PHY 564
Advanced Accelerator Physics
Lecture 3:
Review of Linear Algebra

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Matrix: definition and properties

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix}$$

Addition: $A + B = C \Leftrightarrow a_{ij} + b_{ij} = c_{ij}$

Multiplied by a constant: $kA = B \Leftrightarrow ka_{ij} = b_{ij}$

Equality: $A = B \Leftrightarrow a_{ij} = b_{ij}$

Multiplication (inner product): $AB = C \Leftrightarrow \sum_k a_{ik} b_{kj} = c_{ij}$

* Multiplication demands that A has the same number of columns as B has rows.

$$A(B+C) = AB + AC, \qquad (AB)C = A(BC)$$

* In general $AB \neq BA$

$$(AB)C = \sum_k \left(\sum_j a_{ij} b_{jk} \right) c_{kl} = \sum_{k,j} a_{ij} b_{jk} c_{kl} = \sum_j a_{ij} \sum_k \left(b_{jk} c_{kl} \right) = A(BC)$$

Matrix: special cases I

- Diagonal matrix:

$$a_{ij} = 0 \quad \text{for} \;\; i \neq j$$

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & 0 & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

If A and B are both diagonal matrix, they are commutative:

$$AB = BA$$

- Identity matrix:

$$AI = IA = A \qquad \text{for} \;\; \forall A$$

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & 0 & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Matrix: special cases II

- Block diagonal matrix: A and A_i are square matrix.

$$A = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{bmatrix},$$

$$A = \left[\begin{array}{ccc|c|cc} 1 & 3 & 2 & 0 & 0 & 0 \\ 7 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{array}\right]$$

- Triangular matrix:

Upper diagonal matrix: elements below diagonal are all zero

$$U = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Lower diagonal matrix: elements above diagonal are all zero

$$L = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Matrix : transpose matrix

- A matrix, B, is called the transpose matrix of a matrix A if

$$A_{ij} = B_{ji}$$

  The transpose matrix is often denoted as A^T,
  i.e.
$$A_{ij} = \left(A^T\right)_{ji}$$

- A square matrix A is called an orthogonal matrix if

$$A^T A = AA^T = I$$

In this case, $A^T = A^{-1}$, i.e. the inverse matrix of A

We will talk more about the inverse matrix later...

- A square matrix A is called symmetric matrix if

$A^T = A$ and anti-symmetric if $A^T = -A$

Matrix: trace

- In any square matrix, the sum of the diagonal elements is called the trace.

$$Tr(A) = \sum_i a_{ii}$$

- A useful property: $Tr(AB) = Tr(BA)$

$$Tr(AB) = \sum_i \left(\sum_j A_{ij} B_{ji} \right)$$
$$= \sum_j \left(\sum_i B_{ji} A_{ij} \right) = Tr(BA)$$

- In general, $Tr(ABC) = Tr(BCA) \neq Tr(BAC)$

- Trace is a linear operator:

$$Tr(A + kB) = Tr(A) + k \cdot Tr(B)$$

Matrix: determinant of a matrix

- For a square matrix,

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

- The determinant

$$D=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}=\det(A)$$

is called the determinant of matrix A and is denoted by det(A).

Determinant I

n columns

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

n rows

$$= \sum_{i,j,k} \varepsilon_{ijk\dots} a_{1i} a_{2j} a_{3k} \cdots$$

$$= \sum_{i,j,k} \varepsilon_{ijk\dots} a_{i1} a_{j2} a_{k3} \cdots$$

$\varepsilon_{ijk\cdots}$ is Levi-Civita symbol

$$\varepsilon_{ijk\cdots} = \begin{cases} 1 & \text{if } (i, j, k\dots) \text{ is even permutation of } (1, 2, 3\dots) \\ -1 & \text{if } (i, j, k\dots) \text{ is odd permutation of } (1, 2, 3\dots) \\ 0 & \text{if any of the two indices is repeated} \end{cases}$$

$$\varepsilon_{ijk\cdots l\cdots m\cdots} = -\varepsilon_{ijk\cdots m\cdots l\cdots}$$

Determinant II

A determinant of n dimension can be expanded over a column (or a row) into a sum of n determinants of n-1 dimension:

$$D = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & \dots \end{vmatrix} = \sum_i C_{ij} a_{ij}$$

n-1 columns

$$M_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

n-1 rows

$C_{ij} = (-1)^{i+j} M_{ij}$ is called the ij[th] cofactor of D.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant III

- Multiplied by a constant

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- A determinant is equal to zero if any two columns (rows) are proportional

$$\begin{vmatrix} ka_{12} & a_{12} & a_{13} \\ ka_{22} & a_{22} & a_{23} \\ ka_{32} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$\sum_{i,j,k} \varepsilon_{ijk\dots} a_{i1} a_{j2} a_{k3} \cdots = \sum_{i,j,k} \varepsilon_{ijk\dots} a_{i2} a_{j2} a_{k3} \cdots$$

$$= \sum_{i,j,k} \varepsilon_{ijk\dots} a_{j2} a_{i2} a_{k3} \cdots = \sum_{i,j,k} \varepsilon_{jik\dots} a_{i2} a_{j2} a_{k3} \cdots$$

$$= -\sum_{i,j,k} \varepsilon_{ijk\dots} a_{i1} a_{j2} a_{k3} \cdots$$

$$\Rightarrow \sum_{i,j,k} \varepsilon_{ijk\dots} a_{i1} a_{j2} a_{k3} \cdots = 0; \text{ if } a_{i1} = a_{i2} \text{ for } \forall i$$

- The value of a determinant is unchanged if a multiple of one column (row) is added to another column (row)

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{12} & a_{12} & a_{13} \\ ka_{22} & a_{22} & a_{23} \\ ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Linear equation system

- Existence of non-trivial solution of homogeneous equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{cases}$$

$$x \cdot D \equiv x \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x + a_{12}y + a_{13}z & a_{12} & a_{13} \\ a_{21}x + a_{22}y + a_{23}z & a_{22} & a_{23} \\ a_{31}x + a_{32}y + a_{33}z & a_{32} & a_{33} \end{vmatrix} = 0$$

Similarly, $y \cdot D = 0$ and $z \cdot D = 0$

- Thus a set of homogenous linear equations have non-trivial solutions only if the determinant of the coefficients, D, vanishes.

Matrix: properties of the determinant of a matrix

- Some properties of the determinant of matrices
  - $\det(A^T) = \det(A)$
  - $\det(kA) = k^n \det(A)$
  - $\det(AB) = \det(A)\det(B)$

Proof of det(AB)=det(A)det(B):

(1)
$$\sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} a_{\beta i} a_{\alpha j} a_{\gamma k} \cdots = \sum_{j,i,k\ldots} \varepsilon_{jik\ldots} a_{\beta j} a_{\alpha i} a_{\gamma k} \cdots = \sum_{i,j,k\ldots} \varepsilon_{jik\ldots} a_{\alpha i} a_{\beta j} a_{\gamma k} \cdots = -\sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} a_{\alpha i} a_{\beta j} a_{\gamma k} \cdots$$

(2)
$$\sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} a_{\alpha i} a_{\beta j} a_{\gamma k} \cdots = \varepsilon_{\alpha\beta\gamma\ldots} \sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} a_{1i} a_{2j} a_{3k} \cdots = \varepsilon_{\alpha\beta\gamma\ldots} |A|$$

(3)
$$|AB| = \sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} (AB)_{1i} (AB)_{2j} (AB)_{3k} \cdots$$
$$= \sum_{i,j,k\ldots} \sum_{\alpha,\beta,\gamma} \varepsilon_{ijk\ldots} A_{1\alpha} B_{\alpha i} A_{2\beta} B_{\beta j} A_{2\gamma} B_{\gamma j} \cdots$$
$$= \sum_{\alpha,\beta,\gamma} A_{1\alpha} A_{2\beta} A_{2\gamma} \cdots \left\{ \sum_{i,j,k\ldots} \varepsilon_{ijk\ldots} B_{\alpha i} B_{\beta j} B_{\gamma j} \cdots \right\}$$
$$= |B| \sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma\ldots} A_{1\alpha} A_{2\beta} A_{2\gamma} \cdots$$
$$= |A||B|$$

Matrix: inversion

- Inversion of a square matrix A is to find a square matrix B such that

$$AB = BA = I$$

B is called the inverse matrix of A and often denoted by A^{-1}, i.e.

$$AA^{-1} = A^{-1}A = I$$

- One way to find the inverse matrix is by

$$\left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|}, \quad \text{where } C_{ji} \text{ is the } ji^{th} \text{ cofactor of } A$$

Matrix: similarity transformation and diagonalization

- Two matrix, A and B, are called similar if there exists an invertible matrix P such that

$$B = P^{-1}AP,$$

and the transformation from A to B is called similarity transformation.

- Diagnolization of a matrix, A, is to find a similarity transformation matrix, P, such that $P^{-1}AP$ is a diagonal matrix:

Matrix: diagonalization

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \quad \text{or} \quad AP = P\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

- If we look at the j[th] column of the second equation, it follows

$$\sum_k a_{ik} p_{kj} = \sum_k p_{ik} \lambda_k \delta_{kj} = \lambda_j p_{ij} \qquad (*)$$

Defining a $n \times 1$ matrix (i.e. a column vector) $\left|P^J\right\rangle$ such that $\left|P^j\right\rangle_i = p_{ij}$ (note: j is fixed)

Equation (*) becomes: $A\left|P^j\right\rangle = \lambda_j \left|P^j\right\rangle$

Matrix: eigenvalue and eigenvector

- For a matrix A, a vector matrix X is called an eigenvector of A if

$$A \cdot X = \lambda X$$

where λ is called the eigenvalue associated with the eigenvector X.

- The eigenvalues are found by solving the following polynomial equation

$$(A - \lambda I) \cdot X = 0 \Rightarrow \det(A - \lambda I) = 0$$

Defective Matrix

- Not all square matrix can be diagonalized:

$$A=\begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix};\ \det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -3 \\ 3 & -4-\lambda \end{vmatrix}=0 \Rightarrow (\lambda+1)^2=0 \Rightarrow \lambda=-1$$

$$\begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=-\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 3x_1-3x_2=0 \\ 3x_1-3x_2=0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

We end up with only one eigenvector.

- A square matrix that does not have a complete set of eigenvectors is not diagonalizable and is called a defective matrix.

- If a matrix, A, is defective (and hence is not similar to a diagonal matrix), then what is the simplest matrix that A is similar to?

Jordan form matrix

- Definition: a Jordan block with value λ is a square, upper triangular matrix whose entries are all λ on the diagonal, all 1 on the entries immediately above the diagonal, and zero elsewhere:

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

1D: $[\lambda]$

2D: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

3D: $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

- Definition: a Jordan form matrix is a block diagonal matrix whose blocks are all Jordan blocks

- Theorem: Let A be a $n \times n$ matrix. Then there is a Jordan form matrix that is similar to A.

$$\left[\begin{array}{cc|cc|c|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

Symplectic Matrix

- As long as the system has a Hamiltonian, the Jacobian matrix, $M_{\alpha\beta} = \frac{\partial X_\alpha}{\partial (X_0)_\beta}$, which describe the motion of the particles, satisfies

$$M^T S M = S \qquad (**)$$

$$\delta X(s) = M(s)\,\delta X(0)$$

$$S = \begin{pmatrix} S_{1D} & 0 & \dots & 0 \\ 0 & S_{1D} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & S_{1D} \end{pmatrix}; \qquad S_{1D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- A matrix satisfying condition (**) is called a symplectic matrix.
  - inverse: $M^T SM = S \Rightarrow SM^T SM = S^2 = -I \Rightarrow \left(-SM^T S\right)M = I \Rightarrow M^{-1} = -SM^T S$
  - if M and N are both symplectic, then their product, MN, is also symplectic $(MN)^T S(MN) = N^T M^T SMN = N^T SN = S$
  - if M is symplectic, M^T is also symplectic

$$\left(M^T SM\right)^{-1} = -S \Rightarrow M^{-1}S\left(M^T\right)^{-1} = S \Rightarrow S = MSM^T \Rightarrow \left(M^T\right)^T SM^T = S$$

Symplectic Matrix II

- If λ is an eigenvalue of a symplectic matrix M, then $1/\lambda$ is also an eigenvalue of M and the multiplicity of λ and $1/\lambda$ is the same.
  - It implies that the eigenvalues are coming in pairs $\{\lambda, 1/\lambda\}$.
- As a consequence of above property, the determinant of a symplectic matrix is 1.