0. 0. 3. 0. 0. 1.                                          ! All particles have initial Z=3cm.

```

- substitute the following to SCALING:

```
'SCALING'                                                                                    7
1 1
DIPOLE
-1                                    ! Causes field increase in DIPOLE, in correlation to particle
1.                                                             ! rigidity increase by CAVITE.
1

```

- substitute the following for B_0 (3rd line) under DIPOLE:

```
22.5 5.27283999437 -0.6 0. 0.       ! Reference angle ACENT set to AT/2; Bo field at RM; radial index.

```

The turn-by-turn values of the vertical component of the spins as they precess at fixed energy are displayed in Fig. 20.70. A quick, and accurate enough, approximation to the vertical component of the precession axis is $\langle S_Z \rangle|_{\text{period}} = \frac{1}{2}\{\min[S_Z(\theta)] + \max[S_Z(\theta)]\}$, it yields the $\langle S_Z \rangle(\Delta)$ graph of Fig. 20.71.

A match of the $\langle S_Z \rangle$ values by (Eq. 9.49)

$$S_y(\Delta) = \frac{\Delta}{\sqrt{\Delta^2 + |\epsilon_R|^2}}$$

given $G\gamma_R = 4 - nu_Z$, yield vertical tune and resonance strength values, respectively,

$$\nu_Z = 0.88845 \qquad \text{and} \qquad |\epsilon_R| = 2.77 \times 10^{-4}$$

Satisfactorily, ν_Z is consistent with earlier results, and $|\epsilon_R| = 2.77 \times 10^{-4}$ for $\epsilon_Z/\pi = 79 \times 10^{-6}$ here, is consistent in order of magnitude with $|\epsilon_R| = 2.44 \times 10^{-5}$ for

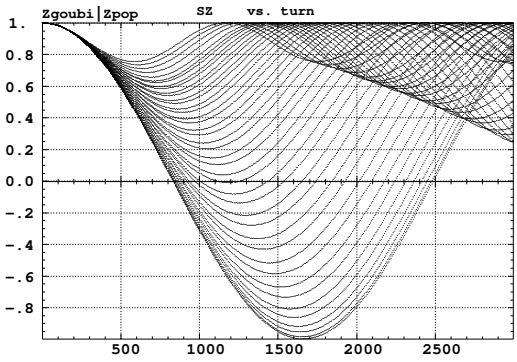

Fig. 20.70 Turn-by-turn value of the vertical component of spins precessing at fixed energy in Saturne I synchrotron, observed at the beginning of the sequence, where spins start vertical ($S_Z = 1$). The greater (respectively smaller) the distance to the resonance, the closer the precession axis to the vertical axis (resp., to the bend plane), and the greater (resp. the smaller) the oscillation frequency

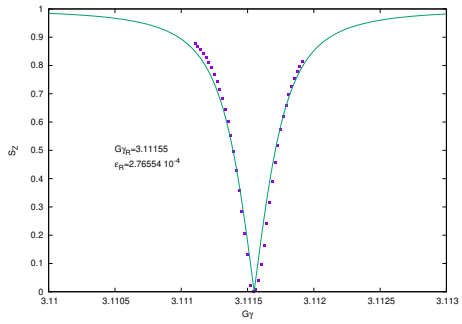

Fig. 20.71 Vertical component of the spin precession axis as a function of $G\gamma$, in the vicinity of the resonance

$\epsilon_Z/\pi = 10^{-6}$ in the previous question (h). The difference deserves further inspection, a possible additional question in this exercise.

9.2

Construct the ZGS. Spin Resonances

A photo taken in the ZGS ring can be found in Fig. 9.4. A schematic layout is given in Fig. 9.24 and a sketch of the double dipole cell in Fig. 9.25. These figures introduce to the geometry and, in complement to Tab. 9.3, display some of the parameters of the synchrotron, and will referred to in building the ZGS ring in the following.

(a) A model of ZGS synchrotron.

DIPOLE is used to simulate the cell dipoles. It is necessary to have Fig. 20.55 at hand (in addition to the users' guide), when filling up the data list under DIPOLE. Some comments regarding these data:

- DIPOLE is defined in a cylindrical coordinate system.
- The bending sector is 45 degrees, however the field region extent AT has to encompass the fringe fields, at both ends of the 45 deg sector. A large 5 deg extension is taken, for a total AT=55 deg which ensures absence of truncation of the fringe fields at the AT sector boundaries, over the all radial excursion of the beam.
- RM is given the curvature radius value, $RM = B\rho/B = 1.035270_{[\mathrm{T\,m}]}/0.04986851_{[\mathrm{T}]} =$ 20.76 m, this makes magnet positioning and closed orbit checks easier (see below). Note that the field and reference rigidity are, as for the simulation input data, for 50 MeV, the injection energy, this is an arbitrary choice.
- ACENT=27.5 deg is the reference azimuth, for the positioning of the entrance and exit effective field boundaries (EFB). It is taken in the middle of the AT range, an arbitrary choice.
- The entrance radius in the AT sector is $RE = RM/\cos(AT - \omega^+) = RM/\cos(5^o)$, with $\omega^+ = 22.5$ deg the positioning of the entrance EFB with respect to ACENT. And similarly for the positioning of the exit reference frame, $RS = RM/\cos(AT - (ACENT - \omega^-)) = RM/\cos(5^o)$ with $\omega^- = -22.5$ deg the positioning of the exit EFB. Note that $\omega^+ - \omega^- = 45^o$, the value of the bend angle.
- The entrance angle TE identifies with the extension to the 45 deg sector, namely, TE=5 deg. And similarly for the positioning of exit frame, 5 deg downstream of the exit EFB, thus TS=5 deg.

In order to build the cell, and in the first place the two cell dipole (the two dipoles are mirror symmetric, thus build one, the other follows straightforwardly), it is a good idea to proceed by steps:

(i) first build a 45 deg deviation sector in the hard edge model (Tab. 20.59). FAISCEAU located next to DIPOLE indicates that a trajectory entering DIPOLE at radius R=RM, normally to the EFB (thus, $Y_0 = 0$ and $T_0 = 0$ in OBJET) exits with Y=0 and T=0. Data validation at this stage can be performed by comparing DIPOLE's transport matrix computed with MATRIX, and the theoretical expectation (after Eq. **??**, Sec. 19.5, numerical values truncated to 4th decimal)

$$T = \begin{pmatrix} \cos\alpha & \rho\sin\alpha & 0 & 0 & 0 & \rho(1-\cos\alpha) \\ -\frac{1}{\rho}\sin\alpha & \cos\alpha & 0 & 0 & 0 & \sin\alpha \\ 0 & 0 & 1 & \rho\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & 0 & 0 & 1 & \rho(\alpha-\sin\alpha) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\rho=20.76]{\alpha=\pi/4,} \begin{pmatrix} 0.7071 & 14.6795 & 0 & 0 & 0 & 6.0804 \\ -0.03406 & 0.7071 & 0 & 0 & 0 & 0.7071 \\ 0 & 0 & 1 & 16.3048 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.7071 & 0 & 0 & 0 & 1 & 1.6253 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

MATRIX computation outcomes from raytracing can be found for comparison in Tab. 20.60.

Table 20.59 Simulation input data file: a 45 degree sector bend in the hard edge model. The reference trajectory has equal entrance and exit position, and opposite sign angles. It coincides with the arc R=RM. MATRIX computes the transport matrix of the dipole (bottom of this Table), for comparison with the fringe field model, and possible comparison with matrix codes outcomes

```
ZGS. Hard edge dipole model. Transport matrix.
'OBJET'
1.03527036749193e3                                              ! Reference Brho: 50 MeV proton.
5                                          ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                                ! Coordinate sampling.
0. 0. 0. 0. 0. 1. 'o'      ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1
'DIPOLE'                                                  ! Analytical modeling of a dipole magnet.
20                    ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
45. 2076.                    ! Field region angle=45; reference radius set to curvature radius value.
22.5 0.4986851481175 0. 0. 0.  ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
.0  0.                                                      ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                  ! Enge coefficients.
22.5 0.  1.E6  -1.E6  1.E6  1.E6                    ! Angle to ACENT; face angle; face is straight.
.0  0.                                                                                ! EFB 2.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 0.  1.E6  -1.E6  1.E6  1.E6
0. 0.                                                                          ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1                                        ! Degree of interpolation polynomial; flying grid sizing.
200.                                      ! Integration step size. It can be large in uniform field.
2 2076. 0. 2076. 0.                                 ! Positioning of entrance and exit frames.
! reference frames.
'FAISCEAU'                                                        ! Local particle coordinates.
'MATRIX'                                    ! Compute transport matrix, from trajectory coordinates.
1 0
'END'
```

(ii) next, add fringe fields, including the 5 deg extensions that add to AT (Tab. 20.61). Note that negative drifts with length $RM\tan(5^o) = 181.62646548$ cm have been added at both ends, this recovers the actual length of the trajectory across the 45 deg sector, for comparison with the hard-edge case, namely, s = 16.30486 ± 10^{-5} m under FAISCEAU in both cases. A FIT procedure finds the field value necessary for recovering the exact deviation, as the latter is changed when fringe fields are introduced. Again, FAISCEAU allows checking the correctness of DIPOLE data: exit coordinates come out to be Y=0 and T=0; however the path across the dipole is changed under the effect of the fringe fields, thus its length (s=1630.459 cm) is slightly different, compared to the hard edge case (an arc of radius radius RM=2076 cm and length 1630.487 cm)

(iii) next, add the EFB angles: the sector is closing (wedge angles $\epsilon_1 > 0$ and $\epsilon_2 > 0$ by convention) thus the EFB tilt angle θ under DIPOLE if positive at entrance, negative at exit (Fig. 20.55). In order to reach proper wave number values (this is addressed below), the wedge angles are taken to be $\epsilon_1 = 13^o$ and $\epsilon_2 = 8^o$. These considerations result in the following:

Table 20.60 Outcomes of the simulation file of Tab. 20.59

An excerpt from zgoubi.res. Coordinates of the first particle (the reference trajectory) and its path length under FAISCEAU, at OBJET on the left hand side below, locally on the right hand side:

```
         3  Keyword, label(s) :  FAISCEAU                                                                    IPASS= 1
                                          TRACE DU FAISCEAU
                                       (follows element #      2)
                                           11 TRAJECTOIRES
                                 OBJET                                          FAISCEAU
            D       Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)      D-1     Y(cm)    T(mr)    Z(cm)    P(mr)       S(cm)
  0  1   1.0000     0.000     0.000     0.000     0.000      0.0000    0.0000   -0.000   -0.000    0.000    0.000   1.630487E+03
```

Transport matrix of a 45 *degree sector, hard edge model, two difference cases of integration step size, namely,* 4 *cm and* 2 *m* (*an excerpt of MATRIX computation, from zgoubi.res*). *It can be checked against matrix transport expectations. The "first order symplectic conditions" are very small in the* 4 *cm step size case, which is an indication of accurate numerical integration of the trajectories across DIPOLE*; *the reference trajectory* (*first one*) *exits better aligned* (*reference coordinates, before change of frame for MATRIX computation, are closer to zero*):

- *Case of* 4 *cm step size*:

```
         4  Keyword, label(s) :  MATRIX                                                                      IPASS= 1

 Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
  0.00000000E+00  -3.25144356E-10  -4.13789229E-10   0.00000000E+00   0.00000000E+00   1.63048659E+03   5.43871783E-02

                 TRANSFER  MATRIX  ORDRE  1  (MKSA units)
          0.707107         14.6795          0.00000          0.00000          0.00000          6.08046
         -3.406102E-02    0.707107          0.00000          0.00000          0.00000         0.707107
          0.00000          0.00000          1.00000          16.3049          0.00000          0.00000
          0.00000          0.00000         7.285552E-16     1.00000          0.00000          0.00000
         0.707107         6.08046          0.00000          0.00000          1.00000          1.62533
          0.00000          0.00000          0.00000          0.00000          0.00000          1.00000

         DetY-1 =       0.0000000025,    DetZ-1 =       0.0000000002
         T12=0 at   -20.76     m,        T34=0 at   -16.30     m
       First order symplectic conditions (expected values = 0) :
          2.5100E-09    2.3381E-10     0.000          0.000          0.000          0.000
```

- *Case of* 2 *m step size*:

```
           4  Keyword, label(s) :  MATRIX                                                                    IPASS= 1

 Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
  0.00000000E+00  -2.01277929E-03  -2.51514609E-03   0.00000000E+00   0.00000000E+00   1.63048722E+03   5.43871994E-02

                 TRANSFER  MATRIX  ORDRE  1  (MKSA units)
          0.707105         14.6795          0.00000          0.00000          0.00000          6.08056
         -3.406102E-02    0.707108          0.00000          0.00000          0.00000         0.707120
          0.00000          0.00000          1.00000          16.3051          0.00000          0.00000
          0.00000          0.00000        1.457135E-17     1.00003          0.00000          0.00000
         0.707109         6.08048          0.00000          0.00000          1.00000          1.62531
          0.00000          0.00000          0.00000          0.00000          0.00000          1.00000

         DetY-1 =      -0.0000010903,    DetZ-1 =       0.0000286273
         R12=0 at   -20.76     m,        R34=0 at   -16.30     m
       First order symplectic conditions (expected values = 0) :
         -1.0903E-06    2.8627E-05     0.000          0.000          0.000          0.000
```

- the entrance (respectively exit) EFB of the upstream dipole of the cell (Fig. 9.25) is tilted with respect to the reference orbit by an angle $\theta = +13^{\circ}$ (resp. $\theta = -8^{\circ}$),
- the entrance (resp. exit) EFB of the downstream dipole is tilted with respect to the reference orbit by an angle $\theta = +8$ deg (resp. $\theta = -13^{\circ}$).

This final step requires again re-adjusting the radial positioning of the dipole (RE and RS, entrance and exit radius respectively), and field. In that aim the FIT procedure in Tab. 20.61 is added a variable: the RE and RS radii, coupled, and a constraint: the reference orbit has zero radial excursion at exit of the dipole. This FIT results in

Table 20.61 Simulation input data file: ZGS 45deg sector bend, with entrance and exit EFBs wedge angles and fringe fields. The reference trajectory has equal entrance and exit position, and opposite sign angles. It runs closely to the arc R=RM, not strictly coinciding with the latter due to the fringe fields. MATRIX computes the transport matrix of the dipole, for comparison with the hard edge model. Negative drifts with length $RM \tan(5^o) = 181.62646548$ cm are added to recover the hard edge path length

```
ZGS. Simplfied model. Find centered orbit in DIPOLE.
'OBJET'
1.03527036749193e3                                                  ! Reference Brho: 50 MeV proton.
5                                        ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                        ! Coordinate sampling.
0. 0. 0. 0. 0. 1. 'o'      ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1
'DRIFT'
-181.62646548
'DIPOLE'                                                ! Analytical modeling of a dipole magnet.
0                      ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
55. 2076.                    ! Field region angle=45; reference radius set to curvature radius value.
27.5 0.49860858 0. 0. 0.       ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
60.  0.                                                     ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                    ! Enge coefficients.
22.5  13.  1.E6  -1.E6  1.E6  1.E6       ! EFB angle to ACENT; 13 deg EFB tilt angle; EFB is straight.
60.  0.                                                     ! EFB 2 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 -8.  1.E6  -1.E6  1.E6  1.E6       ! EFB angle to ACENT; -8 deg EFB tilt angle; EFB is straight.
0. 0.                                                               ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1              ! Degree of interpolation polynomial; flying grid sizing is step, proper for accuracy.
4.0                                                         ! Integration step size.
2 2084.5090 -0.087266462599717 2084.5090 0.087266462599717         ! Positioning of entrance and exit.
'DRIFT'
-181.62646548
'FIT'
2
3 5 0 .1                                                            ! Vary DIPOLE field.
3 64 3.66  .1
2  1e-15  999
3   1 2  #End  0. 1. 0                          ! Request nul trajcory position  at exit of DIPOLE.
3   1 3  #End  0. 1. 0                            ! Request nul trajcory angle  at exit of DIPOLE.

'FAISCEAU'                                                  ! Local particle coordinates.
'MATRIX'                                  ! Compute transport matrix, from trajectory coordinates.
1 0
'END'
```

An excerpt from zgoubi.res. Coordinates of the first particle (the reference trajectory) and its path length, under FAISCEAU, at OBJET on the left hand side, locally on the right hand side:

```
      5  Keyword, label(s) :  FAISCEAU                                                                IPASS= 1
                                        TRACE DU FAISCEAU
                                     (follows element #      4)
                                          11 TRAJECTOIRES
                                OBJET                                          FAISCEAU
          D       Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)      D-1     Y(cm)    T(mr)    Z(cm)    P(mr)      S(cm)
O  1   1.0000     0.000     0.000     0.000     0.000      0.0000    0.0000   -0.000   -0.000    0.000    0.000   1.630459E+03     1
```

Transport matrix of ZGS 45 degree sector with EFB wedge angles and fringe fields (an excerpt of MATRIX computation, from zgoubi.res). It can be checked against matrix transport expectations. The "first order symplectic conditions" are small, which is an indication of accurate numerical integration of the trajectories across DIPOLE:

```
      7  Keyword, label(s) :  MATRIX                                                                  IPASS= 1

 Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
  0.00000000E+00  -2.19331903E-08  -2.24434360E-08   0.00000000E+00   0.00000000E+00   1.63080750E+03   6.65146963E-02

                TRANSFER  MATRIX  ORDRE  1  (MKSA units)
         0.870365         14.6806          0.00000          0.00000          0.00000          6.08068
        -2.030224E-02    0.806503          0.00000          0.00000          0.00000         0.748209
          0.00000          0.00000         0.827040          16.3143          0.00000          0.00000
          0.00000          0.00000       -1.580329E-02     0.897394          0.00000          0.00000
         0.774666          6.08004          0.00000          0.00000          1.00000          1.63006
          0.00000          0.00000          0.00000          0.00000          0.00000          1.00000

         DetY-1 =      -0.0000003451,    DetZ-1 =       0.0000000379
         T12=0 at   -18.20     m,        T34=0 at   -18.18     m
      First order symplectic conditions (expected values = 0) :
        -3.4507E-07    3.7861E-08     0.000          0.000          0.000          0.000
```

re-adjusted magnetic field and RE, RS positioning, with the respective values

$$B_0 = 0.49860858\,kG \quad \text{and} \quad RE = RS = 2084.5090\,cm$$

This is the values used in the ZGS cell simulation in Tab. 20.62,

(iv) and, finally, assemble this dipole and its mirror symmetric, in a cell (Fig. 9.25 and Tab. 20.62). The mirror symmetric is obtained by just permuting the entrance and exit wedge angles. The cell includes a half long-drift at each end, and a short drift between the dipoles. The three have been taken equal for simplification, 3.37 m long.

Table 20.62 Simulation input data file: ZGS cell simplified model, obtained by assembling DIPOLE taken from Tab. 20.61 and its mirror symmetric (which means, permuting entrance and exit EFB tilt angles θ), and adding drift spaces. This input data file defines the ZGS cell sequence segment S_ZGS_cell to E_ZGS_cell, for possible use in INCLUDE statements in subsequent exercises. It also defines, for the same purpose, the dipoles sequence segments S_ZGS-DIP_UP to E_ZGS-DIP_UP (first dipole of the cell) and S_ZGS-DIP_DW to E_ZGS-DIP_DW (second dipole of the cell). In these possible INCLUDE statements, this file is used under the name ZGS_cell.inc

```
File ZGS_cell.INC.dat.
! ZGS. Simplfied model, 8-periodic.
'MARKER'   ZGSCellMATRIX_S                                              ! Just for edition purposes.
'OBJET'
1.03527036749193e3                                                  ! Reference Brho: 50 MeV proton.
5                                        ! Create an 11 particle set, proper for MATRIX computation.
.001 .01 .001 .01 .001 .0001                                                ! Coordinate sampling.
0. 0. 0. 0. 0. 1.          ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
1

'MARKER'  S_ZGS_cell                                                        ! ZGS cell begins here.
'DRIFT' half_longDrift
337.

'MARKER' S_ZGS-DIP_UP                                                ! 1st dipole of cell begins here.
'DRIFT'
-181.62646548
'DIPOLE'  DIP_UP                                                ! Analytical modeling of a dipole magnet.
2                     ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
55. 2076.                  ! Field region angle=45; reference radius set to curvature radius value.
27.5 0.49860858 0. 0. 0.       ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
60.  0.                                                          ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                  ! Enge coefficients.
22.5  13.  1.E6  -1.E6  1.E6  1.E6             ! EFB angle to ACENT; EFB tilt angle; EFB is straight.
60.  0.                                                          ! EFB 2 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 -8.  1.E6  -1.E6  1.E6  1.E6             ! EFB angle to ACENT; EFB tilt angle; EFB is straight.
0. 0.                                                                        ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1            ! Degree of interpolation polynomial; flying grid sizing is step, proper for accuracy.
2.0                                                                   ! Integration step size.
2 2084.5090 -0.087266462599717 2084.5090 0.087266462599717        ! Positioning of entrance and exit.
'DRIFT'
-181.62646548
'MARKER' E_ZGS-DIP_UP                                                  ! 1st dipole of cell ends here.

'DRIFT' shortDrift
337.

'MARKER' S_ZGS-DIP_DW                                                ! 2nd dipole of cell begins here.
'DRIFT'
-181.62646548
'DIPOLE'  DIP_DW                                                ! Analytical modeling of a dipole magnet.
2                     ! IL=2 here, to log trajectory coordinates in zgoubi.plt, at integration steps.
55. 2076.                  ! Field region angle=45; reference radius set to curvature radius value.
27.5 0.49860858 0. 0. 0.       ! Reference angle ACENT set to AT/2; Bo field at RM; indices, all zero.
60.  0.                                                          ! EFB 1 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.                                  ! Enge coefficients.
22.5  8.  1.E6  -1.E6  1.E6  1.E6              ! EFB angle to ACENT; EFB tilt angle; EFB is straight.
60.  0.                                                          ! EFB 2 with fringe field extent.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-22.5 -13.  1.E6  -1.E6  1.E6  1.E6            ! EFB angle to ACENT; EFB tilt angle; EFB is straight.
0. 0.                                                                        ! EFB 3. Unused.
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 1            ! Degree of interpolation polynomial; flying grid sizing is step, proper for accuracy.
2.0                                                                   ! Integration step size.
2 2084.5090 -0.087266462599717 2084.5090 0.087266462599717        ! Positioning of entrance and exit.
'DRIFT'
-181.62646548
'MARKER' E_ZGS-DIP_DW                                                  ! 2nd dipole of cell ends here.

'DRIFT' half_longDrift
337.
'MARKER'  E_ZGS_cell                                                        ! ZGS cell ends here.

'FAISCEAU'                                                           ! Local particle coordinates.
'TWISS'
2  1. 1.
'MARKER'   ZGSCellMATRIX_E                                            ! Just for edition purposes.
'END'
ZGS. Simplfied model, 8-periodic.
```

An excerpt from zgoubi.res. Coordinates of the first particle (the reference trajectory) and its path length, under FAISCEAU, at OBJET on the left hand side, locally on the right hand side:

```
      18  Keyword, label(s) :  FAISCEAU                                                                IPASS= 4
                                        TRACE DU FAISCEAU
                                     (follows element #      4)
                                         11 TRAJECTOIRES
                                OBJET                                          FAISCEAU
           D       Y(cm)     T(mr)     Z(cm)     P(mr)      S(cm)      D-1     Y(cm)     T(mr)     Z(cm)     P(mr)      S(cm)
O  1   1.0000     0.000     0.000     0.000     0.000     0.0000    0.0000   -0.000    -0.000     0.000     0.000   4.272614E+03
```

Lattice parameters

The TWISS command down the sequence (Tab. 20.62) produces the periodic beam matrix results shown in Tab. 20.63

Table 20.63 Results obtained running the simulation input data file of Tab. 20.62, ZGS cell - an excerpt from zgoubi.res

```
   13  Keyword, label(s) :  TWISS                                                                                IPASS= 4

Reference, before change of frame (particle #  1  - D-1,Y,T,Z,s,time)  :
 0.00000000E+00  -1.59240732E-05  -9.81570020E-07   0.00000000E+00   0.00000000E+00   4.27261430E+03   4.53811009E-01

    Beam  matrix  (beta/-alpha/-alpha/gamma) and  periodic  dispersion  (MKSA units)
       28.633680     0.000002     0.000000     0.000000     0.000000    36.853463
        0.000002     0.034924     0.000000     0.000000     0.000000     0.000003
        0.000000     0.000000    37.008846     0.000001     0.000000    -0.000000
        0.000000     0.000000     0.000001     0.027021     0.000000     0.000000
        0.000000     0.000000     0.000000     0.000000     0.000000     0.000000
        0.000000     0.000000     0.000000     0.000000     0.000000     0.000000

                         Betatron  tunes  (Q1 Q2 modes)
               NU_Y =  0.21235913         NU_Z =  0.19286706

                         Momentum compaction :
                        dL/L / dp/p =   1.4126935
 (dp =  0.000000E+00     L(0)   =    4.27261E+03 cm,  L(0)-L(-dp) =    6.03584E-01 cm,  L(0)-L(+dp) =   -6.03595E-01 cm)

                         Transition gamma  =  8.41348710E-01

                         Chromaticities :
               dNu_y / dp/p =  4.70986585E-02          dNu_z / dp/p =  4.45745634E-02
```

The TWISS command also produces a zgoubi.TWISS.out file which details the optical functions along the sequence (at the downstream end of the optical elements. The header of that file details the optical parameters of the structure (Tab. 20.64).

Table 20.64 An excerpt of zgoubi.TWISS.out file resulting from the execution of the ZGS cell simulation input data file of Tab. 20.62. Note that the ring (4-period) wave numbers are 4 times the cell values Q1, Q2 displayed here. Optical functions (betatron function and derivative, orbit, phase advance, etc.) along the optical sequence are listed as part of zgoubi.TWISS.out following the header. The top part and last line of that listing are given below

```
@ LENGTH          %le     42.72614305
@ ALFA            %le     1.412693458
@ ORBIT5          %le                    -0
@ GAMMATR         %le    0.8413487096
@ Q1              %le    0.2123591260      [fractional]
@ Q2              %le    0.1928670550      [fractional]
@ DQ1             %le    0.4709865847E-01
@ DQ2             %le    0.4457456345E-01
@ DXMAX           %le     3.81566835E+01    @ DXMIN           %le     3.68534544E+01
@ DYMAX           %le     0.00000000E+00    @ DYMIN           %le     0.00000000E+00
@ XCOMAX          %le     3.68530296E-01    @ XCOMIN          %le    -1.59240732E-07
@ YCOMAX          %le     0.00000000E+00    @ YCOMIN          %le     0.00000000E+00
@ BETXMAX         %le     3.25272034E+01    @ BETXMIN         %le     2.86307346E+01
@ BETYMAX         %le     3.73198843E+01    @ BETYMIN         %le     3.50936471E+01
@ XCORMS          %le     8.67153286E-04
@ YCORMS          %le     0.
@ DXRMS           %le     6.22665688E-01
@ DYRMS           %le     0.00000000E+00
```

Top and bottom four lines (*truncated*) *of zgoubi.TWISS.out optical functions listing, including the periodic* β_x, β_y (β_Y, β_Z *in zgoubi notations*) *and* D_x (η_Y *in zgoubi notations*) *values at cell ends*:

```
# alfx          btx           alfy          bty           alfl          btl           Dx            Dxp
-2.2668087e-6  2.8636996e+1 -1.0203802e-6  3.7013045e+1  0.0000000e+0  0.0000000e+0  3.6856253e+1  1.2733594e-4  etc.
-2.2589191e-6  2.8636995e+1 -9.9937511e-7  3.7013042e+1  0.0000000e+0  0.0000000e+0  3.6853463e+1  2.6012859e-6  etc.
-2.2589191e-6  2.8636995e+1 -9.9937511e-7  3.7013042e+1  0.0000000e+0  0.0000000e+0  3.6853463e+1  2.6012859e-6  etc.
-1.1768220e-1  2.9033592e+1 -9.1049986e-2  3.7319884e+1  0.0000000e+0  0.0000000e+0  3.6853472e+1  2.6012859e-6  etc.
.............................
1.1775697e-1  2.9027748e+1  9.1140989e-2  3.7313084e+1  0.0000000e+0  0.0000000e+0  3.6853454e+1  2.6012859e-6  etc.
1.1775697e-1  2.9027748e+1  9.1140989e-2  3.7313084e+1  0.0000000e+0  0.0000000e+0  3.6853454e+1  2.6012859e-6  etc.
5.1297527e-5  2.8630735e+1  7.3912348e-5  3.7005690e+1  0.0000000e+0  0.0000000e+0  3.6853463e+1  2.6012859e-6  etc.
5.1297527e-5  2.8630735e+1  7.3912348e-5  3.7005690e+1  0.0000000e+0  0.0000000e+0  3.6853463e+1  2.6012859e-6  etc.
```

(b) Betatron functions of the ZGS cell.

Among the various ways to produce the betatron functions along the sequence (and throughout the DIPOLEs), here are two possibilities, based on the storage of particle coordinates in zgoubi.plt during stepwise raytracing:

1. a direct way consists in using OBJET[KOBJ=5] and transport the 11-particle set so obtained across the sequence. Then, betaFromPlt from zgoubi toolbox [2] can be used to compute the transport matrix, step by step across the sequence, from the coordinate values logged in zgoubi.plt during the stepwise integration;
2. an indirect way consists in launching a few particles on a common invariant (horizontal and/or vertical) and subsequently plot the s-dependent quantities $\hat{Y}^2(s)/\epsilon_Y$ and/or $\hat{Z}^2(s)/\epsilon_Z$. The maximum value of the latter, a function of the distance s, is the betatron function along the sequence, $\beta_{Y,Z}(s)$.

The second method is used here (this is an arbitrary choice. Exercises may be found in the various Chapters, that use the first method and may be referred to, if desired).

The input data file to derive the betatron function following method (2) above is given in Tab. 20.65. The initial ellipse parameters (under OBJET) are the periodic values, namely, $\alpha_Y = \alpha_Z = 0$, $\beta_Y = 28.63\,\text{m}$, $\beta_Z = 37.01\,\text{m}$, they are a subproduct of the TWISS procedure performed in (a), to be found in zgoubi.TWISS.out (Tab. 20.64). The resulting envelopes and their squared value are shown in Fig. 20.72. Note that this raytracing also provides the coordinates of the 60 particles on their common upright invariant

$$x^2/\beta_x + \beta_x x'^2 = \epsilon_x/\pi$$

at start and at the end of the cell (with x standing for either Y or Z, and $\epsilon_{Y,Z}/\pi = 10^{-4}$, here). This allows checking that the initial ellipse parameters (under OBJET, Tab. 20.65) are effectively periodic values, and that the raytracing went correctly, namely by observing that the initial and final ellipses do superimpose (Fig. 20.73).

Table 20.65 Simulation input data file: raytrace 60 particles across ZGS cell to generate beam envelopes. Store particle data in zgoubi.plt, along DRIFTs and DIPOLEs. The INCLUDE file and segments are defined in Tab. 20.62

```
ZGS envelopes.
'OBJET'
1.03527036749193e3                                                    ! Reference Brho: 50 MeV proton.
8                              ! Create a set of 60 particles evenly distributed on the same invariant;
1 60 1    ! case of 60 particles on a vertical invariant; use 60 1 1 instead for horizontal invariant.
0. 0. 0. 0. 0. 1.
0.  2.8637014E+1 0e-4
0.  3.7012633E+1 1e-4
0. 1. 0.

'FAISTORE'                                  ! This logs the coordinates of the particle to zgoubi.fai,
zgoubi.fai  S_ZGS_cell  E_ZGS_cell                                  ! at the two LABEL1s as indicated.
1

'MARKER'  S_ZGS_cell
'DRIFT' half_longDrift                                ! Option 'split' devides the drift in 10 pieces,
337. split 10  2                                    ! 'IL=2' causes log of particle data to zgoubi.plt.

'INCLUDE'
1
./ZGS_cell.inc[S_ZGS-DIP_UP:E_ZGS-DIP_UP]

'DRIFT' shortDrift
337.  split 10  2

'INCLUDE'
1
./ZGS_cell.inc[S_ZGS-DIP_DW:E_ZGS-DIP_DW]

'DRIFT' half_longDrift
337.  split 10  2
'MARKER'  E_ZGS_cell

'FAISCEAU'
'END'
```

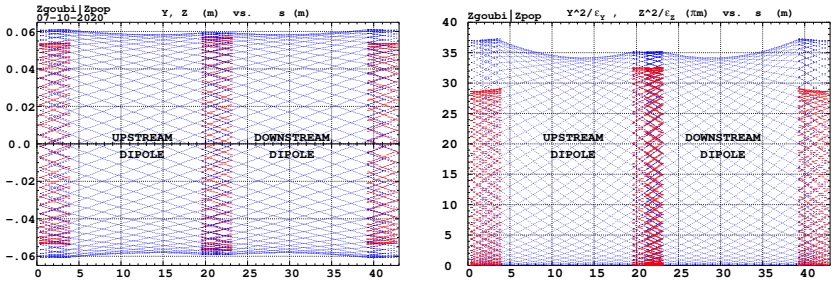

Fig. 20.72 Left: horizontal and vertical envelopes as generated by plotting the coordinates Y(s) (thick lines, red, along the drifts only) or Z(s) (thin lines, blue) across the ZGS cell, of 60 particles evenly distributed on a common $10^{-4}\,\pi\mu$m invariant, either horizontal or vertical (while the other invariant is zero). Right: a plot of $Y^2(s)/\epsilon_Y$ and $Y^2(s)/\epsilon_Y$: the extrema identify with $\beta_Y(s)$ and $\beta_Z(s)$, respectively. The extrema extremorum values are $\hat{\beta}_Y = 32.5$ m and $\hat{\beta}_Z = 37.1$ m, respectively. These plots are obtained using zpop, which reads stepwise particle data from zgoubi.plt

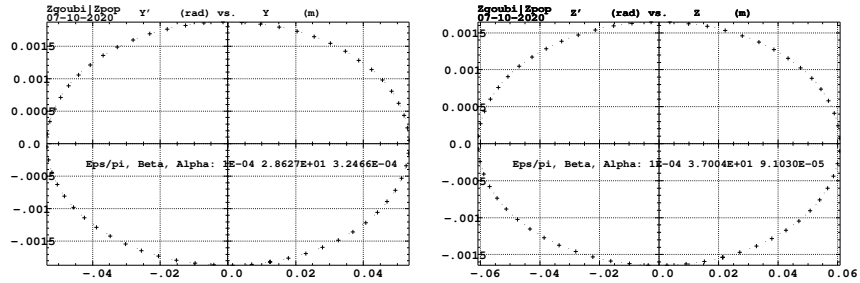

Fig. 20.73 Sixty particles evenly distributed on a common periodic invariant (of value either $\epsilon_Y = 10^{-4}\pi\mu$m and $\epsilon_Z = 0$, left plot, or the reverse, right plot) have been tracked from start to end of the cell. These periodic invariants are defined assuming the periodic ellipse parameters determined from prior TWISS, given in Tab. 20.64; values resulting from an *rms* match of the coordinates are given in the figure, and do agree with those TWISS data. The figure shows the good superposition of the start and end invariants (the start and end *rms* match ellipse parameters show negligible difference), which confirms the correct value of the periodic ellipse parameters, namely, left graph: horizontal phase space at start (crosses) and end (dots) of the cell; right graph: vertical phase space at start (crosses) and end (dots) of the cell

Dispersion function

Raytracing off-momentum particles on their chromatic closed orbit provides the periodic dispersion function. In order to do so, the input data file of Tab. 20.65 can be used, it just requires changing OBJET to the following:

```
'OBJET'
1.03527036749193e3                                                    ! Reference Brho: 50 MeV proton.
2                                                                     ! Create particles individually'
3 1                                                                                ! three particles.
+36.85e-1 0. 0. 0. 0. 1.001 'p' ! Chromatic orbit coordinates Y0 and T0 for D=1.001 relative rigidity.
0. 0. 0. 0. 0. 1.  'o'                                                             ! On-momentum orbit.
-36.85e-1 0. 0. 0. 0. 0.999 'm' ! Chromatic orbit coordinates Y0 and T0 for D=0.999 relative rigidity.
1 1 1
```

The position and angle of the chromatic particles, which are offset by $\Delta p/p = \pm 10^{-3}$, are drawn from the value of the periodic dispersion $\eta_Y = 36.85\,\mathrm{m}$ and its derivative $\eta'_Y \approx 0$ (Tab. 20.64), namely, $Y_0 = \eta_Y\,\Delta p/p = \pm 3.685\,\mathrm{cm}$ and $T_0 = \eta_Y\,\Delta p/p = 0$.

Running Tab. 20.65 simulation file with this new OBJET produces the following coordinates at FAISCEAU, located at the end of the sequence (an excerpt from zgoubi.res):

```
          18  Keyword, label(s) :  FAISCEAU                                                    IPASS= 1
                                          TRACE DU FAISCEAU
                                       (follows element #     17)
                                            3 TRAJECTOIRES
                                 OBJET                                  FAISCEAU
              D       Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)      D-1      Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)
    p  1   1.0010    3.685     0.000     0.000     0.000      0.0000    0.0010    3.685     0.000     0.000     0.000    4.278650E+03
    o  1   1.0000    0.000     0.000     0.000     0.000      0.0000    0.0000    0.000     0.000     0.000     0.000    4.272614E+03
    m  1   0.9990   -3.685     0.000     0.000     0.000      0.0000   -0.0010   -3.685    -0.000     0.000     0.000    4.266579E+03
```

The local coordinates Y, T (under FAISCEAU, right hand side) are equal to the initial coordinates Y_0, T_0 (under OBJET, left hand side), to better than 5 μm, 0.5 μrad accuracy respectively (zgoubi.fai can be consulted for greater precision on these values), so confirming the periodicity of these chromatic trajectories. Figure 20.74 shows the particle trajectories through the two DIPOLEs. A difference between the on- and off-momentum trajectories yields as expected a quasi-constant $\eta_Y \approx 36.8\,\mathrm{m}$ whereas $\eta'_Y \approx 0$. η_Y departs from exactly zero due to the fringe fields and to the wedge focusing.

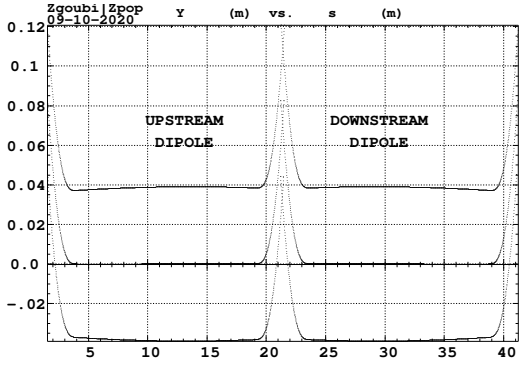

Fig. 20.74 A plot of the radial excursion, within DIPOLE range (namely, AT=55^o extent, Tab. 20.62), of an on-momentum particle (its radial position in the dipole body is $R_0 \approx 20.7628$ m, corresponding to Y=0 in this graph) and two particles at respectively $dp/p = \pm 10^{-3}$. The diverging parts at DIPOLE ends are in the 5 deg fringe field regions. A graph obtained using zpop, which reads stepwise particle data from zgoubi.plt: menu 7; 1/1 to open zgoubi.plt; 2/[6,2] to select Y versus distance; 7 to plot

Orbit difference

The method can be used to compute the dispersion function, just like in machine operation. This requires tracking a particle with +dp/p momentum offset, save its zgoubi.plt data (say, in zgoubi.plt+dpp), and repeat with -dp/p (zgoubi.plt-dpp). A gnuplot script can compute and plot the orbit difference, and normalize to dp/p; the result is the periodic dispersion, displayed in Fig. 20.75.

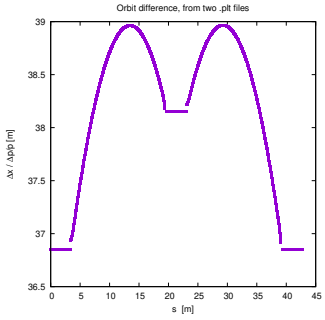

Fig. 20.75 Dispersion function along ZGS cell, obtained by orbit difference. The discontinuities are artifacts, they are located in the overlapping regions between the optical sequence DIPOLEs and DRIFTs (Tab. 20.65)

(c) Some verifications regarding the model.

The field along large excursion orbits can be logged in zgoubi.plt, using option IL=2 (or 20, or 200, etc. for printout every 10, or 100, etc. integration step) under DIPOLE.

The simulation file of Tab. 20.65 is used to raytrace five particles, with OBJET changed to the following:

```
'OBJET'
1.03527036749193e3                                              ! Reference Brho: 50 MeV proton.
2                                                               ! Create particles individually,
5 1                                                                     ! five particles.
+36.85e-1 0. 0. 0. 0. 1.01  'p'          ! Chromatic orbit coordinates for D=1.01 relative rigidity.
0. 0. 0. 0. 0. 1.  '0'                                                  ! On-momentum closed orbit.
-36.85e-1 0. 0. 0. 0. 0.99  'm'          ! Chromatic orbit coordinates for D=0.99 relative rigidity.
0. 0. 5. 0. 0.  1.   'm'                      ! Initial vertial excursion is Z0= 5 cm off-mid-plane.
0. 0. 20. 0. 0.  1.  'm'                      ! Initial vertial excursion is Z0=20 cm off-mid-plane.
1 1 1 1 1
```

Apart from the on-momentum particle (2nd in the list) this OBJET defines two particles on $\Delta p/p = \pm 1\%$ chromatic orbit (1st and 3rd in the list), this is an excursion of a few tens of centimeters, large as requested, as $\Delta x \approx 38 \times dp/p$. OBJET also defines 2 particles launched into the cell at respectively $Z_0 = 5$ cm and $Z_0 = 20$ cm.

The magnetic field as a function of the azimuthal angle in DIPOLE frame, along these trajectories across the upstream DIPOLE of the cell, is shown in Fig. 20.76. The field curves for the first four trajectories essentially superimpose except for the fringe field regions (Fig. 20.76), due to the wedge angles. This behaves as expected. Detail inspection is possible, from the detailed particle coordinate and field data in zgoubi.plt - this is out of the scope of the present question.

The field along the 5th particle trajectory features overshoots (Fig. 20.76), this is due to the very large vertical excursion ($Z \approx 20$ cm in the entrance fringe field region). It looks reasonable, however it may be an artifact in the case that the high order derivatives of the field in that region are large, resulting from the truncated Taylor series method used for off mid-plane field extrapolation [1, Sec. 1.3.3].

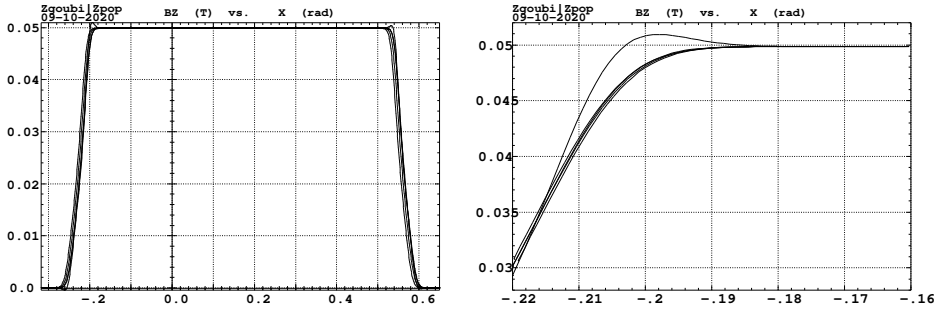

Fig. 20.76 Magnetic field along 5 different trajectories across the upstream DIPOLE, including four large horizontal and vertical excursion cases, and a zoom in on the entrance fringe field region

(d) Sinusoidal approximation of the betatron motion

The approximation

$$y(\theta) = A\ \cos(\nu_Z \theta + \phi)$$

is checked here considering the vertical motion (considering the horizontal motion leads to similar conclusions). The value of the various parameters in that expression are determined as follows:

- the particle raytraced for comparison is launched with an initial excursion $Z_0(\theta = 0) = 5\,\text{cm}$ (4th particle in OBJET, above). At the launch point (middle of the long drift) the beam ellipse is upright (Fig. 20.73), whereas phase space motion is clockwise, thus take

$$\text{A} = 5\,\text{cm} \quad \text{and} \quad \phi = \pi/2$$

- the vertical betatron tune of the 4-cell ring is (Tab. 20.64)

$$\nu_\text{Z} = 4 \times 0.192869 = 0.77147$$

- $\theta = s/R$ and $R = \oint ds/2\pi$ with (Tab. 20.64)

$$2\pi\text{R} = \text{circumference} = 4 \times 42.72614331 = 170.90457\,\text{m}$$

The comparison with a trajectory obtained from raytracing is given in Fig. 20.77 and confirms the validity of the sinusoidal approximation.

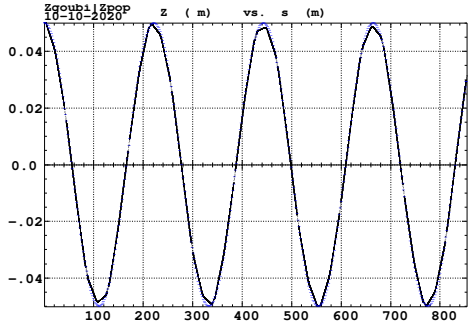

Fig. 20.77 Vertical betatron motion, five turns around the ZGS ring, from raytracing (continuous line), and sine approximation, superimposed (dashed line)

(e) An acceleration cycle. Symplecticity checks.

Eleven particles are launched for 65,000 turn tracking at a rate of

$$\Delta W = q\hat{V}\,\cos\phi_s = 400 \times \sin 150^0 = 200\,\text{keV/turn}$$

($E : 0.05 \rightarrow 13.05\,\text{GeV}$), all evenly distributed on the same initial vertical invariant

$$Z^2/\beta_Z + \beta_Z Z'^2 = \epsilon_Z/\pi \tag{20.17}$$

with $\epsilon_Z/\pi = 10^{-4}$ m, or, normalized, $\beta\gamma\epsilon_Z/\pi = 0.33078 \times 10^{-4}$ m.

The simulation file is given in Tab. 20.66. CAVITE[IOPT=3] is used, it provides an RF phase independent boost

$$\Delta W = q\hat{V}\sin\phi_s$$

as including synchrotron motion is not necessary here, even better, this ensures constant depolarizing resonance crossing speed, so precluding any possibility of multiple crossing (it can be referred to [3] regarding that effect).

Table 20.66 Simulation input data file: track 11 particles launched on the same vertical invariant, with quasi-zero horizontal invariant. The INCLUDE adds the ZGS cell four times, the latter is defined in Tab. 20.62 and Fig. 9.25. An MCOBJET is commented, it is used in a subsequent spin tracking exercise

```
ZGS ring. Polarization landscape.
'MARKER'   ZGSPolarLand_S                                          ! Just for edition purposes.
'OBJET'
1.03527036749193e3                                                 ! Reference Brho: 50 MeV proton.
8                                          ! Create an 11 particle set, proper for MATRIX computation.
1 11 1   ! Define 9 particles, all with ~0 horiz. inveariant, evenly spread on same vertical invariant.
0. 0. 0. 0. 0. 1. 'o'      ! Reference trajectory: all initial coordinates nul, relative rigidity D=1.
0. 28.63 0. ! Horiz. invariant taken zero. Nominal would be 0.14mu_m norm. i.e. 4.6e-8 non-normalized.
0. 37.01 150e-6      ! epsilon_Z/pi = beta.gamma * epsilon_norm, latter =0.05e-6 m, beta.gamma=0.3308.
0. 1. 0.  0.                                                       ! All paricls are on-momentum.

!'MCOBJET'                                                                          ! Commented.
!1.03527036749193e3                                                ! Reference Brho: 50 MeV proton.
!3                                          ! Create an 11 particle set, proper for MATRIX computation.
!200
!2 2 2 2 2 2
!0. 0. 0. 0. 0. 1.
!0. 28.63 25e-6 3                                  ! Periodic alpha_Y, beta_Y, and invariant value;
!0. 37.01 10e-6 3                                  ! Periodic alpha_Z, beta_Z, and invariant value.
!0. 1. 1.e-8 3
!123456 234567 345678

'PARTICUL'
PROTON               ! Necessary data in order to allow  (i) spin trackingand, and  (ii) acceleration.
'SPNTRK'                                                                    ! Switch on spin tracking,
3                                                                  ! all initial spins vertical.
'FAISCEAU'
'FAISTORE'
b_polarLand.fai                    ! Log particle data in b_polarLand.fai, turn-by-turn; "b_" imposes
7                                                  ! binary write, which results in faster i/o.

'SCALING'
1 1
DIPOLE
-1                                     ! Causes field increase in DIPOLE, in correlation to particle
1.                                                                 ! rigidity increase by CAVITE.
1

! 4 cells follow.
'INCLUDE'
1
4* ./ZGS_cell.inc[S_ZGS_cell:E_ZGS_cell]

'CAVITE'
3
0 0
400e3  0.523598775598                                ! Acceleration rate is 400*0.5=200keV/turn.

'REBELOTE'
87000 0.3 99

'FAISCEAU'
'MARKER'   ZGSPolarLand_E                                          ! Just for edition purposes.
'SPNPRT'

'END'
```

Betatron damping

Figure 20.78 shows the damped vertical motion of the individual particles, over the acceleration range, together with the initial and final distributions of the 11 particles on elliptical invariants. Departure from the matching ellipse at the end of the acceleration cycle, 13 GeV (Eq. 20.17 with $\epsilon_Z/\pi = 2.2244\times10^{-7}$ m), is marginal.

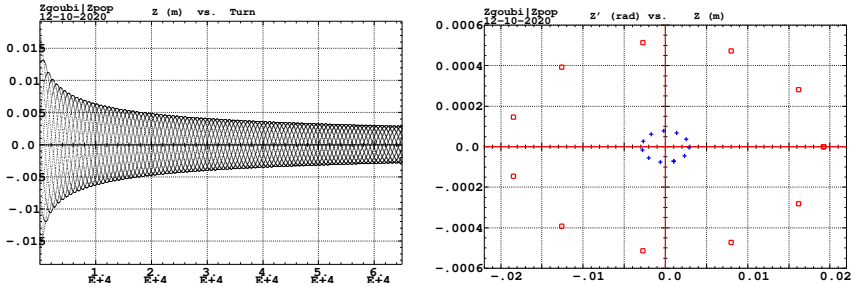

Fig. 20.78 Left: damped vertical motion, from 50 MeV to 13.05 GeV, 65,001 turns. Right: the initial coordinates of the 11 particles (squares) are taken on a common invariant $\epsilon_Z(0) = 10^{-5}\,\pi$m (at 50 MeV, $\beta\gamma = 0.33078$, thus $\beta\gamma\epsilon_Z(0) = 0.33078\times10^{-5}\,\pi$m); the final coordinates after 65,000 turns (crosses) appear to still be (with negligible departure) on a common invariant of value $\epsilon_Z(\text{final}) = 2.2244\times10^{-7}\,\pi$m (at 13 GeV, $\beta\gamma = 14.869842$) thus $\beta\gamma\epsilon_Z(\text{final}) = 0.33076\times10^{-5}\,\pi$m, equal to the initial value

Degree of non-symplecticity of the numerical integration

The degree of non-symplecticity as a function of integration step size is illustrated in Fig. 20.79. The initial motion is taken paraxial, vertical motion is considered as it resorts to off-mid plane Taylor expansion of fields [1, DIPOLE Sec.], a stringent test as the latter is expected to deteriorate further the non-symplecticity inherent to the Lorentz equation integration method (a truncated Taylor series method [1, Eq. 1.2.4]).

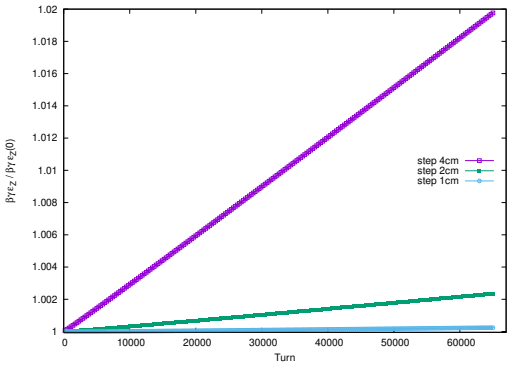

Fig. 20.79 Turn-by-turn evolution of the normalized invariant, $\beta\gamma\epsilon_Z(\text{turn})/\beta\gamma\epsilon_Z(0)$ (initial $\epsilon_Z(0)$ taken paraxial), for four different integration step size values: 1, 2 and 4 cm

Evolution of the wave numbers.

The Fortran tool tunesFromFai_iterate can be used to computes tunes as a function of turn number or energy, it reads turn by turn particle data from zgoubi.fai and computes a discrete Fourier transform over so many turns (a few tens, for instance), every so many turns [4]. Typical results are displayed in Fig. 20.80, tunes have the expected values: $\nu_Y = 0.849$, $\nu_Z = 0.771$. An acceleration rate of 200 keV/turn has been taken (namely, $\hat{V} = 400$ kV and still $\phi_s = 150^0$), to save on computing time. Note that turn-by-turn raytracing allows determining the tune value at all γ along the acceleration cycle (and thus for instance the γ values at which the resonance occurs, see (f)). In these simulations anyway the horizontal and vertical tunes are essentially constant over the all cycle: it is determined by the wedge angle, which won't charge as long as the reference orbit isn't changed. The latter holds here, as SCALING with option NTIM=-1 causes the magnet field to strictly follow the momentum boost by CAVITE.

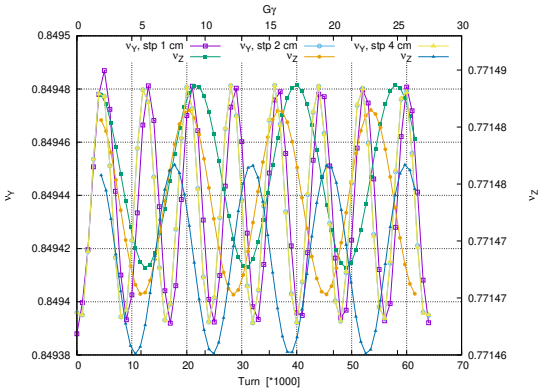

Fig. 20.80 Horizontal ring tune (left vertical axis), $\nu_Y \approx 0.8494$, and vertical ring tune (right vertical axis), $\nu_Z \approx 0.77147$, as a function of turn number, over 65,000 turns ($E : 0.05 \rightarrow 13$ GeV at a rate of 200 keV/turn). The graph displays results for 3 different integration step sizes, namely, 1, 2 and 4 cm, essentially converged

7002 (f) Spin tracking. Bunch polarization.

Spin depolarizing resonances in the ZGS are located at

$$G\gamma_R = kP \pm \nu_Z \quad = 4 - \nu_Z,\ 4 + \nu_Z,\ 8 - \nu_Z,\ 8 + \nu_Z,\ 12 - \nu_Z,\ \text{etc.}$$

7003 with P=4 the superperiodicity of the ring, and $\nu_Z = 0.77147$ taken from Tab. 20.64,
7004 or from Fig. 20.80. $G\gamma_R$ is bounded, in the present simulation, by $G\gamma(17.4\,GeV) =$
7005 $35.0 < 9P - \nu_Z$. Resonances are expected to be stronger at $G\gamma_R = 2 \times 4k \pm \nu_Z =$
7006 $8 - \nu_Z,\ 8 + \nu_Z,\ 16 - \nu_Z$, etc., with the additional factor 2 the number of cells per
7007 superperiod [6, Sec. 3.II].

The simulation data file to track through these resonances is the same as in question (e), Tab. 20.66, except for the substitution of MCOBJET (to be uncommented) to OBJET (to be commented). MCOBJET creates a 200 particle bunch with Gaussian transverse and longitudinal densities, with the following *rms* values at 50 MeV:

$$\epsilon_Y/\pi = 25\,\mu m, \quad \epsilon_Z/\pi = 10\,\mu m, \quad \frac{dp}{p} = 10^{-4}$$

7008 which are presumably close to ZGS polarized proton runs [7]. CAVITE accelerates
7009 that bunch from 50 MeV to 17.4 GeV about, at a rate of $q\hat{V}\sin(\phi_s) = 200$ keV/turn
7010 ($\hat{V} = 400\,kV$, $\phi_s = 30^o$), in 87,000 turns about.
7011 Figure 20.81 shows sample S_Z spin components of a few particles taken among
7012 the 200 tracked. Figure 20.82 displays $\langle S_Z \rangle$, the vertical polarization component of
7013 the 200 particle set. A gnuplot script is used, given in Tab. 20.67.

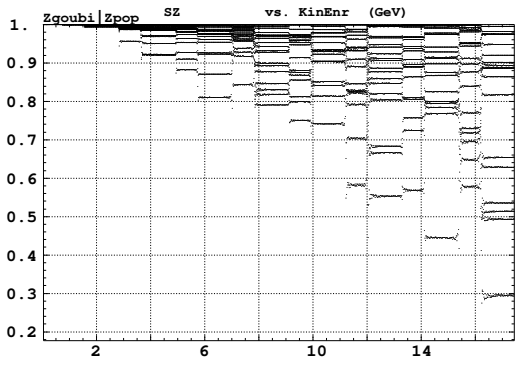

Fig. 20.81 Individual vertical spin component of 20 particles accelerated in ZGS from 50 MeV to 17.4 GeV, at a rate of 200 keV/turn. This plot is obtained using zpop, which reads data from [b_]zgobui.fai: menu 7; 1/2 to open b_zgoubi.fai; 2/[20,23] to select S_Z versus turn; 7 to plot

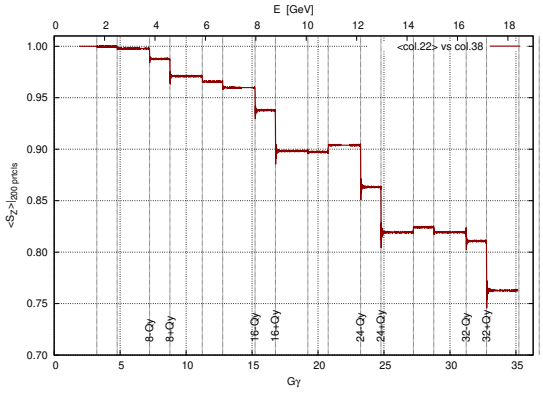

Fig. 20.82 Average vertical component of the polarization vector of a 200 particle bunch, accelerated from 50 MeV to 17.4 GeV. The vertical lines materialize the locations $G\gamma_R = 4k \pm \nu_Z$ of the depolarizing resonances. Resonances are stronger at $G\gamma_R = 8k \pm \nu_Z$ (as labeled)

Table 20.67 A gnuplot script to plot the average vertical spin component of the 200 particle set, along the acceleration ramp (Fig. 20.82). The average is prior computed by an awk script, which reads the necessary data from zgoubi.fai.

```
# gnuplot_avrgFromFai.gnu
set x2label "E  [GeV]"; set xlabel "G{/Symbol g}"; set ylabel "<S_Z>|_{200 prtcls}"
set xtics nomirror; set x2tics; set ytics; set format y '%0.2f'; set grid
M=938.27208; Ei = 50.; G = 1.79284735; Qy = 0.7715; dE = 0.2  # MeV/turn
fName = 'zgoubi.fai'; plotCmd(col_num)=sprintf('< gawk -f average.awk -v col_num=%d %s', col_num,fName)

do for [intgr=1:9] {   set arrow nohead from 4*intgr+Qy, 0.7 to 4*intgr+Qy, 1.01 lw .6 dt 3
                       set arrow nohead from 4*intgr-Qy, 0.7 to 4*intgr-Qy, 1.01 lw .6 dt 3  }

do for [intgr=8:32:8] { set label " ".intgr."-Qy" at intgr-Qy, 0.71 rotate by 90
                        set label " ".intgr."+Qy" at intgr+Qy, 0.71 rotate by 90  }

set x2r [0:19.]; set xr [0:19000./M*G]; set yr [:1.01]
plot  plotCmd(22) u (G/M*(Ei+($1-1.)*dE +M)):2 w l lw 2 lc rgb 'dark-red' t "<col.22> vs col.38"
```

average.awk script to compute $\langle S_Z \rangle$ [5]:

```
function average(x, data){
    n = 0;mean = 0;
    val_min = 0;val_max = 0;
    for(val in data){
        n += 1;
        delta = val - mean;
        mean += delta/n;
        val_min = (n == 1)?val:((val < val_min)?val:val_min);
        val_max = (n == 1)?val:((val > val_max)?val:val_max);
    }
    if(n > 0){
        print x, mean, val_min, val_max;
    }}
{    curr = $38;
    yval = $(col_num);

    if(NR==1 || prev != curr){
        average(prev, data);
        delete data;
        prev = curr;   }
    data[yval] = 1;  }
END{
    average(curr, data);  }
```

(g) Crossing an intrinsic depolarizing resonance.

The simulation data files of question (f) can be used here, Tab. 20.66, *mutatis mutandis*, and the methodology in (f) can be followed. In particular, the following changes are needed:

- Under OBJET:
  - 1st line, change the reference rigidity $BORO$ to the proper value, some distance upstream of the resonance to be crossed,
  - 3rd line, request a single particle ("1 1 1", in lieu of "1 11 1" which distributes 11 particles on the vertical invariant),
  - 6th line, set the invariant ϵ_Z/π to the desired value,
- change the dipole field accordingly under DIPOLE, to maintains the expected curvature radius $\rho_0 = BORO/B = 20.76\,\mathrm{m}$ (Tab. 9.3,
- under CAVITE, provide the desired peak voltage $\hat{V}$,
- under REBELOTE, set the number of turns: a few thousands of turns upstream and downstream of the resonance.

On the other hand, similar simulations are performed in questions (f)-(i) of exercise 9.1. Please refer to the solutions of these Saturne I simulations.

(h) Study of an imperfection depolarizing resonance.

The simulation data files of question (g) can be used here, *mutatis mutandis*, and the methodology in (g) can be followed.

On the other hand, similar simulations are performed in questions (f)-(i) of exercise 9.1, as well as in the "Strong Focusing Synchrotron" Chapter, Sec. 20.5. Please refer to the solutions of these simulations.

References

1. Méot, F.: Zgoubi Users' Guide.
   https://www.osti.gov/biblio/1062013-zgoubi-users-guide Sourceforge latest version:
   https://sourceforge.net/p/zgoubi/code/HEAD/tree/trunk/guide/Zgoubi.pdf
2. A postprocessing tool to transport betatron functions step-by-step, using raytracing data stored in zgoubi.plt.
   https://sourceforge.net/p/zgoubi/code/HEAD/tree/trunk/toolbox/betaFromPlt/
3. Aniel, T., et al.: Polarized particles at SATURNE. Journal de Physique, Colloque C2, supplément au n02, Tome 46, février 1985, page C2-499.
   https://hal.archives-ouvertes.fr/jpa-00224582
4. The Fortran tunesFrmFai_iterate.f, together with a README and an example of its use, can be found at
   https://sourceforge.net/p/zgoubi/code/HEAD/tree/trunk/toolbox/tunesFromFai/
5. https://stackoverflow.com/questions/42677017/plot-average-of-nth-rows-in-gnuplot
6. Lee, S.Y.: Spin Dynamics and Snakes in Synchrotrons. World Scientific, 1997
7. Khoe, T.K., et al.: The High Energy Polarized Beam at the ZGS. Procs. IXth Int. Conf on High Energy Accelerators, Dubna, pp. 288-294 (1974)