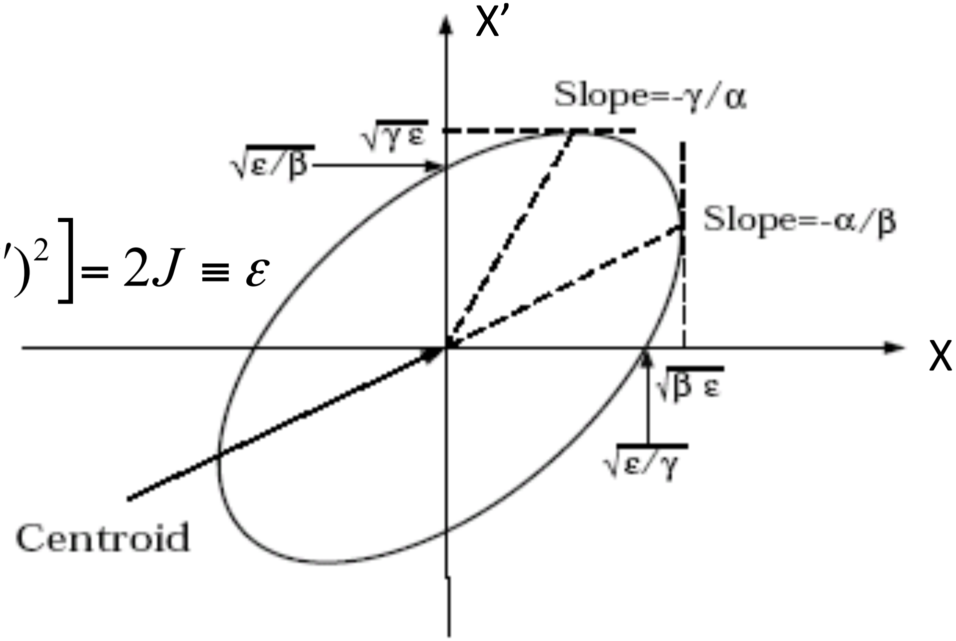
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

What we learned:

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Emittance of a beam

$$\langle X \rangle = \int X \rho(X, X') dX dX', \quad \langle X' \rangle = \int X' \rho(X, X') dX dX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X, X') dX dX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X, X') dX dX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle) \rho(X, X') dX dX' = r \sigma_X \sigma_{X'}$$

$$\varepsilon_{rms} = \sqrt{\sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X \sigma_{X'} \sqrt{1 - r^2}$$

The rms emittance is invariant in linear transport:

$$\frac{d\varepsilon^2}{ds} = 0$$

Normalized emittance $\varepsilon_n = \varepsilon\beta\gamma$ is invariant when beam energy is changed.

Adiabatic damping – beam emittance decreases with increasing beam momentum, i.e. $\varepsilon = \varepsilon_n/\beta\gamma$, which applies to beam emittance in **linacs**.

In storage rings, the beam emittance **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

The Gaussian distribution function

$$\rho(X, P_X) = \frac{1}{2\pi\sigma_X^2} e^{-(X^2+P_X^2)/2\sigma_X^2}$$

$$\rho(\varepsilon) = \frac{1}{2\varepsilon_{rms}} e^{-\varepsilon/2\varepsilon_{rms}}$$

$\epsilon/\epsilon_{\rm rms}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

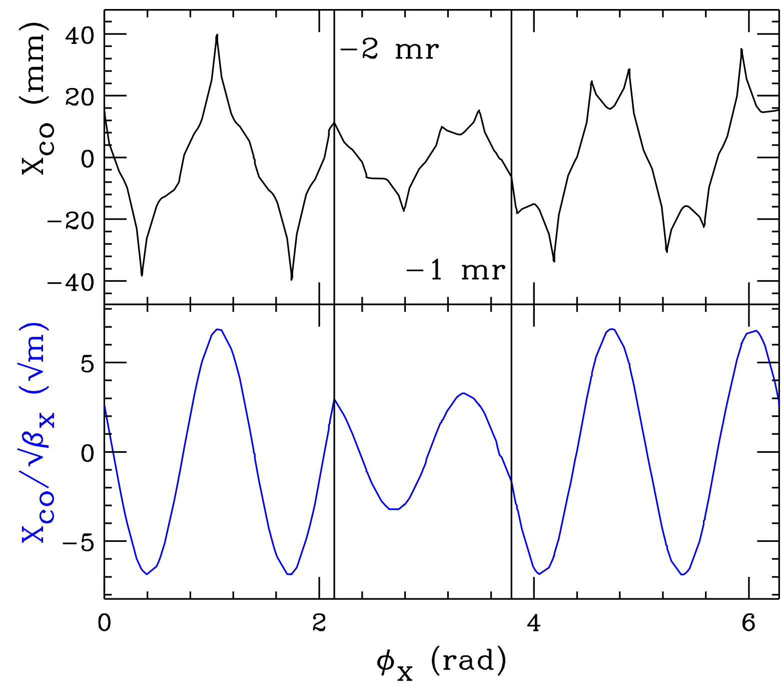
For a localized dipole field error: θ=ΔBℓ/Bρ

$$X'' + K_X(s)X = \theta\delta(s - s_0)$$

$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

$$X_{co}(s) = G(s, s_0)\theta$$



$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

Consider the closed orbit of a distributed dipole field error:

$$X_{\mathrm{co}}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\int_s^{s+C} ds_0\sqrt{\beta(s_0)}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]\frac{\Delta B(s_0)}{B\rho}$$

With coordinate transformation:

$$\varphi(s) = \frac{1}{\nu}\int_{s_0}^{s}\frac{ds}{\beta(s)}, \qquad \psi(s) = \nu\varphi(s)$$

we find

$$X_{\mathrm{co}}(s) = \frac{\nu\sqrt{\beta(s)}}{2\sin\pi\nu}\int_s^{s+C} d\varphi\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]\cos\nu[\pi - |\varphi(s)-\varphi|]$$

Expand the error in Fourier series:

$$\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right] = \sum_{k=-\infty}^{\infty} f_k e^{jk\varphi},$$

$$f_k = \frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi = \frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$X_{\mathrm{co}}(s) = \sqrt{\beta(s)}\sum_{k=-\infty}^{\infty}\frac{\nu^2 f_k}{\nu^2-k^2}e^{jk\varphi(s)} \xrightarrow[\nu\to k_0]{} \sqrt{\beta(s)}\frac{\nu|f_{k0}|\cos(k_0\varphi(s)+\xi_{k0})}{\nu-k_0}$$

Dipole field errors can be decomposed into harmonics. The harmonics nearest to the betatron tunes will produce large closed orbit distortion. Both the harmonic orbit correction and the χ-square correction methods essentially cancel the error harmonics nearest to the betatron tunes. For a distributed δ-dipole field error, we can carry out statistical analysis to the random error and obtain

$$X_{\rm co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\int_s^{s+C} ds_0\sqrt{\beta(s_0)}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]\theta\delta(s-s_0)$$

$$= \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\sum_i \sqrt{\beta(s_i)}\theta_i\cos[\pi\nu - |\psi(s)-\psi(s_i)|]$$

$$\left\langle \left(X_{\rm co}(s)\right)^2\right\rangle^{1/2} = \frac{\sqrt{\beta(s)}}{2\sqrt{2}\sin\pi\nu}\sqrt{\sum_i \beta(s_i)\theta_i^2} \approx \frac{\sqrt{\beta(s)}}{2\sqrt{2}\sin\pi\nu} N\sqrt{\bar{\beta}}\,\theta_{rms}$$

The sensitivity factor of an accelerator is defined as

$$\text{Sensitivity factor} \equiv \frac{\left\langle \left(X_{\rm co}(s)\right)^2\right\rangle^{1/2}}{\theta_{rms}} \approx \frac{\sqrt{\beta(s)}}{2\sqrt{2}\sin\pi\nu} N\sqrt{\bar{\beta}}$$

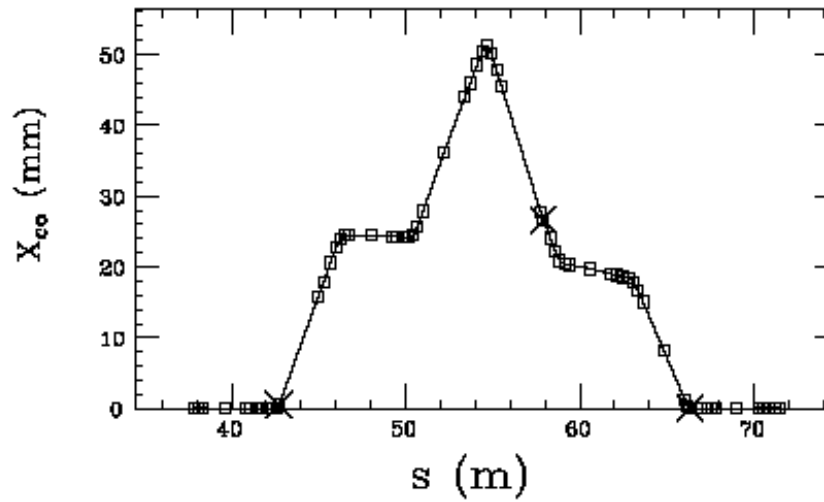
Applications of dipole field error:

closed orbit bump:

$$X_{co}(s) = G(s,s_0)\theta \qquad G(s,s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]$$

$$X_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\sum_{i=1}^{4}\sqrt{\beta(s_i)}\theta_i\cos(\pi\nu - |\psi(s)-\psi(s_i)|)$$

where θi = $(\Delta Bs)_i/B\rho$ and $(\Delta Bs)_i$ are the kick-angle and the integrated dipole field strength of the i-th kicker. The conditions that the closed orbit is zero outside these four dipoles are $X_{co}(s_4) = 0$, $X'_{co}(s_4) = 0$.

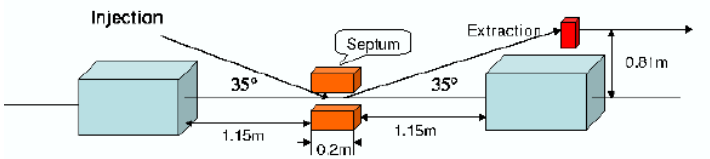


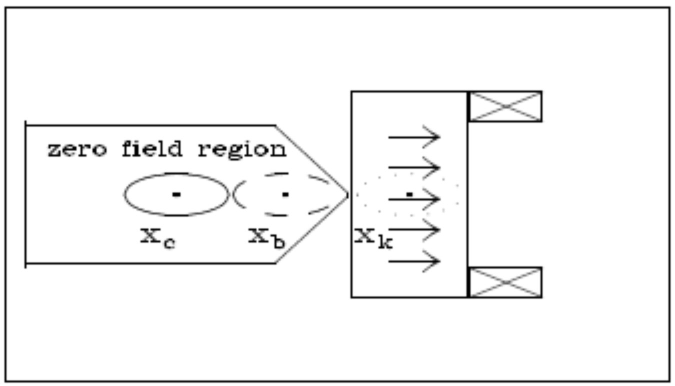
$$\sqrt{\beta_1}\theta_1\cos(\pi\nu-\psi_{41}) + \sqrt{\beta_2}\theta_2\cos(\pi\nu-\psi_{42}) + \sqrt{\beta_3}\theta_3\cos(\pi\nu-\psi_{43}) + \sqrt{\beta_4}\theta_4\cos\pi\nu = 0$$

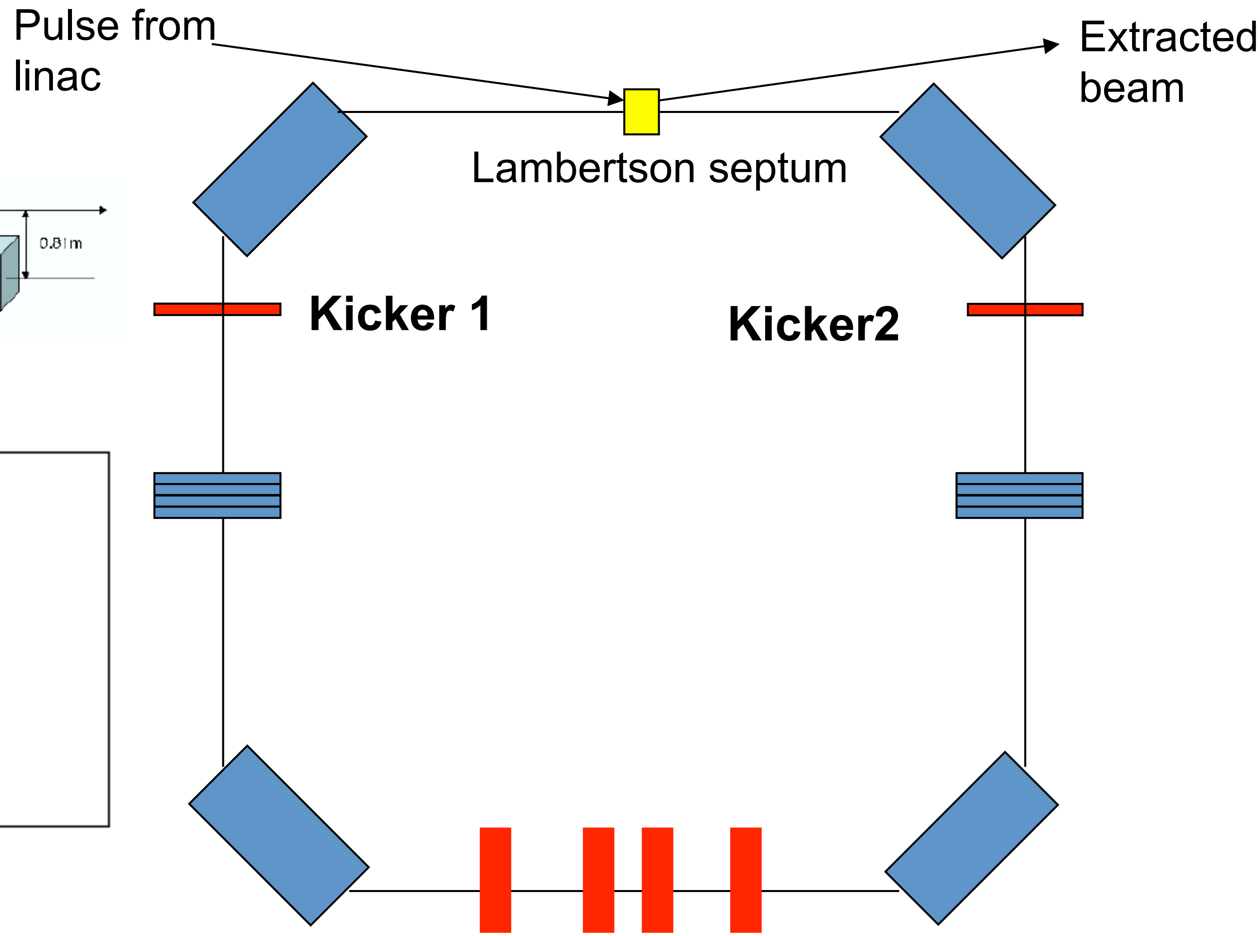
$$\sqrt{\beta_1}\theta_1\sin(\pi\nu-\psi_{41}) + \sqrt{\beta_2}\theta_2\sin(\pi\nu-\psi_{42}) + \sqrt{\beta_3}\theta_3\sin(\pi\nu-\psi_{43}) + \sqrt{\beta_4}\theta_4\sin\pi\nu = 0$$

$$\Rightarrow \quad \sqrt{\beta_3}\theta_3 = -(\sqrt{\beta_1}\theta_1\sin\psi_{41} + \sqrt{\beta_2}\theta_2\sin\psi_{42})/\sin\psi_{43}$$

$$\sqrt{\beta_4}\theta_4 = (\sqrt{\beta_1}\theta_1\sin\psi_{31} + \sqrt{\beta_2}\theta_2\sin\psi_{32})/\sin\psi_{43}$$







$$x_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \sum_{i=1}^{2} \sqrt{\beta_i}\theta_i \cos\left(\pi\nu - |\psi(s) - \psi(s_i)|\right)$$

Condition for localized closed orbit :

$$\sqrt{\beta_1}\theta_1 \cos(\pi\nu) + \sqrt{\beta_2}\theta_2 \cos(\pi\nu - \psi_{21}) = 0$$

$$\sqrt{\beta_1}\theta_1 \sin(\pi\nu) + \sqrt{\beta_2}\theta_2 \sin(\pi\nu - \psi_{21}) = 0$$

$$\psi_{21} = \pi \quad \text{and} \quad \theta_2 = \theta_1 \sqrt{\frac{\beta_1}{\beta_2}}$$

Kicker Strength

Electrostatic kicker:

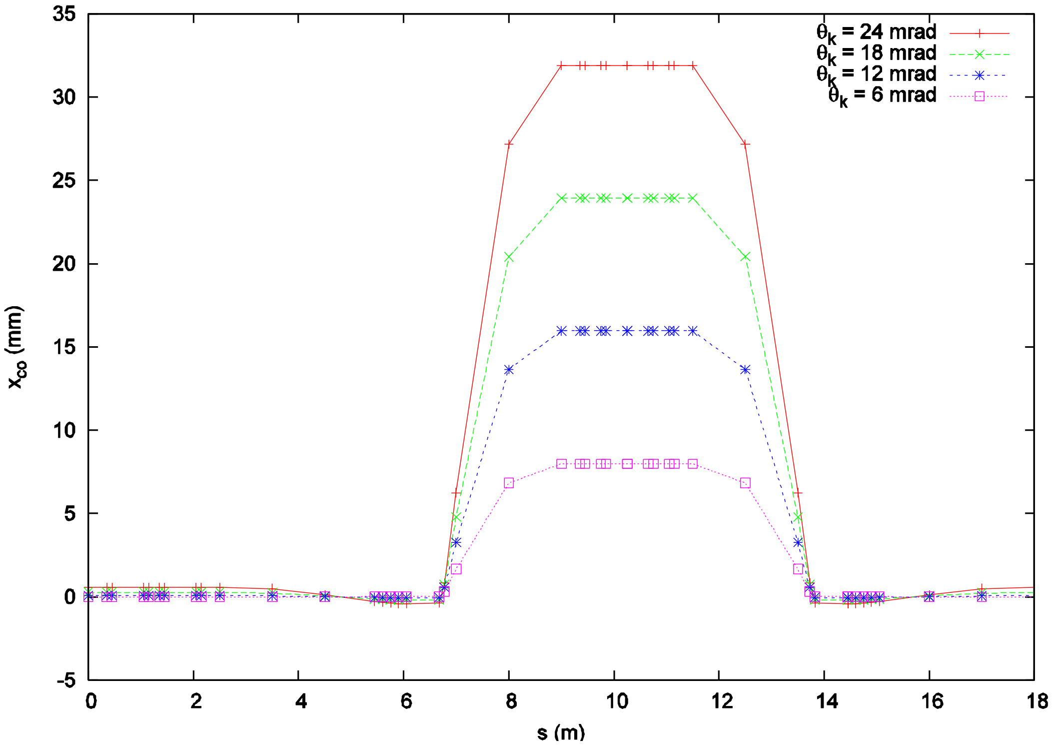
$$\theta_k = \frac{E \cdot L}{c \cdot B\rho} \text{ , where}$$

$B\rho = 0.2[Tm]$ at 60 MeV

L = length of the kicker

c = speed of light

E = gap electric field



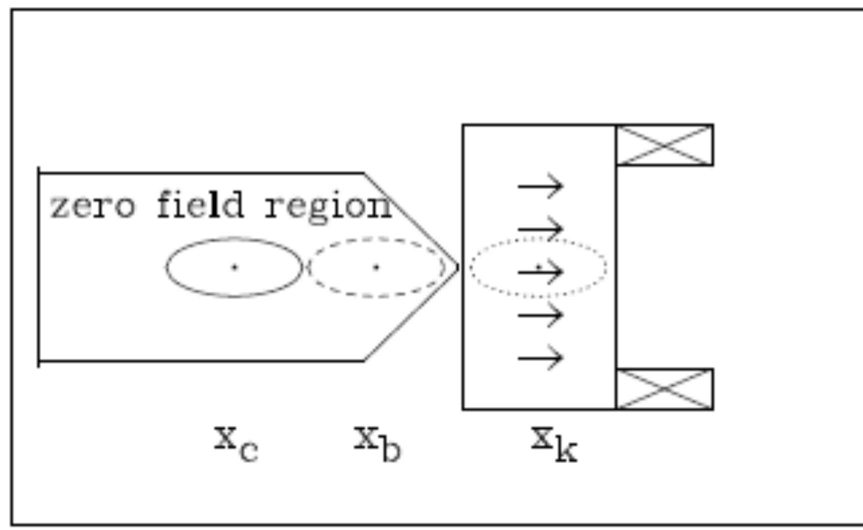
For one turn injection and extraction, the integrated field strength is 0.60 MV at 25 MeV electron beam energy. Choosing a length of L=0.5 m, the applied voltage on two plate is 60 kV.

Injection and extraction kicker

$$\Delta x_{co}(s) = \left\{\sqrt{\beta_x(s_k)\beta_x(s)}\,\sin(\Delta\psi_x(s))\right\}\theta_k$$

$\theta_k = \int B_k ds/B\rho$ is the kicker strength (angle), B_k is the kicker dipole field, $\beta_x(s_k)$ is the betatron amplitude function evaluated at the kicker location, $\beta_x(s)$ is the amplitude function at location s, and $\Delta\phi_x(s)$ is the phase advance from s_k of the kicker to location s. The quantity in curly brackets is called the **kicker lever arm**.

A schematic drawing of the central orbit x_c, bumped orbit x_b, and kicked orbit x_k in a Lambertson septum magnet. The blocks marked with X are conductor-coils, The ellipses marked beam ellipses with closed orbits x_c, x_b, and x_k. The arrows indicated a possible magnetic field direction for directing the kicked beams downward or upward in the extraction channel.



Effect of dipole field error on orbit length

The path length of the reference orbit in the Frenet-Serret coordinate system is

$$C = \oint \sqrt{(1 + x/\rho)^2 + x'^2 + y'^2}\, ds \approx C_0 + \oint \frac{x}{\rho} ds + \cdots$$

C_0 is the orbit length of the unperturbed orbit, and higher order terms associated with betatron motion are neglected. Since a dipole field error gives rise to a closed-orbit distortion, the circumference of the closed orbit may be changed as well. We consider the closed-orbit change due to a single dipole kick at $s = s_0$ with kick angle θ_0, the change in circumference as

$$\Delta C = C - C_0 = \theta_0 \oint \frac{G_x(s, s_0)}{\rho} ds = D(s_0)\theta_0$$

$$D(s_0) = \oint \frac{G_x(s, s_0)}{\rho} ds = \frac{\sqrt{\beta_x(s_0)}}{2 \sin \pi \nu_x} \oint \frac{\sqrt{\beta_x(s)}}{\rho} \cos(\pi \nu_x - |\psi_x(s) - \psi(s_0)|) ds$$

$$\Delta C = \oint D(s_0) \frac{\Delta B_y(s_0)}{B\rho} ds_0$$

Off-momentum closed orbit and dispersion function

We have discussed the closed orbit for a reference particle with momentum p_0, including dipole field errors and quadrupole misalignment. By using closed-orbit correctors, we can achieve an optimized closed orbit that essentially passes through the center of all accelerator components. This closed orbit is called the "golden orbit," and a particle with momentum p_0 is called a **synchronous** particle. However, a beam is made of particles with momenta distributed around a synchronous momentum p_0. What happens to particles with momenta different from p_0? Here we study the effect of off-momentum on the closed orbit. For a particle with momentum p, the momentum deviation is $\Delta p = p - p_0$ and the fractional momentum deviation is $\delta = \Delta p / p_0$, which is typically small of the order of 10^{-6} to 10^{-3}. Since δ is small, we can study the motion of off-momentum particles perturbatively.

$$p = p_0 + \Delta p, \ \delta = \frac{\Delta p}{p_0} \qquad x'' - \frac{\rho + x}{\rho^2} = \left(-\frac{1}{\rho} + Kx\right)\frac{1}{1+\delta}\left(1 + 2\frac{x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}$$

$$x'' + \left(\frac{1}{\rho^2} - K(s)\right)x = \frac{\delta}{\rho} \qquad K(s) = K_1(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_z}{\partial x}$$

The bending angle resulting from a dipole field is different for particles with different momenta. i.e. nonzero δ. The resulting betatron equation of motion is inhomogeneous. The solution of an in-homogeneous linear equation of motion is a linear superposition of the particular solution and the solution of the homogeneous equation, i.e.

$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$

The solution of the homogeneous equation is the betatron oscillation we have discussed earlier. The solution of the inhomogeneous equation is called the dispersion function, or the off-momentum closed orbit.

$$x = x_\beta + x_{\mathrm{co}} = x_\beta + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s)\right)D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

For a pure dipole (K=0):

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathrm{L} & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

When $\theta << 1 \quad i.e. \quad L << \rho$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For quadrupoles:

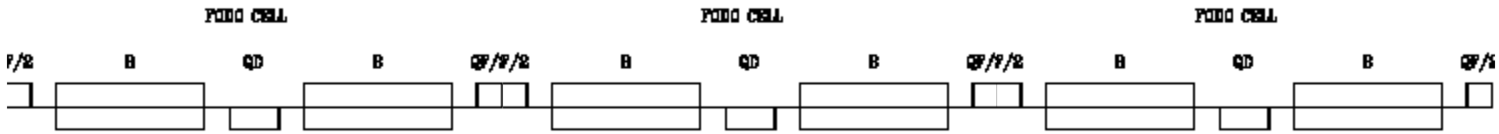
$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell & 0 \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For combined function magnets:

$$\bar{d} = \begin{pmatrix} \frac{1}{\rho K_x}(1-\cos\sqrt{K_x}\ell) \\ \frac{1}{\rho\sqrt{K_x}}\sin\sqrt{K_x}\ell \end{pmatrix}$$

Example: FODO cell



$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{L^2}{2f^2} & 2L(1+\dfrac{L}{2f}) & 2L\theta(1+\dfrac{L}{4f}) \\ -\dfrac{L}{2f^2}+\dfrac{L^2}{4f^3} & 1-\dfrac{L^2}{2f^2} & 2\theta(1-\dfrac{L}{4f}-\dfrac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

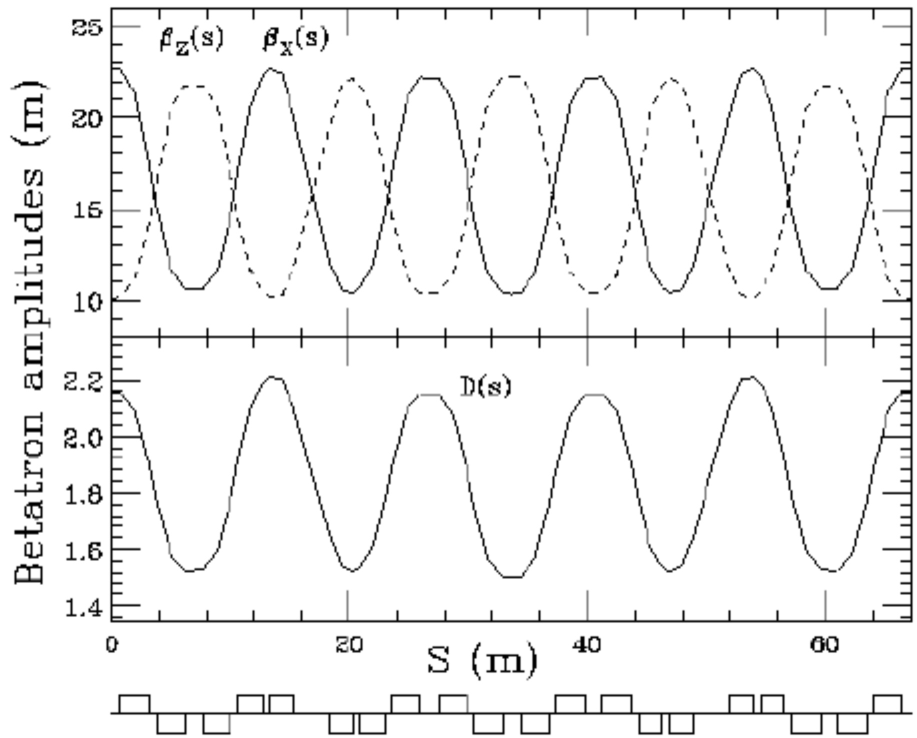
Using the Courant-Snyder parameterization for the transfer matrix, we obtain

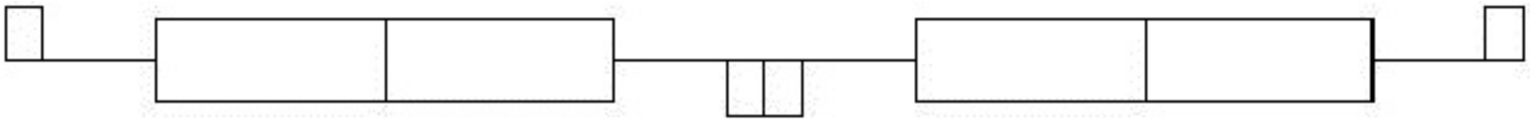
$$\sin\frac{\Phi}{2} = \frac{L}{2f}, \quad \beta_F = \frac{2L(1+\sin\frac{\Phi}{2})}{\sin\Phi}, \quad \alpha_F = 0 \qquad D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$

The dispersion is proportional to the cell length *L* times the bending angle θ, and inversely proportional to the square of the phase advance.
The dispersion at other locations can be obtained by using the 3×3 transfer matrix $M(s_2,s_1)$.

The AGS (33 GeV proton synchrotron built in 1960) is made of 60 (5×12) FODO cells. The CPS (28 GeV) is made of 50 FODO cells.

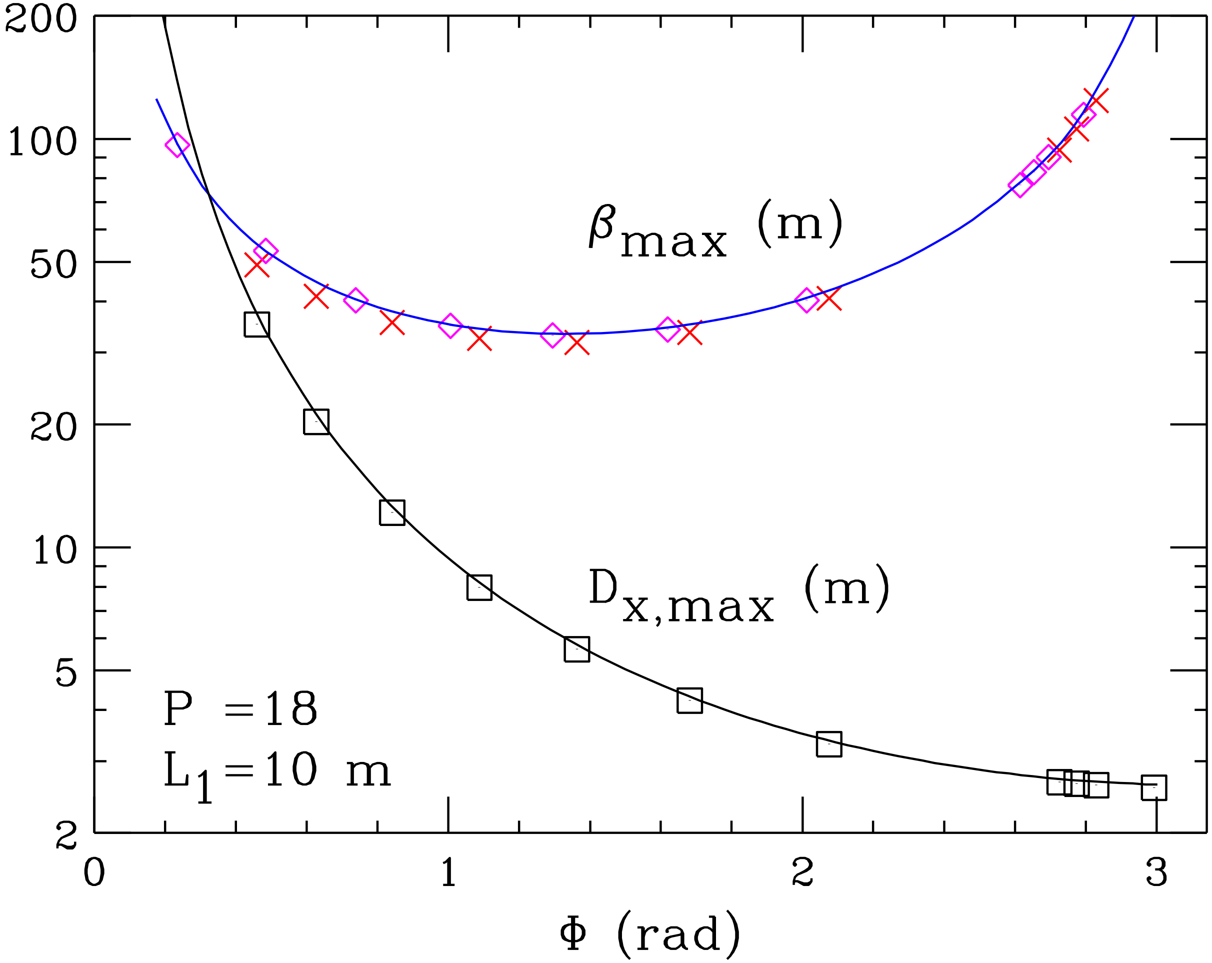
The betatron amplitude functions for one superperiod of the AGS lattice, made of 20 combined-function magnets. The upper plot shows β_x (solid) and β_y (dashed). The middle plot shows the dispersion function D_x. The lower plot shows schematically the placement of combined-function magnets. The superperiod can be approximated by five FODO cells. The phase advance of each FODO cell is about 52.8°.



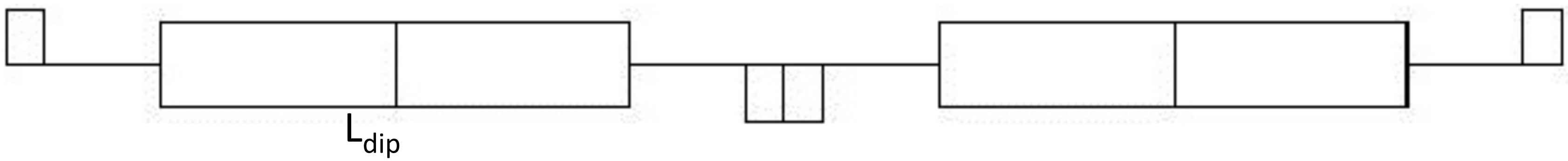


$$\beta_{max} = \frac{2L_1\left(1+\frac{L_1}{2f}\right)}{\sin\Phi} = \frac{2L_1\left(1+\sin\frac{\Phi}{2}\right)}{\sin\Phi}$$

$$D_F = \frac{L\theta\left(1+\frac{1}{2}\sin\frac{\Phi}{2}\right)}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$



What is the effect of **bending radius** on dispersion function?



L=10m, L_Q=1m, P=18

L_{dip} = 2, 4, 6, 8 m

$$D_{F/D} = \frac{L\theta[1 \pm \frac{1}{2}\sin(\Phi/2)]}{\sin^2(\Phi/2)}$$

θ=π/P