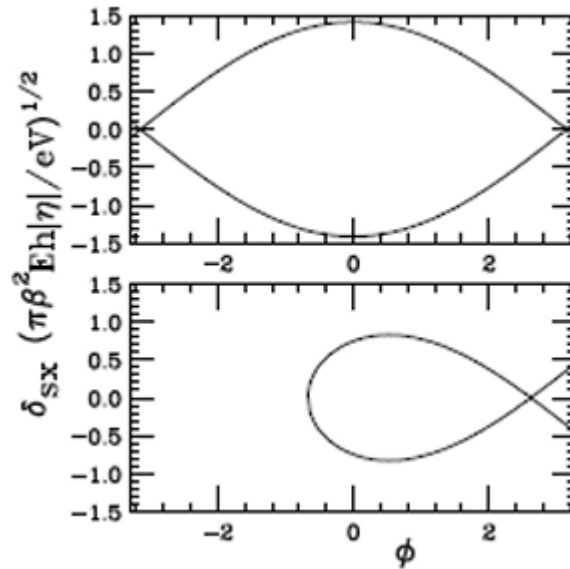


PHY 554

Fundamentals of Accelerator Physics

Lecture 13: Synchrotron motion



Stable conditions for longitudinal motion in accelerators

The longitudinal electric field at an rf gap

$$\mathcal{E} = \mathcal{E}_0 \sin(\phi_{\text{rf}}(t) + \phi_s), \quad \phi_{\text{rf}} = h\omega_0 t$$

where $\omega_0 = \beta_0 c / R_0$ is the angular revolution frequency of a reference (synchronous) particle, $\mathcal{E}_0$ is the amplitude of the electric field, $\beta_0 c$ and R_0 are respectively the speed and the average radius of the reference orbiting particle, h is the harmonic number, and ϕ_s is the phase angle for a synchronous particle.

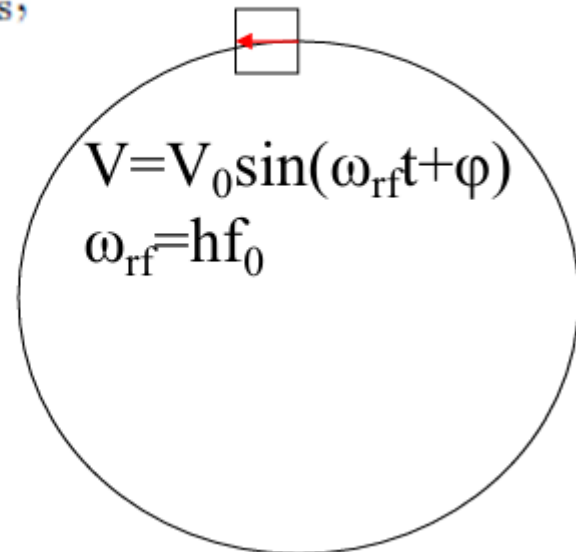
We assume that the reference particle passes through the cavity gap in time $t \in nT_0 + (-g/2\beta_0 c, g/2\beta_0 c)$ ($n = \text{integer}$), where g is the rf cavity gap width. The energy gain for the reference particle per passage is

$$\Delta E = e\mathcal{E}_0\beta_0 c \int_{-g/2\beta_0 c}^{g/2\beta_0 c} \sin(h\omega_0 t + \phi_s) dt = e\mathcal{E}_0 g T \sin \phi_s,$$

$$T = \frac{\sin(hg/2R_0)}{(hg/2R_0)}$$

The effective voltage seen by the orbiting particle is $V = \mathcal{E}_0 g T$. The acceleration rate for a synchronous particle is

$$\dot{E}_0 = \frac{\omega_0}{2\pi} eV \sin \phi_s,$$



We consider a non-synchronous particle with small deviations of rf parameters from the synchronous particle, i.e.

$$\begin{cases} \omega = \omega_0 + \Delta\omega, & \phi = \phi_s + \Delta\phi, & \theta = \theta_s + \Delta\theta, \\ p = p_0 + \Delta p, & E = E_0 + \Delta E. \end{cases}$$

Here $\phi_s, \theta_s, \omega_0, p_0, E_0$ are respectively the rf phase angle, azimuthal orbital angle, angular revolution frequency, momentum, and energy of a synchronous particle, and $\phi, \theta, \omega, p, E$ are the corresponding parameters for an off-momentum particle. The phase coordinate is related to the orbital angle by $\Delta\phi = \phi - \phi_s = -h\Delta\theta$, or

$$\Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h}\frac{d}{dt}\Delta\phi = -\frac{1}{h}\frac{d\phi}{dt}.$$

The energy gain per revolution for this non-synchronous particle is $eV \sin \phi$, where ϕ is the rf phase angle. Thus the acceleration rate of a non-synchronous particle is

$$\dot{E} = \frac{\omega}{2\pi} eV \sin \phi.$$

The equation of motion for the energy difference is $\frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = \frac{1}{2\pi} eV (\sin \phi - \sin \phi_s)$

$$\frac{1}{\omega} \dot{E} - \frac{1}{\omega_0} \dot{E}_0 = \frac{1}{\omega_0} \dot{\Delta E} - \dot{E} \frac{\Delta\omega}{\omega_0^2} \approx \frac{1}{\omega_0} \dot{\Delta E} + \left[\dot{E} \frac{\Delta(1/\omega_0)}{\Delta E} \right] \Delta E + \dots = \frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right)$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = \frac{1}{2\pi} eV (\sin \phi - \sin \phi_s) \quad \delta = \frac{\Delta p}{p_0} = \frac{\omega_0}{\beta^2 E} \frac{\Delta E}{\omega_0}$$

$$\dot{\delta} = \frac{\omega_0}{2\pi \beta^2 E} eV (\sin \phi - \sin \phi_s)$$

The time evolution of the phase angle variable ϕ is

$$\dot{\phi} = -h(\omega - \omega_0) = -h\Delta\omega \quad \Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h} \frac{d}{dt}\Delta\phi = -\frac{1}{h} \frac{d\phi}{dt}$$

$$\omega R / \omega_0 R_0 = \beta / \beta_0 \quad \frac{\Delta\omega}{\omega_0} = \frac{\beta R_0}{\beta_0 R} - 1.$$

$$R = R_0(1 + \alpha_0\delta + \alpha_1\delta^2 + \alpha_2\delta^3 + \dots),$$

$$\alpha_c = \frac{1}{R_0} \frac{dR}{d\delta} = \alpha_0 + 2\alpha_1\delta + 3\alpha_2\delta^2 + \dots \equiv \frac{1}{\gamma_T^2}$$

Let $p = mc\beta\gamma = p_0 + p$ be the momentum of a non-synchronous particle.

The fractional off-momentum coordinate δ is

$$\delta = \frac{\Delta p}{p_0} = \frac{\beta\gamma}{\beta_0\gamma_0} - 1.$$

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = \left(\alpha_c - \frac{1}{\gamma^2}\right) \frac{\Delta p}{p_0} = \eta \delta,$$

$$\eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

$$R = R_0(1 + \alpha_0 \delta + \alpha_1 \delta^2 + \alpha_2 \delta^3 + \dots)$$

$$\alpha_c = \frac{dR}{R_0 d\delta} = \alpha_0 + 2\alpha_1 \delta + 3\alpha_2 \delta^2 + \dots \equiv \frac{1}{\gamma_T^2}$$

$$\frac{\Delta \omega}{\omega_0} = -\eta(\delta) \delta = -(\eta_0 + \eta_1 \delta + \eta_2 \delta^2 + \dots) \delta$$

$$\eta_0 = \left(\alpha_0 - \frac{1}{\gamma_0^2}\right), \quad \eta_1 = \frac{3\beta_0^2}{2\gamma_0^2} + \alpha_1 - \alpha_0 \eta_0, \dots\dots$$

For the moment, we neglect all nonlinear effects in the phase slip factor. The equation of motion is given by

$$\frac{d}{dt}(\phi - \phi_s) = -h\Delta\omega = h\omega_0 \frac{\Delta T}{T_0} = h\eta\omega_0 \frac{\Delta p}{p_0} = \frac{\eta h \omega_0^2}{\beta^2 E_0} \frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = \frac{1}{2\pi} e V_0 (\sin \phi - \sin \phi_s),$$

$$\dot{\delta} = \frac{\omega_0}{2\pi \beta^2 E} e V (\sin \phi - \sin \phi_s)$$

Linearization – Small-amplitude synchrotron oscillation

- Let $\psi = \phi - \phi_s$ and assume $|\psi| \ll 1$.
- Expand the RF voltage around the synchronous phase.
- Combine the two first-order equations into a second-order equation.

$$\sin(\phi_s + \psi) - \sin \phi_s \simeq \psi \cos \phi_s$$

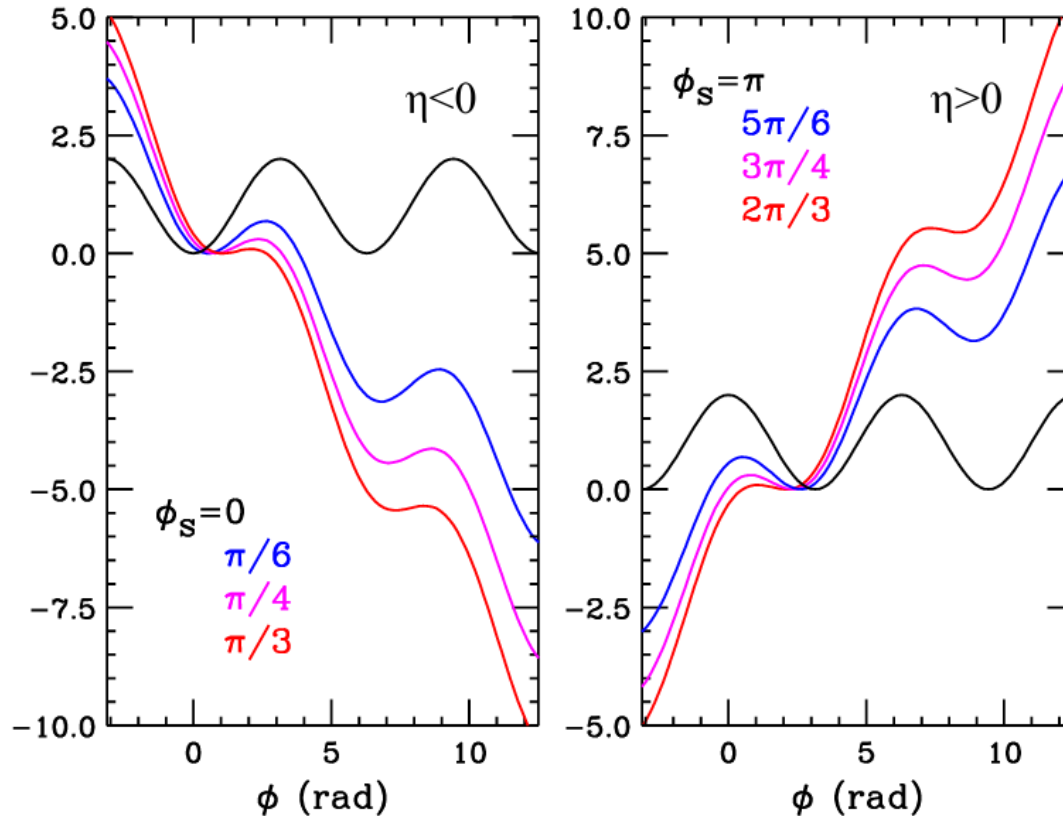
$$\ddot{\psi} = \frac{h\omega_0^2 \eta e V \cos \phi_s}{2\pi\beta^2 E} \psi$$

$$\ddot{\psi} + \omega_s^2 \psi = 0, \quad \omega_s^2 = \frac{h\omega_0^2 e V |\eta \cos \phi_s|}{2\pi\beta^2 E}$$

stability requires $\eta \cos \phi_s < 0$

The sign of η decides where the stable phase is

Potential well for different η



- Below transition: $\gamma < \gamma_T$, so $\eta < 0$.
- Stable acceleration uses $0 < \phi_s < \pi/2$.
- Above transition: $\gamma > \gamma_T$, so $\eta > 0$.
- The stable accelerating phase shifts to $\pi - \phi_s$.

phase stability condition

$$\eta \cos \phi_s < 0$$

$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV \eta_0 \cos \phi_s}{2\pi\beta^2 E}(\phi - \phi_s)$$

The angular synchrotron frequency is $\omega_s = \omega_0 \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}} = \frac{c}{R} \sqrt{\frac{heV|\eta \cos \phi_s|}{2\pi E}}$

where c is the speed of light and R is the average radius of the synchrotron. The synchrotron tune, defined as the number of synchrotron oscillations per revolution, is

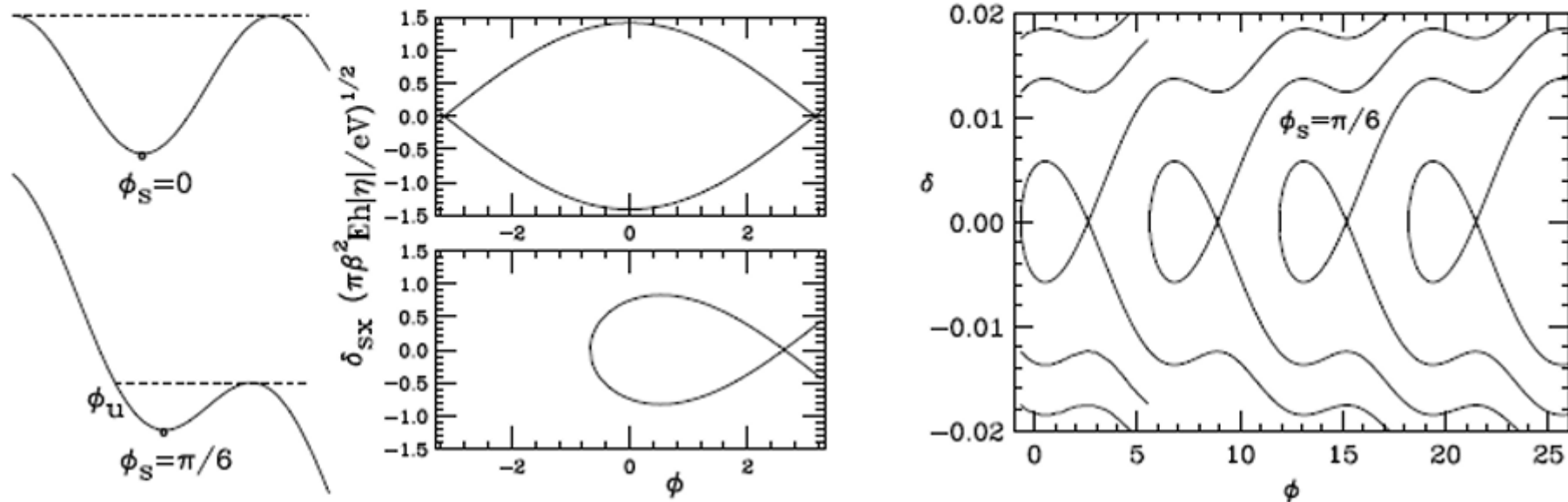
$$Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}}$$

Typically the synchrotron tune is of the order of $\leq 10^{-3}$ for proton synchrotrons and 10^{-1} for electron storage rings.

P-synchrotron	AGS	RHIC	FNAL-MI	FNAL-BST	SSC	Cooler
K.E. [GeV/u]	0.2	28	8	0.2	2000	0.045
V_{rf} [MV]	0.3	0.3	2	0.95	10	0.0001
h	12	342	588	84	17424	1
γ_T	8.5	24.5	21.8	5.446	140	4.6
C [m]	807.12	3833.84	3319.4	474.2	87120	86.8
ν_s	3.23E-02	5.89E-04	1.37E-02	1.50E-01	8.39E-04	3.95E-04

At 0.4 GeV injection, the synchrotron tune becomes 0.0924 for FNAL-Booster

Synchrotron bucket area, separatrix



Left: schematic drawing of the rf potentials for $\phi_s = 0$ and $\pi/6$. The dashed line shows the maximum "energy" for stable synchrotron motion. Middle: the corresponding separatrix orbits in $(\pi h |\eta| / e V \beta^2 E)^{1/2} E_{sx}$ vs ϕ . The phase ϕ_u is the turning point of the separatrix orbit. Right: an example of stable rf buckets, called fish diagram, with $\phi_s = \pi/6$.

In what we derived, the rf electric field is considered to be uniformly distributed in an accelerator. In reality, rf cavities are localized in a short section of a synchrotron, and therefore synchrotron motion is more realistically described by the symplectic mapping equation

$$\begin{cases} \delta_{n+1} = \delta_n + \frac{eV}{\beta^2 E}(\sin \phi_n - \sin \phi_s), \\ \phi_{n+1} = \phi_n + 2\pi h\eta(\delta_{n+1})\delta_{n+1}. \end{cases}$$

The physics of the mapping equation can be visualized as follows. First, the particle gains or loses energy at its n th passage through the rf cavity, then the rf phase ϕ_{n+1} depends on the new off-momentum coordinate δ_{n+1} . It is no surprise that satisfies the symplectic condition:

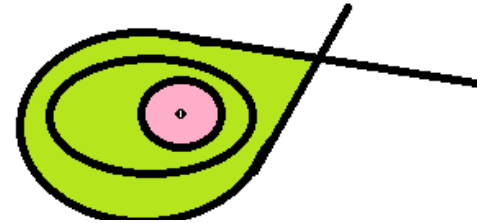
$$\text{Jacobian} = \frac{\partial(\delta_{n+1}, \phi_{n+1})}{\partial(\delta_n, \phi_n)} = 1.$$

The mapping from (ϕ_n, δ_n) to $(\phi_{n+1}, \delta_{n+1})$ preserves phase-space area. The phase-space area enclosed by a trajectory (ϕ, δ) obtained from the above mapping equation is independent of energy. It can not be used in tracking simulations of beam acceleration. During beam acceleration, the phase-space area in $(\phi, E/\omega_0)$ is invariant. The phase-space mapping equation for phase-space coordinates $(\phi, E/\omega_0)$ should be used. The adiabatic damping of phase-space area can be obtained by transforming phase-space coordinates $(\phi, E/\omega_0)$ to (ϕ, δ) .

Fixed points of a Hamiltonian are located at phase space points with zero velocity field:

$$H(q, p, t): \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = 0, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0.$$

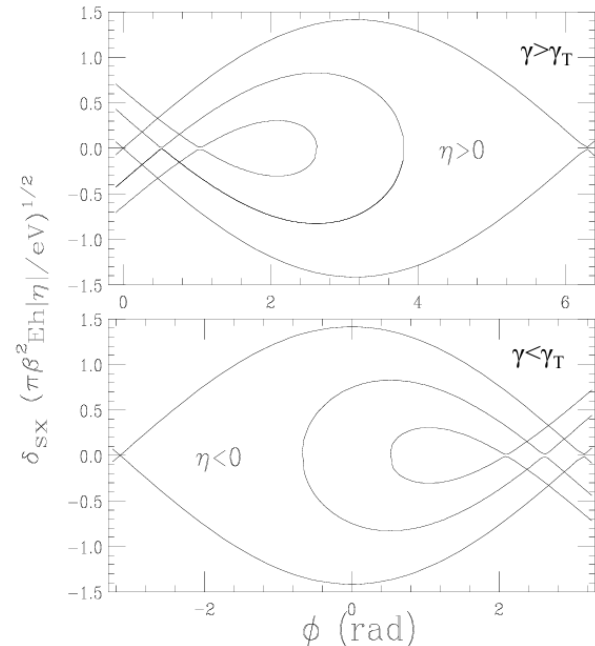
$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\eta\omega_0\delta \quad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 e V_0}{2\pi\beta^2 E} (\sin \phi - \sin \phi_s)$$



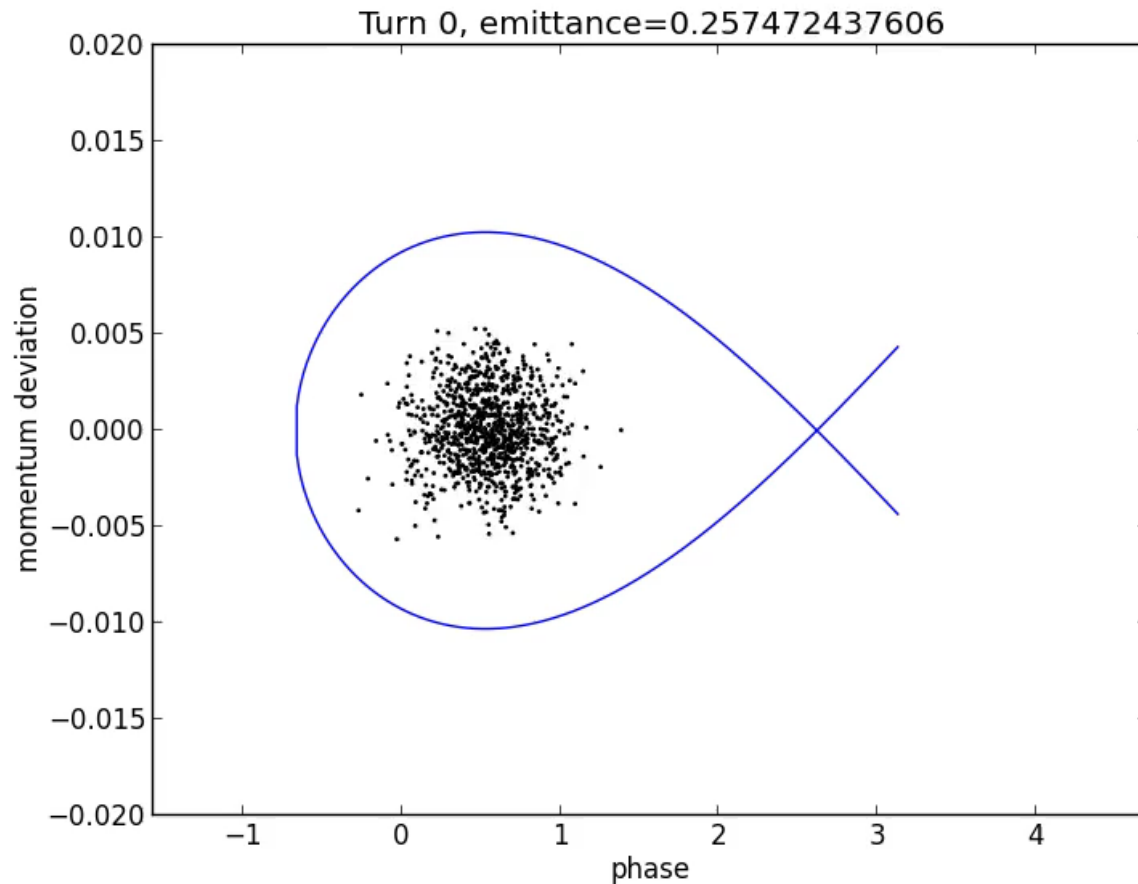
The fixed points of the synchrotron Hamiltonian are at phase space points: $(\phi_s, 0)$ and $(\pi - \phi_s, 0)$. Small amplitude motion around the stable fixed point $(\phi_s, 0)$ is elliptical with synchrotron tune Q_s . Motion near the UFP is hyperbolical.

- The separatrix passes through the unstable fixed point.
- Turning points determine the bucket length.

Separatrix = last stable longitudinal trajectory



Phase Space Area Examples and Evolution I

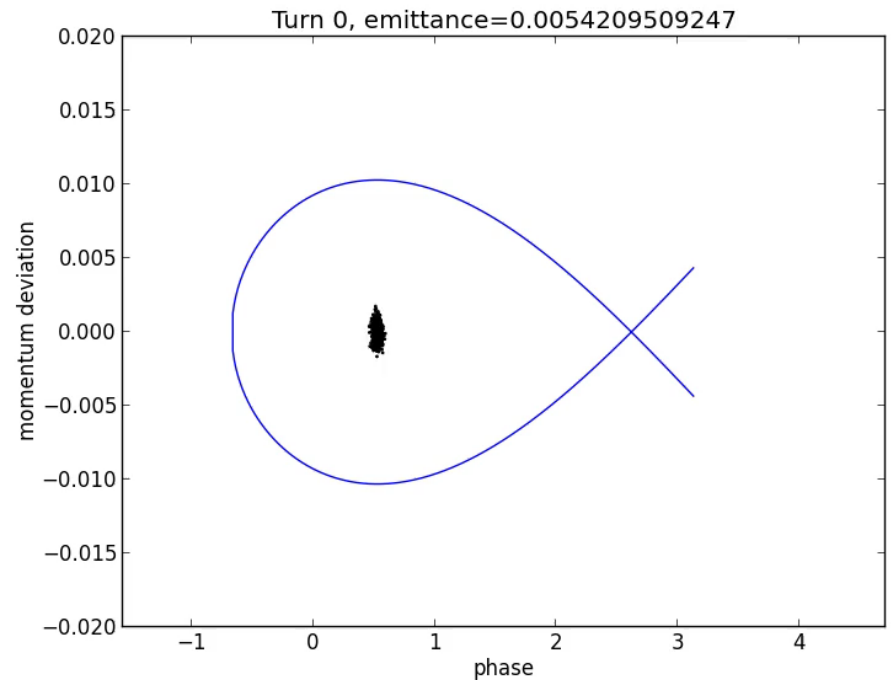
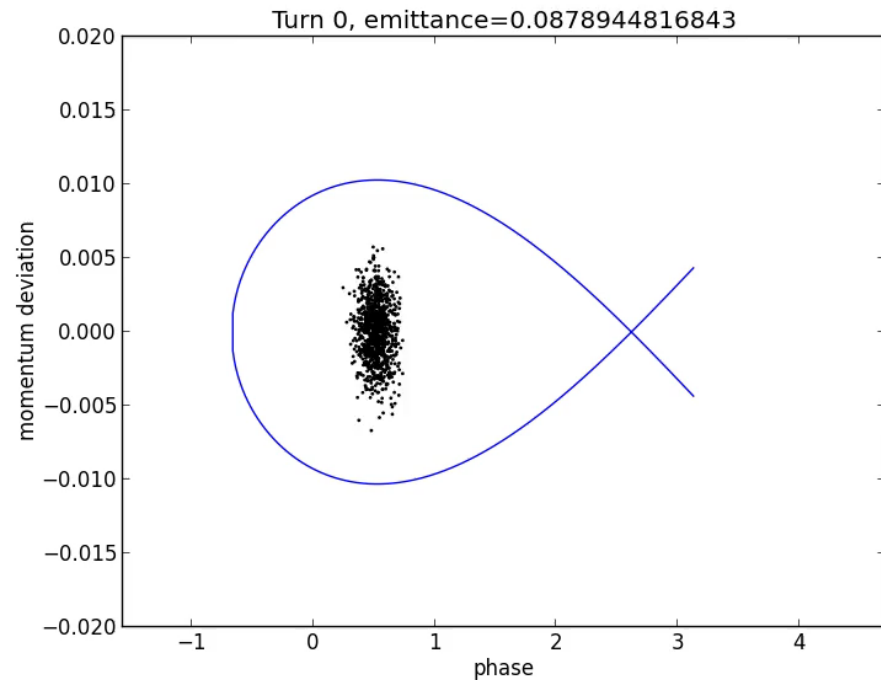


A Matched case
(Perfect injection):

Initial conditions match:

$$\frac{a_{\delta}}{a_{\phi}} = \frac{Q_s}{h|\eta|}$$

Phase Space Area Examples and Evolution II

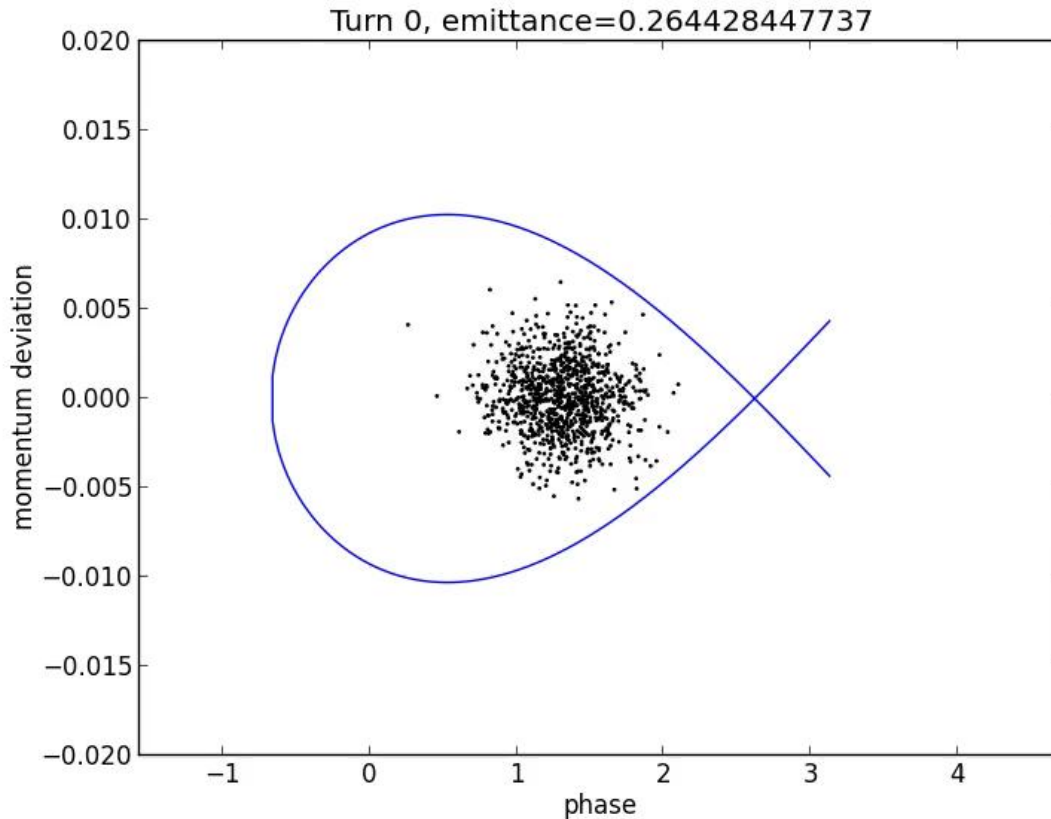


An unmatched case

$$\frac{a_{\delta}}{a_{\phi}} = \frac{3Q_s}{h|\eta|}$$

Phase Space Area

Examples and Evolution III



Time jitter at injection, other wise same as the matched case:

The phase error is:

$$\phi_{err} = \pi / 4$$

What we should remember from Lecture 13

- RF cavities provide phase-dependent energy kicks: $\Delta E \propto \sin\phi$.
- The slip factor η converts energy error into arrival-time error.
- Small-amplitude motion is harmonic when $\eta \cos\phi_s < 0$.
- The synchrotron tune scales like $\sqrt{h V |\eta \cos\phi_s| / E}$.
- Large-amplitude nonlinear motion is organized by RF buckets and separatrices.
- Bucket area and bucket height describe longitudinal acceptance.

core equation

$$\ddot{\psi} + \omega_s^2 \psi = 0$$

only after linearization

RF force + phase slip = longitudinal focusing